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Uncertainty Analysis by Dimension Reduction Integration and Saddlepoint Approximations

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Abstract

Uncertainty analysis, which assesses the impact of the uncertainty of input variables on responses, is an indispensable component in engineering design under uncertainty, such as reliability-based design and robust design. However, uncertainty analysis is an unaffordable computational burden in many engineering problems. In this paper, a new uncertainty analysis method is proposed with the purpose of accurately and efficiently estimating the cumulative distribution function (CDF), probability density function (PDF), and statistical moments of a response given the distributions of input variables. The bivariate dimension reduction method and numerical integration are used to calculate the moments of the response; then Saddlepoint Approximations are employed to estimate the CDF and PDF of the response. The proposed method requires neither the derivatives of the response nor the search of the Most Probable Point (MPP), which is needed in the commonly used First and Second Order Reliability Methods (FORM and SORM) and the recently developed First Order Saddlepoint Approximation (FOSPA). The efficiency and accuracy of the proposed method is illustrated with three example problems. With the same computational cost, this method is more accurate for reliability assessment and much more efficient for estimating the full range of the distribution of a response than FORM and SORM. This method provides results as accurate as Monte Carlo simulation, with significantly reduced computational effort.

1. Introduction

The quantitative assessment of the impact of the uncertainty of input variables on a response is now widely recognized as an important and indispensable component in engineering design under uncertainty, such as reliability based design [1-3], robust design [4-6], design for Six Sigma [7], and decision making under risk and uncertainty [8]. Consider a response Y , which is modeled as

$$Y = g(\mathbf{X}), \quad (1)$$

where the function, g , represents the model under study and is termed a performance function or response function; and \mathbf{X} is a vector of the random input variables, i.e., $\mathbf{X} = [X_1, X_2, \dots, X_n]$. In this paper, X_1, X_2, \dots, X_n are assumed mutually independent. The main task of uncertainty analysis is to evaluate the cumulative distribution function (CDF), probability density function (PDF), and the statistical moments (mean, variance, etc.) of Y , given the distributions of input variables \mathbf{X} . Theoretically, the CDF of Y at y can be evaluated with a multi-dimensional integral,

$$F_Y(y) = P\{Y \leq y\} = \int \cdots \int_{g(\mathbf{X}) \leq y} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}, \quad (2)$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint PDF of \mathbf{X} . In the field of structural reliability, $Y = g(\mathbf{X})$ is also referred to as a limit-state function, and y is treated as a limit state [1]. If the safety is defined by the event of $Y \leq y$, then Eq. (2) defines the reliability. Due to the nonlinear integration boundary, $g(\mathbf{X}) = y$, and the high dimensionality in Eq. (2), in practice it is very difficult or even impossible to obtain an analytical solution to the probability integration [5]. Difficulties in computing this probability have led to the development of various approximation methods.

Traditionally, commonly used methods comprise three major categories: (1) analytical methods, (2) surrogate methods, and (3) simulation methods. Typical analytical methods include

the First Order Reliability Method (FORM) and the Second Order Reliability Method (SORM) [9-11]. FORM or SORM achieves a solution to Eq. (2) by simplifying the performance function, $g(\mathbf{X})$, using first or second order Taylor series expansion at the Most Probable Point (MPP). Each of them requires an optimization (iterative) process to locate the MPP, which involves a nonlinear transformation from non-normal variables into standard normal variables, sensitivity analysis (gradient calculation), and a global optimal solution. Gradient-free methods, such as genetic algorithms, can also be used for MPP search; but the search process may become more expensive. The nonlinear transformation may increase the nonlinearity of the performance function [12]. Sensitivity analysis imposes a restriction on the smoothness of the performance function. Also, because a linear (in FORM) or quadratic (in SORM) approximation may not sufficiently capture the nonlinearity of the performance function, their accuracy may not be satisfactory if the nonlinearity of the performance function in the transformed normal space is high. Since SORM uses the curvature information at the MPP, it is generally more accurate but more expensive than FORM. It should be noted that SORM can also work in the original space without any transformation and it is asymptotically accurate as the estimated probability is close to 1. Moreover, neither FORM nor SORM is suitable for distribution estimation. The surrogate methods [13-15] use a simplified model, which is generally obtained from Design of Experiments or variable screening by means of sensitivity analysis. The accuracy of the surrogate methods is usually not satisfactory, though sometimes they can give a quick solution [16]. Both analytical and surrogate methods may have numerical difficulties because they are computationally expensive when the number of random input variables is large and when numerical methods for derivatives are involved. Simulation and sampling methods [17-20], such as direct Monte Carlo simulation, Quasi-Monte Carlo simulation, Latin hypercube sampling and

importance sampling, are easy and flexible to use and do not exhibit the limitations of analytical and surrogate methods. However, when the probability in Eq. (2) is high, simulation methods become too expensive [5].

Point estimate methods [21-23] are another alternative approach to evaluate the moments and probability distribution of the response. Usually, the first four moments of the response are predicted using a few points per variable, and then the distribution is described in terms of a 4-parameter Beta or Lambda distribution. The computational effort of this approach increases exponentially with the number of random variables. For example, in Seo and Kwak's work [23], three points per variable are applied; 3^n function evaluations are required for calculating the probability in Eq. (2) when there are n random input variables in the response. Obviously, this approach becomes computationally unaffordable if n is larger than 10.

Recently, a new uncertainty analysis method – First Order Saddlepoint Approximation (FOSPA), has been reported [12]. This method uses accurate Saddlepoint Approximations [24] resulting in more accurate and efficient solutions than FORM, and in some cases than SORM. It focuses only on reliability analysis and does not cover response distribution estimation. In FOSPA, the response function is linearly approximated by its first order Taylor expansion at the Most Likelihood Point (MLP) in the original space, and the Cumulant Generating Function (CGF) of the response is then analytically obtained from the linearly approximate function. Since this method also falls into the aforementioned Category 1, similar to FORM or SORM, its main drawbacks are the need for derivatives of the response and the reliance on the existence of a unique MLP, which, in general, may not be the case. Additionally, it may still cause high computational demand when the problem dimension becomes large.

The objective of this work is to develop an efficient and accurate method for uncertainty analysis to estimate the *complete distribution* of a response, given the distributions of random input variables. In the proposed method, all random variables are transformed into a specific normal distribution; and the bivariate dimension reduction and the Gauss-Hermite quadrature techniques are then used to calculate the statistical moments of the response; the CGF of the response is then expressed in a form of power expansion, whose coefficients (cumulants) are associated with the moments from the dimension reduction numerical integration; the accurate Saddlepoint Approximations are finally employed to calculate the CDF and PDF of the response. An overview of the proposed method is presented in Section 2, followed by moment calculation with dimension reduction numerical integration in Section 3 and Saddlepoint Approximations for PDF and CDF in Section 4. Three examples are used to demonstrate the effectiveness of the proposed method in Section 5. Section 6 presents the conclusions.

2. Overview of the Proposed Method

The central principle of the proposed method is to use the accurate Saddlepoint Approximations to estimate the PDF and CDF of a response Y . The only requirement for using the Saddlepoint Approximations is that the Cumulant Generating Function (CGF) of Y , K_Y , has to be known. Since Y is a general function of random variables \mathbf{X} , K_Y will not be readily available. However, K_Y can be estimated with the moments of Y . Therefore, the procedure of the proposed method is composed of two steps: (1) moment calculation of Y , and (2) CDF and PDF estimation. In Step 1, in order to estimate the moments of Y with high efficiency and accuracy, we adopt a recently developed dimension reduction method [25,26] to compute statistical moments with the following modifications:

(1) A transformation from general random variables into normal variables with mean of 0 and variance of 0.5, in order to ensure the applicability of any continuous distributions and ease of the moment estimation; and

(2) The direct use of Gauss-Hermite integration (GHI) to calculate statistical moments.

In Step 2, the CGF of the response Y , K_Y , is estimated in a power expansion of the statistical cumulants, which are derived from the moments obtained in Step 1; and Saddlepoint Approximations are then used to accurately estimate the CDF and the PDF of the response. Figure 1 outlines the procedure. Details of each step are presented in the following sections.

Insert Figure 1 here.

3. Calculation of Moments Using Dimension Reduction Numerical Integration

The purpose of Step 1 is to estimate the moments of response Y , given the distributions of \mathbf{X} . The strategy of a newly developed dimension reduction numerical integration [26] with modifications will be used to fulfill the task.

3.1 Moments about Zero

The j th moment about zero, μ_j' , of $Y = g(\mathbf{X})$ is defined as

$$\mu_j' = E\{[g(\mathbf{X})]^j\} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [g(\mathbf{x})]^j f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}, \quad (3)$$

where $f_{\mathbf{X}}(\mathbf{x}) = f_{x_1}(x_1) \cdots f_{x_n}(x_n)$ in which $f_{x_i}(x_i)$ is the PDF of X_i , and E represents the expectation operator. Equation (3) can also be written as

$$\mu_j' = E\{[g(\mathbf{X})]^j\} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [g(x_1, \dots, x_n)]^j f_{x_1}(x_1) \cdots f_{x_n}(x_n) dx_1 \cdots dx_n. \quad (4)$$

Practically, such an integral cannot be evaluated analytically because of the high dimensionality and the complicated integrand. Direct numerical integrations, such as Gauss-Legendre or Gauss-Hermite quadrature [27], can be applied but are not economically feasible when the number of random variables is larger than three or four [25]. Monte Carlo simulation is another technique to solve the problem, but generally requires a large number of function evaluations to accurately estimate higher-order moments. In this paper, a transformation from general random variables into normal variables of $N\left(0, \frac{1}{2}\right)$ is employed, and a recently developed function approximation method of dimension reduction for multi-dimensional integration [26] is used to approximate the function, $[g(x_1, \dots, x_n)]^j$, in Eq. (4). The central idea of the dimension reduction method is to approximate the integral in Eq. (4) by a summation of a series of lower-dimensional integrals.

3.2 Transformation of Random Variables

Rosenblatt transformation [28] has been widely used in statistics, reliability analysis, and uncertainty analysis [2,5,29,30]. By Rosenblatt transformation, a non-normal variable is transformed to a standard normal variable $N(0,1)$. In this paper, adopting the same idea of Rosenblatt transformation, we employ a transformation of a random variable to a normal variable that follows a distribution of $N\left(0, \frac{1}{2}\right)$. The reasons for this transformation are: (1) to make all the random variables have the same distribution of $N\left(0, \frac{1}{2}\right)$ and therefore ensure a symmetry about zero, which is required by the dimension reduction method [26]; (2) to make the proposed method suitable to any continuous distributions; and (3) to obtain a concise formula of Gauss-

Hermite integration (GHI) for the moment estimation, since two exponential items will be cancelled – one is from GHI and the other is from the PDF of a normal variable that follows $N\left(0, \frac{1}{2}\right)$. The concise formula will be shown in Section 3.4.

The proposed transformation function between the original random variable X_i ($i = 1, \dots, n$) and the normal variable $U_i \sim N\left(0, \frac{1}{2}\right)$ is expressed as

$$u_i = T_i(x_i) = \frac{1}{\sqrt{2}} \Phi^{-1} \left[F_{X_i}(x_i) \right], \quad (5)$$

where $\Phi^{-1}[\cdot]$ is the inverse function of the CDF of a standard normal variable; $F_{X_i}(x_i)$ is the CDF of random variable X_i ; x_i is a realization of X_i ; and u_i is a realization of the normal variable,

$U_i \sim N\left(0, \frac{1}{2}\right)$. The PDF of U_i is given by

$$f_{U_i}(u_i) = \frac{1}{\sqrt{\pi}} e^{-u_i^2}, \quad (6)$$

and the CDF of U_i is given by

$$F_{U_i}(u_i) = \Phi(\sqrt{2}u_i). \quad (7)$$

For example, if $X_i \sim N(\mu_{X_i}, \sigma_{X_i}^2)$, the transformation between U_i and X_i is

$$u_i = T_i(x_i) = \frac{x_i - \mu_{X_i}}{\sqrt{2}\sigma_{X_i}}.$$

Using the inverse transformation of Eq. (5), Eq. (4) can be rewritten as

$$\mu'_j = E \{ [g(\mathbf{X})]^j \} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ g \left[T^{-1}(\mathbf{u}) \right] \right\}^j f_{\mathbf{U}}(\mathbf{u}) d\mathbf{u}, \quad (8)$$

where

$$T^{-1}(\mathbf{u}) = [T_1^{-1}(u_1), T_2^{-1}(u_2), \dots, T_n^{-1}(u_n)], \quad (9)$$

in which $T_i^{-1}(\cdot)$ ($i=1, \dots, n$) is the inverse function of the transformation function in Eq. (5);

$f_{\mathbf{U}}(\mathbf{u})$ is the joint probability density function of \mathbf{U} and is given by

$$f_{\mathbf{U}}(\mathbf{u}) = \prod_{i=1}^n f_{U_i}(u_i). \quad (10)$$

3.3 Dimension Reduction

Dimension reduction is a technique of function approximation for the purpose of moment estimation. It approximates an n -dimensional function by the summation of a series of, at most, D -dimensional functions ($D < n$). In this paper, the *bivariate* dimension reduction ($D = 2$) is used. Let $Z(\mathbf{u}) = Z(u_1, \dots, u_n) = \left\{ g [T^{-1}(\mathbf{u})] \right\}^j$ in Eq. (8). By the bivariate dimension reduction method [26],

$$Z(\mathbf{u}) \cong \widehat{Z}(\mathbf{u}) = \sum_{i=0}^2 (-1)^i \binom{n+i-3}{i} Z_{2-i}, \quad (11)$$

where

$$Z_0 = Z(0, \dots, 0), \quad (12)$$

$$Z_1 = \sum_k Z(0, \dots, 0, u_k, 0, \dots, 0) \quad (k = 1, \dots, n), \quad (13)$$

and

$$Z_2 = \sum_{k_1 < k_2} Z(0, \dots, 0, u_{k_1}, 0, \dots, 0, u_{k_2}, 0, \dots, 0), \quad (14)$$

in which $k_1, k_2 = 1, 2, \dots, n$ and $k_1 < k_2$. It is noted that Z_1 is a summation of n one-dimensional functions and Z_2 is a summation of $\binom{n}{2}$ two-dimensional functions. From Eqs. (11) through (14),

$Z(\mathbf{u})$ is additively decomposed into functions of, at most, two variables; in other words, $Z(\mathbf{u})$ is approximated by a summation of functions of, at most, two variables. For example, if $Z(\mathbf{u}) = Z(u_1, u_2, u_3)$, the bivariate dimension reduction approximation of $Z(\mathbf{u})$ is

$$Z(\mathbf{u}) \cong \widehat{Z}(\mathbf{u}) = Z(0, u_2, u_3) + Z(u_1, 0, u_3) + Z(u_1, u_2, 0) - Z(u_1, 0, 0) - Z(0, u_2, 0) - Z(0, 0, u_3) + Z(0, 0, 0) \quad (15)$$

Xu and Rahman [26] have proved that $\widehat{Z}(\mathbf{u})$ in Eq. (11) is an accurate approximation of $Z(\mathbf{u})$ by comparing the Taylor series expansions of both $Z(\mathbf{u})$ and $\widehat{Z}(\mathbf{u})$ at $\mathbf{0}$. Their conclusion is that the second Taylor series expansion of $\widehat{Z}(\mathbf{u})$ at $\mathbf{0}$ is more accurate than the second order Taylor series expansion of $Z(\mathbf{u})$ at $\mathbf{0}$. Interested readers can refer to [26] for more details.

Since $f_U(\mathbf{u})$ is the joint PDF of the n independent random variables of $N\left(0, \frac{1}{2}\right)$, substituting Eq. (11) into Eq. (8) reduces the n -dimensional integral of Eq. (8) into a summation of, at most, two-dimensional integrals,

$$\begin{aligned} \mu'_j &= E\{[Z(\mathbf{U})]\} \cong \sum_{i=0}^2 (-1)^i \binom{n+i-3}{i} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} Z_{2-i} f_U(\mathbf{u}) d\mathbf{u} \\ &= \binom{n-1}{2} \sum_{k_1 < k_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Z(0, \dots, 0, u_{k_1}, 0, \dots, 0, u_{k_2}, 0, \dots, 0) f_{U_{k_1}}(u_{k_1}) f_{U_{k_2}}(u_{k_2}) du_{k_1} du_{k_2} \\ &\quad + (-1) \binom{n-2}{1} \sum_{k=1}^n \int_{-\infty}^{\infty} Z(0, \dots, 0, u_k, 0, \dots, 0) f_{U_k}(u_k) du_k + Z(0, \dots, 0). \end{aligned} \quad (16)$$

It is noted that Eq. (16) only involves two- and one-dimensional integrals. The computational effort in numerically computing the two- and one-dimensional integrals in Eq. (16) is less than that in evaluating the n -dimensional integral in Eq. (4) or Eq. (8). The formulas for computational efforts will be given in Section 3.5.

3.4 Calculation of Moments by Gauss-Hermite Integration

As shown in Section 3.3, the moment calculation in Eq. (16) only involves one- and two-dimensional integrals. One term of the one-dimensional integrals in Eq. (16), without leading coefficients, is given by

$$\mathbf{I}_{n1} = \int_{-\infty}^{\infty} Z(0, \dots, 0, u_k, 0, \dots, 0) f_{U_k}(u_k) du_k, \quad k \in [1, 2, \dots, n]. \quad (17)$$

Gauss-Hermite integration (GHI) approximates the one-dimensional integral in Eq. (17) by summing up some terms of the weighted integrand evaluated at, so-called, Gauss points (abscissas) as follows [27],

$$\mathbf{I}_{n1} \cong \sum_{l=1}^r w_l e^{u_l^2} Z(0, \dots, 0, u_l, 0, \dots, 0) f_{U_k}(u_l) = \frac{1}{\sqrt{\pi}} \sum_{l=1}^r w_l Z(0, \dots, 0, u_l, 0, \dots, 0), \quad (18)$$

where r is the quadrature order (the number of abscissas); and u_l and w_l are abscissas (Gauss points) and weights (Gauss weights), respectively, which are listed in Table 1 (for $r = 1, 2, 3$ and 4). For more weight and abscissa information of higher quadrature orders, refer to [31].

Table 1 Weights and abscissas for Gauss-Hermite integration

Order (r)	Abscissa (u_l)	Weight (w_l)
1	0	1.772453
2	± 0.707107	0.886227
3	0	1.181635
	± 1.22474	0.295409
4	± 0.524648	0.804914
	± 1.65068	0.081312

Similarly, the GHI formula for a two-dimensional integral in Eq. (16), without the leading coefficients, is given by

$$\begin{aligned}
\mathbf{I}_{n2} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Z(0, \dots, 0, u_{k_1}, 0, \dots, 0, u_{k_2}, 0, \dots, 0) f_{U_{k_1}}(u_{k_1}) f_{U_{k_2}}(u_{k_2}) du_{k_1} du_{k_2} \\
&\cong \left(\frac{1}{\sqrt{\pi}} \right)^2 \sum_{I_1=1}^{r_1} \sum_{I_2=1}^{r_2} w_{I_1} w_{I_2} Z(0, \dots, 0, u_{I_1}, 0, \dots, 0, u_{I_2}, 0, \dots, 0),
\end{aligned} \tag{19}$$

where r_1, r_2 denote the quadrature order (the number of abscissas) used in the u_{k_1}, u_{k_2} directions; (u_{I_1}, u_{I_2}) are Gauss points; and w_{I_1}, w_{I_2} are the corresponding weights.

3.5 Computational Effort

As mentioned previously, Eq. (16) is a summation of a series of one- and two-dimensional integrals. The computational effort in computing all the one- and two-dimensional integrals in Eq. (16) is generally much less than that in evaluating the original n -dimensional integrals in Eq. (4) or Eq. (8). In this paper, *three* Gauss points per variable are employed for the numerical integration. When the bivariate dimension reduction method is used, there are $\binom{n}{2}$ two-dimensional integrals and $\binom{n}{1}$ one-dimensional integrals in Eq. (16), and the computational effort can be measured by the number of function evaluations, which is given by

$$CE = \binom{n}{2}(3)^2 + \binom{n}{1}(3) + 1 = \frac{9n(n-1)}{2} + 3n + 1. \tag{20}$$

For a performance function with 10 random input variables, the total number of function evaluations is 436. This is significantly less than that of the direct GHI, which is $3^{10} = 59049$. It should be noted that after the moments are obtained, the following Saddlepoint Approximations for estimating CDF and PDF do not need to evaluate the performance function any more. Therefore, Eq. (20) gives the *total* computational cost of the proposed method.

4. Saddlepoint Approximations

Saddlepoint Approximations, originally introduced in the statistics literature by Daniels [24], is an important and powerful tool for obtaining accurate PDF and CDF [32,33]. Discussions and explanations of their applications for a range of distributional problems are given by Reid [34], Goutis and Casella [35], and Huzurbazar [36]. Although the theory of Saddlepoint Approximations is quite complex, simple formulas for calculation of CDF and PDF have been derived; consequently, its use is fairly straightforward [36].

4.1 Cumulant Generating Function

Saddlepoint Approximations for estimating the CDF and PDF of Y rely on the Cumulant Generating Function (CGF) of Y , which is defined as

$$K_Y(t) = \log \left[\int_{-\infty}^{\infty} e^{ty} f_Y(y) dy \right], \quad (21)$$

where \log is the natural logarithm, and $f_Y(y)$ is the PDF of the random response Y . In engineering practice, it is very difficult, or even impossible, to obtain the exact CGF of a performance function from Eq. (21), due to the complicated response function $Y = g(\mathbf{X})$ and the unknown density function $f_Y(y)$. Du and Sudjianto [12] proposed a method to estimate the CGF of Y by approximating the $g(\mathbf{X})$ using first order Taylor expansion at the Most Likelihood Point (MLP). Their method relies on a unique MLP. If there are multiple MLPs, their method may not give an accurate result.

In the proposed method, the power expansion of the CGF of Y is used. The power expansion is given by [37]

$$K_Y(t) = \sum_{j=1}^{\infty} \kappa_j \frac{t^j}{j!}, \quad (22)$$

where κ_j are j th cumulant of Y . Wang [32] and Gatto and Ronchetti [38] proposed to use up to the fourth cumulant item, and showed that the approximate form can yield remarkably good results. In our work, the first four moments are calculated by the dimension reduction numerical integration method, described in Section 3; and the first four cumulants are then obtained through the relations between the cumulants and moments as follows [37],

$$\begin{cases} \kappa_1 = \mu_1' \\ \kappa_2 = \mu_2' - \mu_1'^2 \\ \kappa_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 \\ \kappa_4 = \mu_4' - 4\mu_3'\mu_1' - 3\mu_2'^2 + 12\mu_2'\mu_1'^2 - 6\mu_1'^4 \end{cases} \quad (23)$$

where μ_i' ($i = 1, 2, 3$ and 4) are the first four moments about zero. For higher order cumulants, refer to [37]. The first four cumulant items, $\kappa_1, \kappa_2, \kappa_3$ and κ_4 , are then used to approximate the CGF of Y in Eq. (22), and finally, the Saddlepoint Approximations are applied to obtain the CDF and PDF of Y .

4.2 Saddlepoint Approximations for Calculation of CDF and PDF

Once the CGF of Y is obtained, it is easy to approximate the CDF and PDF of the performance function. Daniels [24] developed a simple formula for computing the PDF of the random variable Y ,

$$f_Y(y) = \left\{ \frac{1}{2\pi K_Y''(t_s)} \right\}^{\frac{1}{2}} e^{[K_Y(t_s) - t_s y]}, \quad (24)$$

where $K_Y(\cdot)$ is the CGF of Y ; $K_Y''(\cdot)$ is the second order derivative of the CGF of Y ; and t_s is the *saddlepoint*, which is the solution to the equation,

$$K_Y'(t) = y, \quad (25)$$

where $K_Y'(\cdot)$ is the first order derivative of the CGF.

Luannani and Rice [39] gave a very concise formula for calculating the CDF of Y ,

$$F_Y(y) = P\{Y \leq y\} = \Phi(w) + \phi(w) \left(\frac{1}{w} - \frac{1}{v} \right), \quad (26)$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the CDF and the PDF of the standard normal distribution, respectively,

$$w = \text{sgn}(t_s) \left\{ 2[t_s y - K_Y(t_s)] \right\}^{1/2}, \quad (27)$$

and

$$v = t_s \left[K_Y''(t_s) \right]^{1/2}, \quad (28)$$

where $\text{sgn}(t_s) = +1, -1$ or 0 , depending on whether the saddlepoint, t_s , is positive, negative or zero.

When r cumulant terms are used, according to Eq. (22), the CGF of Y is given by

$$K_Y(t) = \kappa_1 t + \frac{\kappa_2 t^2}{2!} + \dots + \frac{\kappa_r t^r}{r!}. \quad (29)$$

Since an analytical formulation of CGF for the performance function exists, as in Eq. (29), the use of Saddlepoint Approximations becomes straightforward. Because the CGF is in a polynomial form in terms of variable t , an analytical solution of the saddlepoint and the derivatives of the CGF can be easily obtained. From Eqs. (25) and (29), the equation for the saddlepoint is derived as

$$K_Y'(t) = \kappa_1 + \sum_{j=2}^r \kappa_j \frac{t^{j-1}}{(j-1)!} = y. \quad (30)$$

Solving Eq. (30), we obtain the saddlepoint t_s .

The CGF and its second order derivative at the saddlepoint t_s are then given by

$$K_Y(t_s) = \sum_{i=1}^r \frac{\kappa_i t_s^i}{i!}, \quad (31)$$

and

$$K_Y''(t_s) = \kappa_2 + \sum_{j=3}^r \kappa_j \frac{t_s^{j-2}}{(j-2)!}, \quad (32)$$

respectively.

Thereafter, PDF and CDF can be calculated by Eqs. (24) and (26), respectively.

During the numerical calculations, two problems may arise. One is that Eq. (30) possesses $r-1$ roots and the saddlepoint solution may have multiple real values. This problem was properly considered by Wang [32], who proposed a simple modification to Eq. (30) to ensure that the approximate $K_Y'(t)$ is monotonically increasing. The other problem is the singularity problem, when a square root of a negative value occurs in Eqs. (27) and (28). This problem can be overcome by reversing the sign of the performance function.

5. Examples

In this section, one mathematical example and two engineering problems are presented to demonstrate the effectiveness of the proposed method. For the two engineering problems, the bivariate dimension reduction method is employed to approximate the performance function for calculating moments, and three Gauss points per variable are used for numerical integration of moment generation after the dimension reduction. Comparisons are made with FORM, SORM and MCS to evaluate the accuracy and efficiency of the proposed method, whenever necessary.

Example 1: A Mathematical Problem

Consider a performance function with n independent random variables [10] given by

$$Y = g(\mathbf{X}) = \frac{\sum_{i=1}^n X_i - n}{\sqrt{n}}, \quad (33)$$

where X_i are random variables following standard exponential distributions with the PDF,

$$f_{X_i}(x_i) = e^{-x_i}. \quad (34)$$

This example is selected because Y is gamma-distributed, hence a theoretical solution exists. FORM, SORM, and the proposed method are used to estimate the CDF of the performance function, $Y = g(\mathbf{X})$, over a range of $[-3.5, 4.0]$ for two cases: $n = 4$ and $n = 20$. The results from FORM and the proposed method, and the exact theoretical solutions are depicted in Figs. 2 and 3, respectively. SORM encountered singularity in the left tail of the distribution; therefore, the results from SORM are not included here. For both cases, the results of the proposed method match the exact solution uniformly over the whole range of $[-3.5, 4.0]$. The accuracy of FORM is not acceptable. The inaccuracy comes from the fact that after the transformation from exponential variables to standard normal variables, the original linear function in Eq. (33) becomes a highly nonlinear function [12].

Insert Figure 2 here.

Insert Figure 3 here.

Since, for this simple problem, analytical (closed-form) solutions are available for the MPP, the cumulents, and the saddlepoint, no comparison is made for the efficiency. We will compare the efficiency in the following two engineering examples.

Example 2: Burst Margin of Disk

The burst margin, M_b , of a disk is defined as [40]

$$Y = M_b = g(\mathbf{X}) = \sqrt{\frac{f S}{3 \times 385.82 \delta \left(N \frac{2\pi}{60} \right)^2 (R^3 - R_0^3)(R - R_0)}}, \quad (35)$$

where $\mathbf{X} = [f, S, \delta, N, R, R_0]^T$; and in which f is the material utilization factor, S is the ultimate tensile strength, δ is the density, N is the rotor speed, R is the outer radius, and R_0 is the inner radius. Their distributions are shown in Table 2.

Table 3 shows the estimated moments about zero, $\mu'_j = E\{[g(\mathbf{X})]^j\}$, $j = 1, 2, 3$ and 4, of the performance function from the dimension reduction technique used in the proposed method and MCS with 1,000,000 simulations. The results indicate that the proposed method provides very accurate estimations.

Table 2 Distribution details of random variables

Variable	Distribution	Parameter 1	Parameter 2
F	Weibull	0.958	25.508
S	Normal	220000 lb/in^2	5000 lb/in^2
δ	Uniform	0.28 lb/in^3	0.30 lb/in^3
N	Normal	21000 rpm	1000 rpm
R	Normal	24 in	0.5 in
R_0	Normal	8 in	0.3 in

* Parameter 1 is the scale parameter for Weibull distribution, the mean for normal distribution, and the lower bound for uniform distribution, respectively.

+ Parameter 2 is the shape parameter for Weibull distribution, the standard deviation for normal distribution, and upper bound for uniform distribution, respectively.

Table 3 Estimated moments about zero of M_b from the proposed method and MCS

Moment	Proposed method	Monte Carlo simulation
1 st	2.50788×10^{-5}	2.50796×10^{-5}
2 nd	6.32468×10^{-10}	6.32518×10^{-10}
3 rd	1.60399×10^{-14}	1.60419×10^{-14}
4 th	4.09072×10^{-19}	4.09148×10^{-19}

Table 4 and Table 5 give the probabilities of $M_b < 3.2 \times 10^{-5}$ and $M_b < 2.6 \times 10^{-5}$, respectively, and the corresponding computational effort of FORM, SORM, the proposed method, and MCS. Since a sufficiently large number of simulations is used, the result of MCS is considered an accurate reference. The number of function evaluations used by FORM and SORM include the finite difference derivative calculation and the MPP search. For this problem, FORM, SORM and the proposed method provide accurate results for the case of $M_b < 3.2 \times 10^{-5}$. However, for the case of $M_b < 2.6 \times 10^{-5}$, where the estimated probability is smaller, FORM and SORM are much less accurate than the proposed method. SORM is more accurate than FORM because SORM uses curvature information at the MPP. This phenomenon from Tables 4 and 5 conform to the fact that SORM is asymptotically accurate [10]. The proposed method is more efficient than FORM and SORM, since the number of function evaluations is less than that of FORM or SORM.

Table 4 Probability of $M_b < 3.2 \times 10^{-5}$ and computational effort

	FORM	SORM	Proposed method	MCS
$P\{M_b < 3.2 \times 10^{-5}\}$	0.99932	0.99941	0.99948	0.99940
Function evaluations	262	297	154	1,000,000

Table 5 Probability of $M_b < 2.6 \times 10^{-5}$ and computational effort

	FORM	SORM	Proposed method	MCS
$P\{M_b < 3.2 \times 10^{-5}\}$	0.68808	0.69369	0.69947	0.69954
Function evaluations	264	299	154	1,000,000

Insert Figure 4 here.

Figure 4 shows the CDFs of M_b obtained from the proposed method and from MCS with 1,000,000 simulations. It is noted that the two CDF curves are nearly identical to each other over the entire range. The curves of the corresponding PDFs of M_b are also almost indistinguishable as shown in Fig. 5.

Insert Figure 5 here.

If FORM or SORM were used to generate the CDF or PDF curve, the MPP at each realization of Y (each of the dot points in Fig. 4 or Fig. 5) would be identified first. Assuming that the computational cost were the same at each point as that at point $y = 3.2 \times 10^{-5}$, which is 262 or 297 (see Table 4) for FORM or SORM, respectively, since there are 41 points in Fig. 4 or Fig. 5, the total number of function evaluations by FORM or SORM would be $262 \times 41 = 10742$ or $297 \times 41 = 12177$, respectively. Therefore, the proposed method is much more efficient than FORM or SORM, in generating CDF and PDF curves.

Example 3: An I Beam

An I beam [40], depicted in Fig. 6, is used to further demonstrate the effectiveness of the new method.

Insert Figure 6 here.

The performance function is given by

$$Y = g(\mathbf{X}) = \sigma_{\max} - S, \quad (36)$$

where

$$\sigma_{\max} = \frac{P a(L-a)d}{2LI}, \quad (37)$$

and

$$I = \frac{b_f d^3 - (b_f - t_w)(d - 2t_f)^3}{12}. \quad (38)$$

The eight random variables are $\mathbf{X} = [P, L, a, S, d, b_f, t_w, t_f]^T$. Table 6 provides the distribution information of all eight random variables.

Table 6 Distribution information of random variables

Variable	Type	Mean	Standard deviation
P	Normal	6070	200
L	Normal	120	6
a	Normal	72	6
S	Normal	170000	4760
d	Normal	2.3	1/24
b_f	Normal	2.3	1/24
t_w	Normal	0.16	1/48
t_f	Normal	0.26	1/48

Table 7 shows the estimated moments about zero, $\mu'_j = E\{[g(\mathbf{X})]^j\}$, $j = 1, 2, 3$ and 4, from the proposed method and from MCS with 1,000,000 simulations. It is noted that all four moments calculated by the proposed method are very close to those from MCS.

Table 7 Estimated moments about zero of Y

Moment	Proposed method	MCS
1 st	-1.98188×10^4	-1.98176×10^4
2 nd	7.13728×10^8	7.14110×10^8
3 rd	-2.74970×10^{13}	-2.75457×10^{13}
4 th	1.30143×10^{18}	1.30720×10^{18}

Table 8 gives the probability $Y < 0.0$ (the reliability) and the computational effort of FORM, SORM, the proposed method, and MCS. Again, the result of MCS is considered an accurate reference, since a sufficiently large number of simulations (10^6) is used. The results in Table 8 indicate that the proposed method provides a more accurate solution than FORM or SORM.

Table 8 Probability of $Y < 0.0$ and computational effort

	FORM	SORM	Proposed method	MCS
$P\{Y < 0.0\}$	0.8558	0.8658	0.8716	0.8711
Function evaluations	171	225	277	1,000,000

The estimated CDF and PDF curves of Y are shown in Fig. 7 and Fig. 8, respectively. The proposed method provides CDF and PDF results as accurate as those from MCS.

Insert Figure 7 here.

Insert Figure 8 here.

6. Conclusions

The method of dimension reduction integration and Saddlepoint Approximations estimates a response distribution function through the CGF of the response with the first four moments, whose accuracy and efficacy is the key to overall performance of the proposed method. To achieve an accurate and efficient approximation, the proposed method involves a dimension reduction numerical integration technique for calculating moments of a response and Saddlepoint Approximations for estimating CDF and PDF of the response. This method does not have many of the restrictions that existing methods have, such as the existence and search of the unique MPP, and gradient calculation. The new method can provide very accurate moments as well as PDF and CDF curves over the whole range of the distribution. In the two engineering example problems, the method produced accurate results with significantly reduced computational costs compared to MCS with a large number of simulations (10^6). Results also indicate that the proposed method is generally more accurate than FORM and SORM, and much more efficient for distribution generation. Since there is no need for derivative information, the proposed method is suitable to the situation where the derivative of a response function is difficult to obtain or the derivative does not exist. Because of its good features, the proposed method is an alternative to FORM or SORM when: 1) the performance function does not have a derivative, 2) the MPP is hard to identify, 3) there are multiple MPPs, or 4) the probability is close to the

median where FORM or SORM cannot provide accurate solutions. The method can also replace MCS when: 1) the CDF (or reliability) is extremely high, or 2) the number of random variables is not large. It should be noted that the computational costs will increase when the number of random variables, n , becomes large, since the total computation cost is $\frac{9n(n-1)}{2} + 3n + 1$ (Eq. (20)). In this case, MCS may be the method of choice if the CDF to be evaluated is not high. To improve the efficiency of the proposed method when there are many variables, variable-screening by Design of Experiments can be used to eliminate unimportant random variables. Finally, the accuracy of the proposed method can be further improved with increased cost, such as applying higher-variate dimension reduction [26] or higher-order Gauss-Hermite quadrature.

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