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The First Order Saddlepoint Approximation for Reliability Analysis

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Abstract

In the approximation methods of reliability analysis, non-normal random variables are transformed into standard normal random variables. This transformation tends to increase the nonlinearity of a limit-state function and hence results in less accurate reliability approximation. The First Order Saddlepoint Approximation for reliability analysis is proposed to improve the accuracy of reliability analysis. By approximating a limit-state function at the Most Likelihood Point in the original random space and employing the accurate saddlepoint approximation, the proposed method reduces the chance of increasing nonlinearity of the limit-state function. This approach generates more accurate reliability approximation than the First Order Reliability Method without increasing the computational effort. The effectiveness of the proposed method is demonstrated with two examples and is compared with the First and Second Order Reliability Methods.

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Nomenclature

E	=	expectation
F	=	cumulative distribution function
f	=	probability density function
g	=	limit-state function
$\mathbf{H}$	=	Hessian matrix
k	=	main curvature of the limit-state function at $\mathbf{u}^*$
K	=	cumulant generating function
n_t	=	number of tractable random variables
$n_{\sim t}$	=	number of intractable random variables
P	=	probability
p_f	=	probability of failure
R	=	reliability
t	=	saddlepoint
x	=	realization of random variable $\mathbf{X}$
$\mathbf{X}$	=	vector of random variables
X	=	random variable
$\mathbf{x}^*$	=	Most Probable Point or Most Likelihood Point in x-space
Y	=	system response
u	=	realization of random variable $\mathbf{U}$
$\mathbf{U}$	=	vector of standard normal random variables
U	=	standard normal random variable
$\mathbf{u}^*$	=	Most Probable Point or Most Likelihood Point in u-space
β	=	reliability index
Φ	=	cumulative distribution function of standard normal distribution
Φ^{-1}	=	inverse cumulative distribution function of standard normal distribution
ϕ	=	probability density function of standard normal distribution
∇	=	gradient

I. Introduction

Numerical simulations are routinely used to capture the physical phenomena in detail to predict engineering system behaviors and to reduce the number of physical testing. Since the performance and reliability of engineering systems are directly affected by the uncertainties of model parameters and model structures, it is necessary to consider uncertainties with the computational simulations in the design process in order to ensure high reliability. Typical applications include reliability-based design[1-4] and integrated design for reliability and

robustness[5-8]. Due to higher reliability requirements of an engineering system, the accuracy of the calculation of reliability or the probability of failure becomes very critical. The traditional Monte Carlo Simulation[9] is generally accurate if a sufficient number of simulations are used. However, for high reliability, an excessively large number of simulations are often needed. This high computational demand is often prohibitive for complex engineering simulations such as Finite Element Analysis and Computational Fluid Dynamics. To overcome the shortcoming of the expensive computational cost, approximation methods have been developed[10-18] such as the First Order Reliability Method (FORM) and the Second Order Reliability Method (SORM) to reduce the number of function evaluations (simulation runs). Compared to Monte Carlo Simulation, both FORM and SORM are much more efficient, especially when the reliability is extremely high. Generally, SORM is more accurate than FORM but needs more computations than FORM. In spite of its usefulness, FORM is often not accurate enough in many cases. This arouses a trade-off consideration between the efficiency and accuracy and leads to the need for a more accurate reliability analysis method without large computational demand. To meet this need, we propose a new approximation method for reliability analysis – First Order Saddlepoint Approximation (FOSPA). FOSPA is generally more accurate than FORM, and in some cases more accurate than SORM, while maintaining the same order of magnitude of computational effort as FORM.

In the next section, we will introduce the theoretical and mathematical background of this paper, including FORM, SORM, and the Saddlepoint Approximation. Thereafter, we will present the proposed FOSPA in detail and examples to demonstrate its effectiveness. The discussion and conclusion will be given in more depth at the end of this paper.

II. Methods for Probability Evaluation

Essentially, the evaluation of reliability or the probability of failure by FORM and SORM is to estimate a probability, or the Cumulative Distribution Function (CDF) of a random variable which is a function (i.e., a limit-state function) of other random variables (basic variables) provided that the distributions of the later variables are given. Saddlepoint Approximation[19] was originally developed for related purpose, i.e., to approximate CDF of statistics of a random variable (e.g., mean of random variable). In the following discussion, we will briefly review FORM, SORM, and the Saddlepoint Approximation. Thereafter, we will discuss the need to extend Saddlepoint Approximation to reliability analysis.

A. SORM and FORM

The reliability is defined as

$$R = P\left\{g(\mathbf{X}) \geq 0\right\} \tag{1}$$

The probability of failure is given by

$$p_f = 1 - R = P\left\{g(\mathbf{X}) < 0\right\} \tag{2}$$

If the joint Probability Density Function (PDF) of $\mathbf{X}$ is $f_{\mathbf{x}}$, the probability of failure is evaluated with the integral

$$p_f = P\{g(\mathbf{X}) < 0\} = \int_{g(\mathbf{x})<0} f_{\mathbf{x}}(\mathbf{x})d\mathbf{x} \tag{3}$$

The limit-state function $g(\mathbf{X})$ is usually a nonlinear function of $\mathbf{X}$; therefore, the integration boundary is nonlinear. Since the number of random variables in practical applications is usually high, multidimensional integration is involved. Due to these complexities, there is rarely a closed-form solution to Eq. (3); it is also often difficult to evaluate the probability with

numerical integration methods. When the computation cost of the limit-state function is relatively cheap, Monte Carlo integration is often applied to the problem. However, when Monte Carlo simulation is not computationally affordable, approximation methods such as First Order Reliability Method (FORM)[10] and Second Order Reliability Method (SORM)[11] have become the methods of choice in practical applications. These approximation methods involve the following steps:

1. Transformation of random variables form their original random space into a standard normal space
2. Optimization process to find the Most Probable Point (MPP) – the design point with the highest contribution to the integral calculation in Eq. (3)
3. Linear (in FORM) or quadratic approximation (in SORM) of the limit-state function in the standard normal space at the MPP
4. Calculation of probability using normal distribution tail approximation

In the first step, the original random variables $\mathbf{X} = \{X_1, X_2, \cdots, X_n\}$ (in *x*-space) are transformed into a set of random variables $\mathbf{U} = \{U_1, U_2, \cdots, U_n\}$ (in *u*-space) whose elements follow a standard normal distribution. The transformation is given by[20]:

$$u_i = \Phi^{-1}\left\{F_{x_i}(x_i)\right\}, \tag{4}$$

The probability integration is then rewritten as

$$\mathrm{P}\{g(\mathbf{X}) < 0\} = \int_{g(\mathbf{u})<0} f_{\mathbf{u}}(\mathbf{u})d\mathbf{u} \tag{5}$$

It is noted that after the transformation, the integration in Eq. (5) in *u*-space is identical to the integration in Eq. (3) in *x*-space without any loss of accuracy, and the contours of the integrand $f_{\mathbf{u}}(\mathbf{u})$ become concentric hyper spheres. The motivation for using the transformation

formulation in Eq. (5) instead of Eq. (3) to calculate probability of failure will become clear in the following discussion.

In order to make the integration calculation in Eq.(5) easier, in addition to making the integrand more regular (concentric hyper circle contours), the integral boundary $g(\mathbf{u})$ is also approximated linearly with the first order Taylor expansion as

$$g(\mathbf{U}) \approx g(\mathbf{u}^*) + \nabla(\mathbf{u}^*)(\mathbf{U} - \mathbf{u}^*) \tag{6}$$

or with the second order Taylor expansion as

$$g(\mathbf{U}) \approx g(\mathbf{u}^*) + \nabla(\mathbf{u}^*)(\mathbf{U} - \mathbf{u}^*) + \frac{1}{2}(\mathbf{U} - \mathbf{u}^*)^T \mathbf{H}(\mathbf{u}^*)(\mathbf{U} - \mathbf{u}^*) \tag{7}$$

where $\mathbf{u}^*$ is the expansion point. Eq. (6) is used in FORM and Eq. (7) is used in SORM.

To reduce the loss of accuracy to a minimum degree, it is natural to expand the function $g(\mathbf{U})$ at a point that has the highest contribution to the probability integration. Therefore, the Most Probable Point (MPP) is considered as the expansion point. The MPP is the point on the surface of $g(\mathbf{U}) = 0$ for which PDF of $\mathbf{U}$ is at its maximum. Maximizing the joint PDF of $\mathbf{U}$ on the surface of $g(\mathbf{U}) = 0$, noting that the $f_{\mathbf{u}}(\mathbf{u})$ is a concentric hyper sphere, we have the following formulation for locating the MPP,

$$\begin{cases} \min\limits_{\mathbf{u}} \|\mathbf{u}\| \\ \text{subject to} \quad g(\mathbf{u}) = 0 \end{cases} \tag{8}$$

where $\|\cdot\|$ stands for the norm (length) of a vector.

Geometrically, the MPP is the shortest distance point from surface $g(\mathbf{u}) = 0$ to the origin in *u*-space and the minimum distance $\beta = \|\mathbf{u}^*\|$ is called "*reliability index*". From Eqs. (5) and (6), the probability of failure is approximated by FORM as

$$p_f = P\{g(\mathbf{X}) < 0\} = \Phi(-\beta) \tag{9}$$

From Eqs. (5) and (7), the Second Order Reliability Method (SORM)[11] gives the following approximation,

$$p_f = P\{g(\mathbf{X}) < 0\} = \Phi(-\beta)\prod_{i=1}^{n-1}\left(1+\beta\kappa_i\right)^{1/2} \tag{10}$$

Generally, since the approximation of limit-state in SORM (see Eq. (7)) is better than that in FORM, the accuracy of SORM is higher than that of FORM (see Eq. (6)).

B. Saddlepoint Approximation

Daniels[19] introduced the Saddlepoint Approximation technique for approximating distribution of statistics (e.g., mean) by integration of its density estimate. Since Daniels' work, especially after 1980, research and applications in this area have vastly increased[21~30]. Instead of directly approximating the probability integration in Eq. 2, Saddlepoint Approximation uses a Fourier inversion formula (in an integral form) to approximate a Probability Density Function (PDF). Let Y be a random variable distributed according to the density function $f(y)$. The Moment Generating Function of Y is defined as,

$$M(\xi) = \int_{-\infty}^{+\infty} e^{\xi y}\, f\left(y\right)dy \tag{11}$$

and the Cumulant Generating Function (CGF) of Y is defined as,,

$$K(\xi) = \log\left\{M\left(\xi\right)\right\} \tag{12}$$

To restore $f(y)$ from $K(\xi)$, we can apply the inverse Fourier formula

$$f(y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} M(i\xi)e^{-i\xi y}\,d\xi$$
$$=\frac{1}{2\pi}\int_{-i\infty}^{+i\infty} \exp\{K(\xi)-\xi y\}\,d\xi \tag{13}$$

Using exponential power series expansions to evaluate the integral in Eq. (13) and Hermite polynomials approximation, Daniels[19] arrived to the so-called saddlepoint approximation to $f(y)$ as,

$$f(y)=\left\{\frac{1}{2\pi K''(t)}\right\}^{\frac{1}{2}} \exp\{K(t)-ty\} \tag{14}$$

where $K''(t)$ is the second derivative of the CGF with respect to t, where t is the saddlepoint corresponding to the solution to the following equation

$$K'(t)=y\,. \tag{15}$$

The central idea of deriving Eq. (14) is to choose the integral path passing through the saddlepoint of the integrand, where the integrand is approximated. Since the saddlepoint is an extreme point, the function of integrand falls away rapidly as we move from this point. Thus, the influence of neighboring points on the integral in Eq. (13) is diminished[29]. Interested reader should consult Goutis and Casella[29] for a good explanation of saddlepoint approximation. For the comprehensive methodology, one can refer to Ref. 30.

Although the theory of Saddlepoint Approximation is quite complex, its use, especially the CDF approximation version, is fairly straightforward[21]. The approximation of CDF of Y by the saddlepoint approximation derived by Lugananni and Rice[23] is,

$$F_Y = P\{Y \le y\} = \Phi(w) + \phi(w)\left(\frac{1}{w}-\frac{1}{v}\right) \tag{16}$$

or alternatively by Barndorff-Nielsen[24],

$$F_Y = P\{Y \le y\} = \Phi\left\{w + \frac{1}{w}\log\left(\frac{v}{w}\right)\right\} \tag{17}$$

where

$$w = sign(t)\left\{2\left[ty - K(t)\right]\right\}^{1/2}, \tag{18}$$

and

$$v = t\left\{K''(t)\right\}^{1/2} \tag{19}$$

in which $sign(t) = +1$, -1, or 0, depending on where t is positive, negative, or zero.

Daniels[19] discusses the existence and properties of the real roots to Eq. (15), upon which the saddlepoint approximation depends, and concluded that the saddlepoint approximation can be used whenever t lies with the restricted range assumed by $K'(t)$ where Eq. (15) has a unique real root.

From Eqs. (16) and (17) we see that the CDF of Y is approximated using standard normal distribution as shown by the use of CDF and PDF of the standard normal distribution in Eqs. (16) and (17). Wood, et al.[24] derived a general saddlepoint formula where the normal-base distribution is replaced by a general-base distribution.

As indicated by many previous researches (for example, Ref. 19), the saddlepoint approximation yields extremely good accuracy for CDF, especially for the tail area of a distribution, while it requires only the process of finding one saddlepoint without any integration. In terms of accuracy and efficiency, there is a great potential to extend this technique to reliability analysis and eventually to probabilistic engineering design.

Since the Saddlepoint Approximation method involves the CGF and its derivatives, the major requirement for applications of the technique is the tractability (i.e., the existence of a CGF) of the distribution of random variable Y. For an engineering application, Y is a system

performance (i.e., limit-state function) which is dependent on basic random variables **X**, i.e, $Y = g(\mathbf{X})$. The key to apply the saddlepoint approximation to a general performance Y is to find the CGF of Y provided that distributions of **X** are given. In this article, a general First Order Saddlepoint Approximation (FOSPA) method is developed with the capability of evaluating the CDF of a limit-state function accurately for any continuous distributions of basic variables.

III. The First Order Saddlepoint Approximation Reliability Method

The calculation error of probability of failure of FORM comes from the linear approximation (Eq. (6)) to the limit-state state function in *u*-space. The error of SORM comes from two sources, one is the quadratic approximation (Eq. (7)) to the limit-state function in *u*-space and the other is the approximation of probability integration for the approximated limit-state function in the quadratic form. For detailed discussion on the error of FORM and SORM, please refer to Ref. 31. Even though FORM gives an accurate solution to the probability integration for the approximated limit-state function (a linear function), it is generally less accurate than SORM because of the linear approximation. The fact that SORM is generally more accurate than FORM implies that the accuracy of the limit-state function approximation is very important to ensure highly accurate reliability estimation.

Though the non-normal to normal transformation makes it possible and easy to calculate the probability of failure analytically (without simulations), the transformation generally increases the nonlinearity of a limit-state function because the transformation in Eq. (4) is nonlinear. For example, if a limit-state is a linear function of non-normal random variables, after the transformation using Eq. (4), it will become a nonlinear function of standard normal random variables. If the approximation to the limit-state function at the MPP in *u*-space cannot capture

the nonlinearity well, the accuracy of the probability approximation will become unacceptable. To reduce the accuracy loss to the minimum extent, we need to avoid or reduce the chance of increasing the nonlinearity due to the transformation of random variables. In other word, we may consider approximating a limit-state function in the original *x*-space or avoid unnecessary transformation as much as possible.

To address the aforementioned concerns, we propose the First Order Saddlepoint Approximation Method (FOSPA) to improve the accuracy of reliability analysis while maintaining the same efficiency as FORM. In FOSPA, the limit-state function is linearized in the original random space at the so-called Most Likelihood Point (MLP) if all the random variables are tractable, then the Saddlepoint Approximation can be directly applied. If some of the random variables do not have CGF, they are transformed into other random variables that have CGF before the linearization. In the following, we will discuss the FOSPA in three cases: 1) all the random variables are tractable, 2) some of the random variables are tractable, and 3) none of the random variables is tractable. Strictly speaking, by tractable we mean that a random variable has a closed-form of CGF; otherwise, we call the random variable intractable. At the end of this section, we will present a general procedure and computational aspect of FOSPA implementation.

A. Case 1 – All the Random Variables Are Tractable

The limit-state function $g(\mathbf{X})$ is first linearized at some point $\mathbf{x}^*$, namely, the interval boundary of Eq. (3) is approximated by a hyper plane at $\mathbf{x}^*$. Similar to the concept of the MPP, the expansion point $\mathbf{x}^*$ is chosen such that the joint PDF of $\mathbf{X}$ is at its maximum value on the boundary of the limit-state $g(\mathbf{X}) = 0$; this point is called the *Most Likelihood Point* (MLP). In

other words, the MLP is the point on the boundary $g(\mathbf{X})=0$, which has the highest contribution to the probability of failure $p_f = \int_{g(\mathbf{x})<0} f_{\mathbf{x}}(\mathbf{x})d\mathbf{x}$.

The following model is used to identify the MLP $\mathbf{x}^*$,

$$\begin{cases} \max\limits_{\mathbf{x}} \quad \prod\limits_{i=1}^{n} f_i(x_i) \\ \text{subject to} \quad g(\mathbf{x})=0 \end{cases} \tag{20}$$

The linear form of $g(\mathbf{X})$ at $\mathbf{x}^*$ is

$$g(\mathbf{X}) \approx \nabla(\mathbf{x}^*)(\mathbf{X}-\mathbf{x}^*) \tag{21}$$

Then the CGF of $g(\mathbf{X})$ is given by

$$K(t) = \sum_{i=1}^{n} K_i(t) \tag{22}$$

where $K_i(t)$ is the CGF of $\nabla_i(\mathbf{x}^*)(\mathbf{X}_i - \mathbf{x}_i^*)$.

The first and second derivatives of $K(t)$ are

$$K'(t) = \sum_{i=1}^{n} K_i^{'}(t) \tag{23}$$

and

$$K''(t) = \sum_{i=1}^{n} K_i^{''}(t) \tag{24}$$

respectively.

According to Eq. (25), the saddlepoint t is identified by the solution to the following equation

$$K'(t) - y = \sum_{i=1}^{n} K_i^{'}(t) - y = 0 \tag{25}$$

Once the saddlepoint t is identified, the probability $P\{g(\mathbf{X}) \le y\}$ can be calculated from Eq. (16) with the following equations

$$w = sign(t)\left\{2\left[ty - K(t)\right]\right\}^{1/2} = sign(t)\left(2\left[ty - \sum_{i=1}^{n} K_i(t)\right]\right)^{1/2}, \tag{26}$$

and

$$v = t\left\{K''(t)\right\}^{1/2} = t\left\{\sum_{i=1}^{n} K_i''(t)\right\}^{1/2} \tag{27}$$

The CGFs of some common distributions are listed in Table 1. For more details, please refer to Ref. 32.

Table 1 CGF of Some Distributions

Distribution	PDF	CGF
Normal	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$K(t) = \mu t + \frac{1}{2}\sigma^2 t^2$
Exponential	$f(x) = \beta e^{-\beta x}$	$K(t) = -\ln\left(1 - \frac{t}{\beta}\right)$
Uniform	$f(x) = \frac{1}{b-a}$	$K(t) = \ln\left(e^{bt} - e^{at}\right) - \ln\left(b-a\right) - \ln\left(t\right)$
Type I Extreme Value (Gumbel)	$f(x) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}} \exp\left(-e^{-\frac{x-\mu}{\sigma}}\right)$	$K(t) = \mu t + \log\Gamma(1-\sigma t)$
χ^2	$f(x) = \frac{1}{\Gamma(n/2)2^{n/2}} x^{n/2-1} e^{-\frac{1}{2}x}$	$K(t) = -\frac{1}{2} n \ln\left(1-2t\right)$
Gamma	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$	$K(t) = \alpha\left\{\ln\left(\beta\right) - \mathrm{n}\left(\beta - t\right)\right\}$

B. Case 2 – Some of the Random Variables Are Tractable

Some random variables may not have a closed-form (i.e., intractable) CGF, for example, Weibull distribution and lognormal distribution. There are two ways to approach intractable CGF: 1) Approximate the CGF using polynomial expansions[33] or 2) transform the random

variable into another random variable with tractable CGF. The later approach is adopted in this paper for the purpose of simplicity. One possible transformation is similar to the one used in FORM and SORM as shown in Eq. (4) which is the transformation from a random variable with intractable CGF to a standard normal variable. In general, any distribution with tractable CGF can be used for the transformation.

Let the set of variables which have tractable CGF be $\mathbf{X}^t = \{ X_i^t ; i = 1, 2, \ldots, n_t\}$ and the set of variables without tractable CGF be $\mathbf{X}^{\sim t} = \{ X_j^{\sim t} ; j = 1, 2, \ldots, n_{\sim t}\}$. After the non-normal – normal transformation, $\mathbf{X}^{\sim t}$ is transformed into a set of standard normal variables $\mathbf{U} = \{U_j ; j = 1, 2, \ldots, n_{\sim t}\}$. Then, the formulation for searching the MLP $\left\{\mathbf{x}_t^*, \mathbf{u}^*\right\}$ becomes

$$\begin{cases} \max\limits_{\mathbf{x}^t, \mathbf{u}} \quad \prod\limits_{i=1}^{n_t} f_i(x_i^t) \prod\limits_{j=1}^{n_{\sim t}} \phi(u_j) \\ \text{subject to} \quad g(\mathbf{x}^t, \mathbf{u}) = 0 \end{cases} \tag{28}$$

After linearization, the limit-state function at the MLP $\left\{\mathbf{x}^{t*}, \mathbf{u}^*\right\}$ is given by

$$g(\mathbf{X}) \approx q\left(\mathbf{X}^t, \mathbf{U}\right) = \sum_{i=1}^{n_t} \left.\frac{\partial g}{\partial x_i^t}\right|_{\mathbf{x}^{t*}, \mathbf{u}^*} \left(X_i^t - x_i^{t*}\right) + \sum_{j=1}^{n_{\sim t}} \left.\frac{\partial g}{\partial u_j}\right|_{\mathbf{x}^{t*}, \mathbf{u}^*} \left(U_j - u_j^*\right) \tag{29}$$

Because the limit-state in Eq. (29) is a linear combination of tractable random variables, the saddlepoint approximation method in Eqs. (22) - (27) can be applied in conjunction with Eq. (29) to evaluate the probability of failure.

C. Case 3 – None of the Random Variable Is Tractable

When all random variables are intractable, they must be transformed into selected tractable random variables such as standard normal variables. If all the random variables are

transformed into standard normal variables, after the transformation, the model of searching the MLP becomes

$$\begin{cases} \max\limits_{\mathbf{u}} \ \prod_{i=1}^{n} \phi(u_i) \\ \text{subject to} \quad g(\mathbf{u}) = 0 \end{cases} \tag{30}$$

which is equivalent to the model in Eq. (8) for the MPP search. Therefore, the solution $\mathbf{u}^*$ to the model in Eq. (30) is exactly the MPP defined in the model in (8). At the MLP, the linearization of the limit-state function is given by

$$g(\mathbf{X}) = \sum_{i=1}^{n} \left. \frac{\partial g}{\partial u_i} \right|_{\mathbf{u}^*} (U_i - u_i^*) \tag{31}$$

Appendix 1 shows that the calculated probability of failure from Saddlepoint Approximation based on Eq. (31) is the same result as that of FORM. In other words, FORM is identical to FOSPA when all random variables are transformed into standard normal variables. Therefore, FORM is a special case of FOSPA.

D. The General Procedure and Computation Implementation of FOSPA

The procedure of FOSPA is summarized as follows.

a. Determine whether a random variable has tractable or intractable CGF and form two sets of random variables, one set with tractable CGF, $\mathbf{X}^t$, and the other set without tractable CGF, $\mathbf{X}^{\sim t}$. Transform the later set into standard normal variables $\mathbf{U}$.

b. Solve the model in Eq. (28) to identify the MLP $\left\{\mathbf{x}^{t*}, \ \mathbf{u}^*\right\}$.

c. Linearize the limit-state function at the MLP as shown in Eq. (29).

d. Formulate the saddlepoint equation and solve it to obtain the saddlepoint t.

e. Use Eqs. (15)-(19) to find the probability of failure.

It is noted that if all the random variables are tractable, $\mathbf{X}^{\sim t}$ will be an empty set and the problem belongs to Case 1 and if none of the random variables is tractable, $\mathbf{X}^{t}$ will be an empty set and the problem belongs to Case 3 where the same result as FORM will be obtained.

To make the numerical computation process of FOSPA more stable, several practical measures may be considered and some of them are briefly discussed here. The variables in Eqs. (20) and (28) for MLP search are normalized by the means and standard deviations of the random variables. This normalization makes the design variables in the same scales. Note that this normalization is a linear transformation and will not affect the nonlinearity of the limit-state function but will help the convergence of the iterative process of finding the MLP. To avoid the objective functions of MLP search in Eqs (20) and (28) becoming too small, one may choose to use the natural logarithm of the objective functions. To avoid singularities in Eqs. (18) and (19), one may use the reverse sign of the limit-sate function when a square root of a negative value occurs.

Considering that there is a strong need to minimize the number of limit-state evaluations so that the technique is practical for computationally expensive engineering simulation models (e.g., Finite Element Analysis and Computational Fluid Dynamics), we compare the efficiency of the methods by counting the number of function evaluations of limit-state function. Since FOSPA uses similar optimization formulation to find the MLP as FORM for the MPP, and uses less nonlinear constraint functions, the computational effort (measured by the number of function evaluations) of FOSPA is less than or at most the same as that of FORM.

IV. Numerical Examples

In this section, two examples are used to demonstrate the effectiveness of the proposed method. The first example is associated with a linear limit-state function and the other with a nonlinear limit-state function. We will compare the accuracy and efficiency among FOSPA, FORM, and SORM. If no theoretical solution exists, we will use the result of Monte Carlo simulation with a relatively large sample size as a reference. In the following examples, the first order and second order derivatives are evaluated numerically with finite difference method. Because of this finite difference calculation, SORM, which requires second order derivative information, has an inherent inefficiency in terms of the number of limit-state evaluations.

A. Example 1: Linear Limit-State Function

A linear limit-state function is given by a sum of independent random variables[11] as follows

$$g(\mathbf{X}) = (n + a\sqrt{n}) - \sum_{i=1}^{n} X_i \tag{32}$$

where a is a constant and X_i are n independent random variables.

<u>Case 1: all random variables are tractable</u>

Let each of the random variables follows a standard exponential distribution with CDF

$$F(x_i) = 1 - \exp(-x_i) \tag{33}$$

For this specific example, the theoretical solution can be found. The probability of failure $p_f = P\{g(\mathbf{X}) < 0\}$ is listed in Table 2 and depicted in Fig. 1 for n = 2.

Table 2 Probability $p_f = P\{g(\mathbf{X}) < 0\}$ for *n* = 2

a	FORM	SORM	FOSPA	Exact
0.0001	0.3166	0.3612	0.4068	0.4060
0.5	0.1795	0.2301	0.2482	0.2474
1.0	0.0990	0.1393	0.1459	0.1452
1.5	0.0536	0.0816	0.0835	0.0831
2.0	0.0286	0.0466	0.0469	0.0466
2.5	0.0152	0.0262	0.0260	0.0258
3.0	0.0079	0.0145	0.0142	0.0141
3.5	0.0041	0.0079	0.0077	0.0076
4.0	0.0021	0.0043	0.0041	0.0041
4.5	0.0011	0.0023	0.0022	0.0022
5.0	.0006	0.0012	0.0012	0.0012
5.5	0.0003	0.0007	0.0006	0.0006
6.0	0.0001	0.0003	0.0003	0.0003

Insert Fig. 1 here

Fig. 1 Probability of failure when *n*=2

Fig. 1 shows the probability of failure for different values of *a*. The probability of failure changes in the range roughly between 0.4 and 0 as *a* varies. The curves of FOSPA and the exact solution almost overlap each other over the whole range of the probability. This indicates that FOSPA is evenly good over the range of probability of failure. SORM is more accurate than FORM, but when the probability of failure is high (for example 0.4), SORM is not accurate as shown in Fig. 1. The accuracy of solution from SORM increases as the probability of failure becomes lower. This phenomenon conforms to the fact that SORM is only accurate at the tail of a distribution due to its asymptotic approximation to the probability integration[11]. In this example with linear limit-state function and tractable CGF random variables, the results show that FOSPA is the most accurate method.

Fig. 2 shows that when *a* = 3.5, the original linear limit-state function becomes highly nonlinear after the transformation to standard normal distributions required by both FORM and SORM. The linear approximation of FORM is far away from the transformed nonlinear

limit-state function in *u*-space and even the quadratic approximation in SORM cannot very well capture the nonlinearity of the transformed limit-state function. Therefore, both FORM and SORM are not as accurate as FOSPA in this example. Since FOSPA uses the original linear limit-state function without the increase of nonlinearity and Saddlepoint Approximation results a high accuracy approximation. That is, the overall accuracy of FOSPA is superior to FORM and SORM.

Insert Fig. 2 here

Fig. 2 Limit-state function in *x* and *u* spaces

The result for higher dimension with *n*=10 is listed in Table 3 and depicted in Fig. 3. The result still shows that the FOSPA is much more accurate than FORM and SORM. The related detailed equations used in this example are given in Appendix 2.

Table 3 Probability $p_f = P\{g(\mathbf{X}) < 0\}$ for *n*=10

a	FORM	SORM	FOSPA	Exact
0.0001	0.1429	0.4683	0.4580	0.4579
0.5000	0.0628	0.3392	0.2810	0.2809
1.0000	0.0253	0.2131	0.1554	0.1554
1.5000	0.0094	0.1195	0.0786	0.0786
2.0000	0.0033	0.0610	0.0369	0.0369
2.5000	0.0011	0.0288	0.0162	0.0162
3.0000	0.0004	0.0127	0.0067	0.0067
3.5000	0.0001	0.0053	0.0027	0.0027
4.0000	0.0000	0.0021	0.0010	0.0010
4.5000	0.0000	0.0008	0.0004	0.0004
5.0000	2.85e-6	2.96e-4	1.29e-4	1.29e-4
5.5000	8.03e-7	1.05e-4	4.43e-5	4.44e-5
6.0000	2.22e-7	3.63e-5	1.47e-5	1.47e-5

Insert Fig. 3 here

Fig. 3 Probability of failure when *n*=10

Case 2: Some random variables are not tractable

In the following case, we choose X_3 to follow a Weibull distribution which does not have a closed-form CGF. The distribution information is shown in Table 4.

Table 4 Information of random variables

Variable	Parameter 1	Parameter 2	Distribution
X_1	1.2	–	Exponential [a]
X_2	1.2	–	Exponential
X_3	2	1.5	Weibull [b]

[a] For an exponential distribution, Parameter 1 is the mean.
[b] For a Weibull distribution, Parameters 1 and 2 are parameters *a* and *b*, respectively, in the PDF of a Weibull distribution $f(x) = a\,b\,x^{b-1}e^{-ax^b}$.

Since X_3 is not tractable, it is transformed into a standard normal variable before the Saddlepoint Approximation is applied. Monte Carlo Simulation (MCS) is employed and its result is used as a reference for comparison of the accuracy of other methods. The number of simulations in the Monte Carlo is 10^6. The calculated probability of failure is shown in Table 5. It is noted that FOSPA is the most accurate method and SORM is more accurate than FORM. With FORM and SORM, the transformation of $\{X_1, X_2, X_3\}$ into a standard normal variable $\{U_1, U_2, U_3\}$ makes the original linear limit-state function become nonlinear in terms of $\{U_1, U_2, U_3\}$. On the other hand, FOSPA only involves the transformation of X_3 into a standard normal variable U_3. That is, the original limit-state is only nonlinear in terms of U_3 and the remaining terms of X_1 and X_2 are kept linear. As a result of the minimum increase of nonlinearity of the limit state, FOSPA is more accurate than FORM and SORM. The numbers of function evaluations used by FOSPA, FORM, and SORM (including finite difference calculation and iterations to find MLP/MPP) are 25, 37, and 57, respectively. In this case, the minimum increase

of nonlinearity also helps FOSPA to be the most efficient method for finding the MLP while SORM is the least efficient method for this specific case.

Table 5 Probability of failure for case 2

	a	FORM	SORM	FOSPA	MCS
$P\{g(\mathbf{X})<0\}$	4.3	1.224×10^{-4}	3.025×10^{-4}	2.770×10^{-4}	2.289×10^{-4}
N[a]		25	37	25	10^6

[a] Number of function evaluations

<u>Case 3: All the random variables are not tractable</u>

In the following case, all random variables follow Weibull distributions as shown in Table 6. Since a Weibull distribution does not have tractable CGF, the transformation from $\{X_1, X_2, X_3\}$ to a standard normal variable $\{U_1, U_2, U_3\}$ is required.

Table 6 Information of random variables

Variable	Parameter 1	Parameter 2	Distribution
X_1	2	1.5	Weibull
X_2	2	1.5	Weibull
X_3	2	1.5	Weibull

As expected, FOSPA has the same result as FORM as shown in Table 7. The MLP from FOSPA and the MPP from the FORM are identical, i.e., $\{x_1^{MLP}, x_2^{MLP}, x_3^{MLP}\}=\{x_1^{MPP}, x_2^{MPP}, x_3^{MPP}\}=\{1.2887, 1.2887, 1.2887\}$. In this case SORM is the most accurate method because the second order approximation in SORM provides a better approximation to the limit-state function in *u*-space.

Table 7 Probability $P\{g(\mathbf{X})<0\}$ for case 3

	a	FORM	SORM	FOSPA	MCS
$P\{g(\mathbf{X})<0\}$	0.5	0.0026	0.0038	0.0026	0.0040
N		21	27	21	10^6

B. Example 2: Nonlinear Limit-State Function

Consider the limit-state function of a shaft in a speed reducer defined as

$$g(\mathbf{X}) = S - \frac{32}{\pi D^3}\sqrt{\frac{F^2 L^2}{16} + T^2} \tag{34}$$

where S is the material strength, D is the diameter of the shaft, F is the external force, T is the external torque, and L is the length of the shaft. The limit-state function represents the difference between the strength and the maximum stress.

The variable information is given in Table 8.

Table 8 Distributions of Random Variable

Variables	Parameter 1	Parameter 2	Distribution
Diameter D	39 *mm*	0.1 *mm*	Normal [a]
Span L	400 *mm*	0.1 *mm*	Normal
External force F	1500 *N*	350 *N*	Gumbel [b]
Torque T	250 *Nm*	35 *Nm*	Normal
Strength S	70 *MPa*	80 *MPa*	Uniform [c]

[a] For normal distribution, Parameters 1 and 2 are mean and standard deviation respectively.
[b] For Gumbel distribution, Parameters 1 and 2 are mean and standard deviation respectively.
[c] For a uniform distribution, Parameters 1 and 2 are lower and upper bounds respectively

This problem belongs to Case 1 where all the random variables are tractable. The results of probability of failure as compared with MCS (10^6 simulations) are shown in Table 9. Referenced to MCS, FOSPA generate the most accurate solution with the least computational demand.

Table 9 Probability $P\{g(\mathbf{X}) < 0\}$

	FORM	SORM	FOSPA	MCS
$P\{g(\mathbf{X}) < 0\}$	7.007×10^{-7}	4.3581×10^{-7}	6.1754×10^{-4}	7.850×10^{-4}
N	1472	1514	102	10^6

The above result indicates that FOSPA provides accurate CDF estimate at the right tail of the distribution of the limit-state function. To illustrate the accuracy of FOSPA over the whole

distribution range, the CDF of the limit-state function at the left tail and near the median are also calculated and given in Tables 10 and 11, respectively. From Tables 9 and 10, it is noted that FOSPA is also superior to FORM and SORM at both tails in terms of accuracy and efficiency. Table 11 shows that FOSPA also produces reasonably accurate CDF estimate around the median of the distribution while both FORM and SORM have very large errors. This example demonstrates that FOSPA is evenly accurate over the whole distribution and therefore beneficial for generating a complete distribution of a performance (limit-state function).

Table 10 Probability at the tails of distribution

	FORM	SORM	FOSPA	MCS
$P\{g(\mathbf{X}) < 4.5\times10^7\}$	0.96798	0.97406	0.99927	0.99938
N	212	254	55	10^6

Table 11 Probability near the median

	FORM	SORM	FOSPA	MCS
$P\{g(\mathbf{X}) < 2.48\times10^7\}$	0.1538	0.1406	0.4825	0.5038
N	93	135	43	10^6

V. Discussion

In this section, we summarize the proposed FOSPA method with detailed discussion on its accuracy and efficiency in comparison to FORM and SORM. Based on the discussion, recommendations for selecting the reliability analysis methods under various circumstances will be provided in the next section.

Saddlepoint approximation is an accurate method for estimating CDF of a random variable if its CGF is known. The central idea of the proposed FOSPA is to approximate the CGF of a general limit-state function through linearization of limit-sate function. The linearization is

conducted at the Most Likelihood Point (MLP)—the point where the joint PDF of the random variables is at its maximum value for a given limit-state value. If a random variable does not have a closed form CGF (intractable), it is transformed to another random variable with a tractable CGF before the linearization. In this paper, an intractable random variable is transformed to a standard normal variable. It is worthwhile noting that other types of random variables with tractable CGF can also be used for the transformation. Once the limit-state function is in the form of a linear combination of tractable variables, the CGF of the limit-state function is easily obtained. The saddlepoint is the solution to the equation of the first derivative of the CGF equal to the limit-state value. Thereafter, the saddlepoint approximation solution is used to approximate the probability of failure or the reliability.

In contrast to FORM that conducts linearization of the transformed standard normal space (which imposes nonlinear transformations), FOSPA linearizes the limit-state function in the original space of tractable random variables. As a consequence to minimizing random variable transformation, FOSPA reduces the chance of increasing the nonlinearity of the limit-state function. Therefore, the linearization of the limit-state function in FOSPA gives a more accurate approximation than that of FORM. Generally, FOSPA is more accurate than FORM except in the following cases where they are equivalent 1) all random variables have intractable CGF and they are transformed into standard normal variables; 2) all tractable random variables are normally distributed and all the intractable random variables are transformed into standard normal variables; and 3) all random variables are normally distributed. In the aforementioned three cases, the MLP from FOSPA is identical to the MPP from FORM and, therefore, both methods have the same accuracy. In this sense, FORM is a special case of FOSPA.

It is generally recognized that SORM is more accurate than FORM although there are few counterexamples; however, there is no such direct conclusion about the comparison between FOSPA and SORM in terms of their accuracy. One method is more accurate than the other depending on the problem under consideration. Generally speaking, when the limit-state function is less nonlinear in terms of original random variables or the non-normal to normal transformation increases nonlinearity of the limit-state function significantly, FOSPA may have a higher accuracy than SORM.

The search of the MLP needs an iterative process where the limit-state function is evaluated repeatedly. Since searching an MLP is a similar task as searching an MPP, it is expected that FOSPA has at most the same order of magnitude of computational demand as that of FORM. In many cases, searching the MLP is more efficient than searching the MPP since the constraint function in the optimization model of the MLP is more linear than that of the MPP. It should be noted that the search of the saddlepoint does not consume any limit-state function evaluations. Because SORM needs the second order derivative of a limit-sate function, it is generally much less efficient when the derivative is evaluated numerically.

Considering the same computational effort and higher accuracy of FOSPA compared to FORM, one may choose FOSPA for a reliability analysis. When higher accuracy is needed, one should also consider the fact that depending on the linearity of the limit-state and random distribution, SORM is not always better than FOSPA in terms of accuracy. The computational efficiency, accuracy, and implementation simplicity of the proposed method make it attractive for real world reliability analysis. One of the authors has extensively applied the proposed method to various computationally intensive simulation models used in automotive engine design (for example, Hoffman et al.[34]). As indicated in the Example 2, FOSPA can also be used

to generate accurate CDF associated with a range of limit-state values. This is accomplished by enumerating the limit-state values, perform linearization at all MLP's associated with the limit state values, and then calculate the probability using Eq. (16) or (17). Using this approach, FOSPA can accurately calculate the CDF at both tails as well as around the median (or mean) of a distribution.

To further improve the accuracy, the Second Order Saddlepoint Approximation can be considered and the key to the new development is how to identify the CGF of second order approximation of a limit-state function.

VI. Conclusion

In summary, the proposed First Order Saddlepoint Approximation method for reliability analysis is an attractive alternative to the existing reliability analysis methods FORM and SORM. One may consider the following facts when selecting the reliability methods: FORM is a special case of FOSPA and the later is more accurate than the former with less or at most the same computational effort. If the limit-state function in the original space is less nonlinear than that of standard normal transformed space, FOSPA may be more accurate than SORM. SORM is less efficient (i.e., it requires more function evaluations) than FOSPA and FORM.

Appendix 1: FORM is a Special Case of FOSPA

If none of the random variables is tractable, FORM produces the same result as FOSPA when standard normal transformation is employed. After a limit-state function $g(\mathbf{X})$ is approximated by a linear function in Eq. (31), the CGF of $q(\mathbf{U})$ is given by

$$K(t) = -\sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*} u_i^* t + \frac{1}{2}\sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*}\right)^2 t^2 \tag{A1}$$

and its derivative is

$$K^{'}(t) = -\sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*} u_i^* + \sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*}\right)^2 t \tag{A2}$$

The saddlepoint is obtained from $K^{'}(t) = 0$

$$t = \frac{\sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*} u_i^*}{\sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*}\right)^2} \tag{A3}$$

The CGF at the saddlepoint becomes

$$K(t) = -\frac{1}{2}\frac{\left(\sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*} u_i^*\right)^2}{\sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*}\right)^2} \tag{A4}$$

and its second order derivative with respect to the saddlepoint is given by

$$K^{''}(t) = \sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*}\right)^2 \tag{A5}$$

Substituting (A4) and (A5) into Eqs. (18) and (19) yields

$$w = \frac{\sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*} u_i^*}{\sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*}\right)^{1/2}} = -\beta \tag{A6}$$

and

$$v = \frac{\sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*} u_i^*}{\sum_{i=1}^{n} \left(\left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*} \right)^2} \left\{ 2\sum_{i=1}^{n} \left(\left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*} \right)^2 \right\}^{1/2} = \frac{\sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*} u_i^*}{\sum_{i=1}^{n} \left(\left.\frac{\partial g}{\partial u_i}\right|_{\mathbf{u}^*} \right)^{1/2}} = -\beta \tag{A7}$$

respectively.

Combining Eqs. (A6), (A7), and Eq. (16) results in

$$P\{g < 0\} = \Phi(-\beta) \tag{A8}$$

which is the same result of FORM.

Appendix 2: Case 1 of Example 1

1) FOSPA

The CGF of the limit-state function of Example 1 is given by

$$K(t) = -n\ln(1-t) - (n + a\sqrt{n})t \tag{A9}$$

and its derivatives are

$$K^{'}(t) = \frac{n}{1-t} - (n + a\sqrt{n}) \tag{A10}$$

and

$$K^{''}(t) = \frac{(n + a\sqrt{n})^2}{n} \tag{A11}$$

respectively.

Solving $K^{'}(t) = 0$ produces the saddlepoint

$$t = \frac{a\sqrt{n}}{n + a\sqrt{n}} > 0 \tag{A12}$$

Combining Eqs. (A9) through (A12), we obtain

$$w = sign(t)\left\{2(ty - K(t)\right\}^{1/2} = 2\left\{n\ln(\frac{n}{n+a\sqrt{n}}) + a\sqrt{n}\right\}^{1/2} \tag{A13}$$

$$v = t\left\{K''(t)\right\}^{1/2} = a \tag{A14}$$

and the probability of failure

$$p_f = \Phi\left\{w + \frac{1}{w}\ln(\frac{v}{w})\right\} = \Phi\left\{2\left\{c + a\sqrt{n}\right\}^{1/2} + \frac{1}{2\left\{c + a\sqrt{n}\right\}^{1/2}}\ln(\frac{a}{2\left\{c + a\sqrt{n}\right\}^{1/2}})\right\} \tag{A15}$$

where

$$c = n\ln\left(\frac{n}{n + a\sqrt{n}}\right) \tag{A16}$$

2) FORM

The MPP $\mathbf{u}^* = \left\{u_i^*\right\},\ i = 1,\ 2,\ \cdots n$ is given by

$$u_i^* = -\Phi^{-1}\left(\exp\left(-\frac{n + a\sqrt{n}}{n}\right)\right) \tag{A17}$$

and the reliability index is calculated by

$$\beta = \sqrt{n}u_i^* \tag{A18}$$

Then, the probability of failure is

$$p_f = \Phi\left(-\beta\right) = \Phi\left(-\sqrt{n}u_i^*\right) \tag{A19}$$

3) SORM

The probability of failure by SORM is given by

$$p_f = \Phi^{-1}(-\sqrt{n}u_i^*)\left\{1 + u_i^*\left(\frac{u_i^*\Phi(-u_i^*) - \phi(-u_i^*)}{\Phi(-u_i^*)}\right)\right\}^{-\frac{n-1}{2}} \tag{A20}$$

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