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A Robust Design Method Using Variable Transformation and Gauss-Hermite Integration

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SUMMARY

Robust design seeks an optimal solution where the design objective is insensitive to the variations of input variables while the design feasibility under the variations is maintained. Accurate robustness assessment for both design objective and feasibility usually requires an intensive computational effort. In this paper, an accurate robustness assessment method with a moderate computational effort is proposed. The numerical Gauss-Hermite integration technique is employed to calculate the mean and standard deviation of the objective and constraint functions. To effectively use the Gauss-Hermite integration technique, a transformation from a general random variable into a normal variable is performed. The Gauss-Hermite integration and the transformation result in concise formulas and produce an accurate approximation to the mean and standard deviation. This approach is then incorporated into the framework of robust design optimization. The design of a two-bar truss and an automobile torque arm is used to demonstrate the effectiveness of the proposed method. The results are compared with the commonly used Taylor expansion method and Monte Carlo simulation in terms of accuracy and efficiency.

KEY WORDS: robust design; optimization; uncertainty; Gauss-Hermite integration

1. INTRODUCTION

Deterministic optimization has been widely and successfully applied to engineering design. However, deterministic approaches do not consider the impact of unavoidable uncertainties associated with design parameters and design variables in any engineering system. Hence deterministic optimal solutions may be infeasible or conservative with the presence of uncertainty. Examples of uncertainty include variations in loading, material properties, modeling parameters, and fluctuations in design variables caused by various reasons, such as manufacturing processes, operating conditions, and computational errors. Robust design is one of the powerful tools that assist designers to make reliable decisions under uncertainty [1, 2]. Robust design [1], originally proposed by Taguchi, improves product quality through minimizing the effect of causes of variation without eliminating the causes. With the introduction of the nonlinear programming framework to robust design, both the robustness of design objectives and the robustness of design constraints (call design feasibility) are considered. Robust design optimization makes the objective inert to the variations of design variables and design parameters and satisfies design constraints under variations simultaneously [3].

Despite the benefits of robust design, one of the most challenging issues is the trade-off between efficiency and accuracy [4]. To accurately capture the probabilistic characteristics of the response variables, a number of deterministic analyses are required, which are usually very computationally intensive in practice. Many methods have been developed in modeling robust design problems. Chen *et al*. [5] developed a robust design procedure by using a response surface method to model a decision support system. They used Taylor expansion to approximate the mean and variance of a response variable and

employed the worst case scenario to formulate a constraint function. Ramakrishnan and Rao [6] formulated the robust design problem as a nonlinear optimization problem with Taguchi's quality loss function in which an objective function and constraint function are constructed with the mean and variance. Sundaresan *et al*. [7] incorporated a sensitivity index into the optimization procedure to determine a robust design solution. They defined an objective function as a weighted linear combination of the mean of a response function and a sensitivity index. Jung and Lee [3] developed a simple and efficient approach where an objective function is defined as a performance index related to a probability function, and probabilistic constraints are evaluated with the advanced first-order second moment method.

To improve the efficiency of robust design optimization, another type of method that separates the deterministic optimization from uncertainty analysis has been developed. Parkinson *et al*. [8] proposed a two-step robust design optimization method. The first step is to solve the deterministic optimization problem and obtain the nominal optimum, and then the robust optimal solution is acquired by considering the variations and using the nominal optimum as a starting point. Du and Chen [9] developed a probabilistic design strategy with sequential single loops of deterministic optimization and uncertainty assessment.

Previous work has mainly focused on modeling a robust design problem. Few computational tools have been developed for implementation. The first order Taylor expansion is widely used, and it is efficient but inaccurate. Monte Carlo simulation is another choice, and it is accurate but inefficient. The aim of this work is to develop an accurate and efficient computational method for robust design. In this method, the Gauss-Hermite numerical integration technique is used to calculate the mean and variance (or

standard deviation) of the objective and constraint functions (response variables). Two engineering examples are used to demonstrate the effectiveness of the proposed method. The results are compared with those from deterministic optimization, Taylor expansion method, and Monte Carlo simulation in terms of accuracy and efficiency.

2. ROBUSTNESS ASSESSMENT AND ROBUST DESIGN OPTIMIZATION

In this section, we will review approximation methods for robustness assessment and the common model of robust design optimization.

2.1 Deterministic optimization and robust design optimization models

A typical optimization problem is modeled as

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x},\mathbf{p}) \\ \text{subject to} \quad & g_i(\mathbf{x},\mathbf{p}) \geq 0, \ i = 1,2,\cdots,l \\ & x_j^L \leq x_j \leq x_j^U, \ j = 1,2,\cdots,n \end{aligned} \tag{1}$$

where $\mathbf{x} = [x_1,\cdots,x_n]^{\mathrm{T}}$ is a vector of design variables which are to be determined in the optimization process, and $\mathbf{p} = [p_1,\cdots,p_m]^{\mathrm{T}}$ is a vector of design parameters whose values are known as part of the problem specifications. x_j^L and x_j^U are the upper and lower bounds of the design variables, respectively. This deterministic approach does not consider variations in the design variables and design parameters. Because variations are ubiquitous

in real engineering environments, the solution to a deterministic design may be infeasible or conservative due to those variations.

In robust design optimization, both design variables and parameters are considered as random variables. Robust design optimization minimizes the mean and the variation of the objective function simultaneously while satisfying probabilistic constraints [3]. Therefore, a robust design problem is essentially a multi-objective optimization problem. How to deal with multi-objective problems is beyond the scope of this paper, though there exists a vast amount of literature for multi-objective optimization. Herein we only consider the commonly used weighting factor method that can sufficiently demonstrate our computational method. With the weighting factor method, a robust design problem is modeled as a single objective optimization problem [4, 9, 10] given by

$$\begin{aligned} \underset{\boldsymbol{\mu}_{\mathbf{X}}}{\text{minimize}} \quad & w_1 \frac{\mu_f(\mathbf{X},\mathbf{P})}{\mu_f^*} + w_2 \frac{\sigma_f(\mathbf{X},\mathbf{P})}{\sigma_f^*} \\ \text{subject to} \quad & G_i(\mathbf{X},\mathbf{P}) = \mu_{g_i}(\mathbf{X},\mathbf{P}) - k\sigma_{g_i}(\mathbf{X},\mathbf{P}) \geq 0, \ i = 1,2,\cdots,l \\ & \mu_j^L \leq \mu_j \leq \mu_j^U, \ j = 1,2,\cdots,n \end{aligned} \tag{2}$$

where w_1 and w_2 are weight factors, and $w_1 + w_2 = 1$; $\boldsymbol{\mu}_{\mathbf{X}}$ are the mean values of random design variables **X**; **P** is the vector of random design parameters. μ_f^* is the best achievable (ideal) optimal solution for the mean of the objective function, μ_f, when $w_1 = 1$ and $w_2 = 0$. σ_f^* is the best achievable (ideal) optimal solution for the standard deviation of the objective function, σ_f, when $w_1 = 0$ and $w_2 = 1$. μ_{g_i} and σ_{g_i} are the mean and standard deviation of constraint function g_i, respectively. k is a constant which represents the probability of the constraint satisfaction. For example, k = 3 indicates that the constraint

will be satisfied with a probability of 99.87% under the assumption that the constraint function is normally distributed. In this paper, the following symbol convention is mainly used. An *uppercase* letter denotes a random variable or parameter, a *lowercase* letter denotes an observation (or a realization) of a random variable, or a deterministic variable; and a *bold* letter denotes a vector. For instance, X stands for a random variable; x stands for a realization of X; $\mathbf{X}$ stands for a vector of random variables, $\left[X_1, X_2, \cdots, X_n\right]^{\mathrm{T}}$; and $\mathbf{x}$ is an observation of $\mathbf{X}$.

2.2 Typical methods for robustness assessment

As shown in the previous section, a robust design requires the mean and standard deviation of the objective and constraint functions. Let a random variable Y be a response variable that serves either an objective function or a constraint function, and let the function be in the form of $Y = y(\mathbf{X}, \mathbf{P})$. The mean μ_Y and variance σ_Y^2 of Y are then given by

$$\mu_Y = E[y(\mathbf{X}, \mathbf{P})] = \int_{-\infty}^{\infty} y(\mathbf{x}, \mathbf{p}) f_{\mathbf{X},\mathbf{P}}(\mathbf{x}, \mathbf{p}) d\mathbf{x} d\mathbf{p} \tag{3}$$

and

$$\sigma_Y^2 = E[Y^2] - \mu_Y^2 = \int_{-\infty}^{\infty} \left[y(\mathbf{x}, \mathbf{p})\right]^2 f_{\mathbf{X},\mathbf{P}}(\mathbf{x}, \mathbf{p}) d\mathbf{x} d\mathbf{p} - \mu_Y^2 \tag{4}$$

respectively. In the above equations, $f_{\mathbf{X},\mathbf{P}}$ is the joint probability density function of the random variables $\mathbf{X}$ and parameters $\mathbf{P}$. Practically, it is very difficult or even impossible to obtain an exact solution to Equations (3) and (4) because of the complicated integrand and the high dimensionality.

Various methods have been developed to approximate the mean and variance. A comprehensive review is provided by Du and Chen [4]. The commonly used methods include Taylor expansion approximation and Monte Carlo simulation. Taylor expansion approximation is a simple approach which uses the first order approximation of Y at the mean values $(\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\mu}_{\mathbf{P}})$. The approximations of the mean and variance of Y are given by

$$\mu_Y \cong y(\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\mu}_{\mathbf{P}}) \tag{5}$$

and

$$\sigma_Y^2 \cong \sum_{i=1}^{n} \left(\left. \frac{\partial y}{\partial X_i} \right|_{\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\mu}_{\mathbf{P}}} \right)^2 \sigma_{X_i}^2 + \sum_{k=1}^{m} \left(\left. \frac{\partial y}{\partial P_k} \right|_{\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\mu}_{\mathbf{P}}} \right)^2 \sigma_{P_k}^2 \tag{6}$$

respectively.

Monte Carlo simulation method gives very straight formulations as

$$\mu_Y \cong \frac{1}{N} \sum_{i=1}^{N} y(\mathbf{x}_i, \mathbf{p}_i) \tag{7}$$

and

$$\sigma_Y^2 \cong \frac{1}{N-1} \sum_{i=1}^{N} \left[y(\mathbf{x}_i, \mathbf{p}_i) - \mu_Y \right]^2 \tag{8}$$

respectively, where N is the number of simulations (samples); $\mathbf{x}_i$, $\mathbf{p}_i$ are the samples of random variables, which are drawn from the distributions of **X** and **P**.

Taylor expansion approximation provides an efficient solution to the means and variances. However, its accuracy may not be good enough when the response functions are highly nonlinear and the variations are large. Moreover, it requires accurate gradient calculations that can impose additional restrictions such as the differentiability of the

functions and the accurate numerical differentiation methods. Monte Carlo simulation is suitable for any function forms and is dimension-independent; but in order to ensure an acceptable accuracy, it needs a sufficiently large number of simulations (often thousands of function evaluations). Thus, Monte Carlo simulation is not practical for engineering applications.

Point estimate method, originally proposed by Rosenblueth [11], is another method for estimating statistical moments and has been used in engineering applications [12, 13]. The point estimate methods use the first few moments of each of random variables for obtaining the points and weights for each random variable to compute the moments of a response. The points and weights are different for different random variables.

In this paper, we use Gauss-Hermite integration technique to develop a simple point-estimate-type method for accurately and efficiently evaluating the means and variances of response functions, in which abscissas (points) and weights are fixed for all random variables. This approach is then incorporated into the framework of the robust design optimization to achieve an accurate robust solution.

3. ROBUSTNESS ASSESSMENT AND ROBUST DESIGN USING GAUSS - HERMITE INTEGRATION

3.1 Gauss -Hermite integration

For convenience, we use a vector $\mathbf{Z}$ to represent the vector of all the random variables $\mathbf{X}$ and $\mathbf{P}$, namely, $\mathbf{Z} = [\mathbf{X}, \mathbf{P}]^T$. Gauss-Hermite integration (GHI) [14] is a

numerical method for estimating integrals. For a one-dimensional integration, it approximates an integral by summing up some items of weighted *integrand* evaluated at so-called Gauss points (abscissas) as follows:

$$\int_{-\infty}^{\infty} y(z)dz \cong \sum_{I=1}^{r} w_I e^{z_I^2} y(z_I) \tag{9}$$

where r is the quadrature order (equal to the number of abscissas), z_I are abscissas (Gauss points), and w_I are weights (Gauss weights).

The GHI formula for an n-dimensional integral is given by

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} y(z_1,\cdots,z_n)dz_1\cdots dz_n \cong \sum_{I_1=1}^{r_1}\cdots\sum_{I_n=1}^{r_n} w_{I_1}\cdots w_{I_n} e^{z_{I_1}^2+\cdots+z_{I_n}^2} y(z_{I_1},\cdots,z_{I_n}) \tag{10}$$

where $r_1,\cdots,r_n$ denote the quadrature order (the number of abscissas) used in the $z_1,\cdots,z_n$ directions, $(z_{I_1},\cdots,z_{I_n})$ are the Gauss points, and $w_{I_1},\cdots,w_{I_n}$ are the corresponding weights.

The weights and abscissas for the GHI, for r = 1, 2, 3 and 4, are listed in Table I. For the weights and abscissas of higher quadrature orders, refer to [15].

Table I. Weights and abscissas for Gauss -Hermite integration.

Order (r)	Abscissa (z_I)	Weight (w_I)
1	0	1.772453
2	±0.707107	0.886227
3	0	1.181635
	±1.22474	0.295409
4	±0.524648	0.804914
	±1.65068	0.081312

3.2 Transformation of random variables

Before applying Gauss-Hermite integration to Equations (3) and (4) to calculate the mean and variance, we propose to transform all the random variables into a set of *identical* normal variables. Rosenblatt transformation [16] has been widely used in statistics, structural reliability, and uncertainty analysis. By Rosenblatt transformation, dependent, non-normal random variables are converted into mutually independent and standard normal variables [4, 16-18]. In this paper, adopting the same principle as Rosenblatt transformation, we transform all the random variables to the normal variables that follow a distribution of $N\left(0,\frac{1}{2}\right)$ (with a mean of 0 and a variance of $\frac{1}{2}$), instead of the standard normal distribution $N\left(0,1\right)$ in Rosenblatt transformation.

Without loss of generality, assume that Z_i $(i=1,2,\cdots,m+n)$ are independent. The proposed transformation is given by

$$u_i = t(z_i) = \frac{1}{\sqrt{2}}\Phi^{-1}\left[F_{Z_i}(z_i)\right],\ i=1,2,\cdots,m+n \tag{11}$$

where $\Phi^{-1}[\cdot]$ is the inverse function of the cumulative density function (CDF) of the standard normal distribution, $F_{Z_i}(z_i)$ is the CDF of random variable Z_i, z_i is a realization of Z_i, and u_i is a realization of the normal variable U_i. The probability density function (PDF) of U_i is given by

$$f_{U_i}(u_i) = \frac{1}{\sqrt{\pi}}e^{-u_i^2},\ i=1,2,\cdots,m+n \tag{12}$$

and the CDF of U_i is

$$F_{U_i}(u_i) = \Phi(\sqrt{2}u_i),\ i = 1, 2, \cdots, m+n \tag{13}$$

For example, if $Z_i \sim N\left(\mu_{Z_i}, \sigma_{Z_i}^2\right)$, the transformation between u_i and z_i is $u_i = t(z_i) = \frac{z_i - \mu_{Z_i}}{\sqrt{2}\sigma_{Z_i}}$.

It should be pointed out that if Z_i's are dependent, there is a slightly difference in Equations (11) and (13) [17-18].

The reasons for using this transformation are multifold. First, as shown in Table I, all abscissas are small values *around* zero. To efficiently and accurately estimate mean and variance, GHI requires that the PDF of the random integration variables should not be equal or very close to zero at the abscissas. The proposed approach can satisfy such a requirement. For illustration, Figure 1(a) gives the PDF of $X \sim N(10,1)$, which has the PDF $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-10)^2}{2}}$ and is almost equal to zero when X takes the abscissas given in Table I. After the transformation, the PDF of U is not equal to zero when U takes the abscissas, as shown in Figure 1(b).

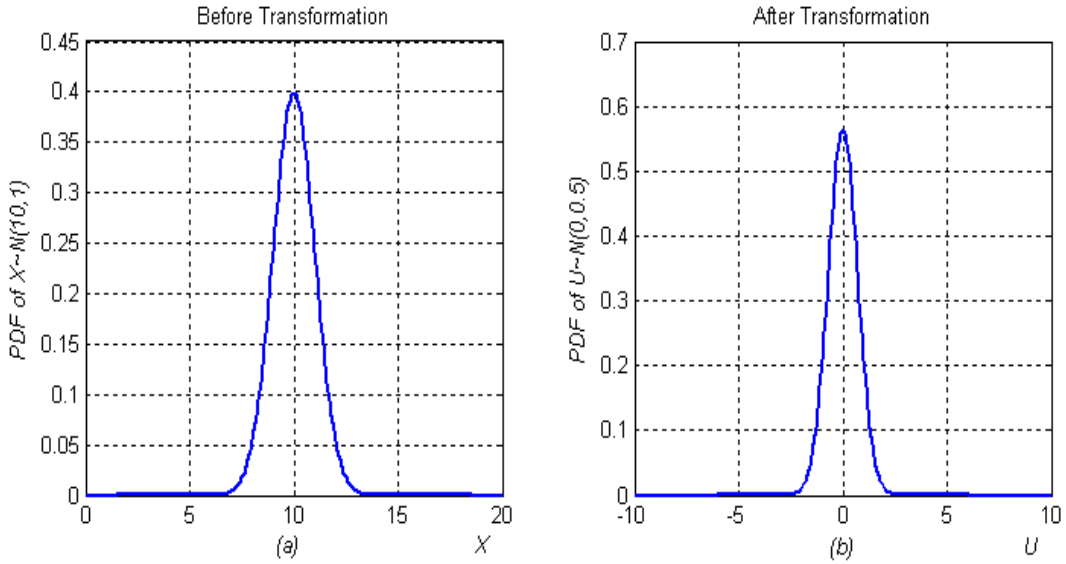

Figure 1. An example of the PDF curves before and after the transformation.

If the PDFs of integration variables are equal or very close to zero when the integration variables take the abscissas, GHI cannot provide an efficient and accurate estimation of the mean and variance. To demonstrate this, let us use GHI to approximate the mean of the response variable $Y = X$, $X \sim N(10,1)$ with four abscissas. As given, the exact mean of X is 10. However, without the transformation, according to Equations (3) and (9), GHI estimates the mean of X as

$$\mu_Y = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-10)^2}{2}} dx = \sum_{I=1}^{4} w_I e^{x_I^2} \left[x_I \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_I-10)^2}{2}} \right]$$
$$= (0.081312) e^{(-1.65068)^2} \left[(-1.65068) \frac{1}{\sqrt{2\pi}} e^{-\frac{(-1.65068-10)^2}{2}} \right]$$
$$+ (0.804914) e^{(-0.524648)^2} \left[(-0.524648) \frac{1}{\sqrt{2\pi}} e^{-\frac{(-0.524648-10)^2}{2}} \right]$$
$$+ (0.804914) e^{(0.524648)^2} \left[(0.524648) \frac{1}{\sqrt{2\pi}} e^{-\frac{(0.524648-10)^2}{2}} \right]$$
$$+ (0.081312) e^{(1.65068)^2} \left[(1.65068) \frac{1}{\sqrt{2\pi}} e^{-\frac{(1.65068-10)^2}{2}} \right] = 5.9502 \times 10^{-16} \neq 10$$

(14)

GHI fails to approximate the mean. The reason is that the $f_X(x)$ in the integrand is almost equal to zero at each of the abscissas (see Figure 1 and Table I). After the transformation, GHI approximates the mean of *Y* as

$$\mu_Y = \int_{-\infty}^{\infty} \left(\sqrt{2}u + 10\right) f_U(u)du = \int_{-\infty}^{\infty} \left(\sqrt{2}u + 10\right) \frac{1}{\sqrt{\pi}} e^{-u^2} du = \sum_{I=1}^{4} w_I e^{u_I^2} \left[\left(\sqrt{2}u_I + 10\right) \frac{1}{\sqrt{\pi}} e^{-u_I^2} \right]$$
$$= \sum_{I=1}^{4} w_I \left[\left(\sqrt{2}u_I + 10\right) \frac{1}{\sqrt{\pi}} \right] = \frac{1}{\sqrt{\pi}} \times 10 \times 2 \times (0.804914 + 0.081312) = 10.0$$

(15)

which is identical to the exact value. Therefore, the transformation ensures that GHI provides an accurate estimation.

The second reason for the transformation is that non-normal random variables can be converted into dependent, mutually independent normal variables. This makes our approach suitable to random variables with any distributions.

The third reason is that this specific transformation results in a concise GHI formula for estimating means and variances because the two exponential items are cancelled: one is from GHI in Equation (9) or (10) and the other is from the PDF of the normal random variables given in Equation (12). This cancellation reduces a complex expression into a more concise one as shown in Equation (15). This will be further demonstrated in the next subsection.

Using the *inverse* function of the proposed transformation function given in Equation (11), Equations (3) and (4) can be rewritten as

$$\mu_Y = \int_{-\infty}^{\infty} y\left[t^{-1}(\mathbf{u})\right] f_{\mathbf{U}}(\mathbf{u}) d\mathbf{u} \tag{16}$$

and

$$\sigma_Y^2 = \int_{-\infty}^{\infty} \left\{ y\left[t^{-1}(\mathbf{u})\right] \right\}^2 f_{\mathbf{U}}(\mathbf{u}) d\mathbf{u} - \mu_Y^2 \tag{17}$$

respectively, where

$$t^{-1}(\mathbf{u}) = \left[t^{-1}(u_1), t^{-1}(u_2), \cdots, t^{-1}(u_{m+n})\right]^{\mathrm{T}} \tag{18}$$

in which $t^{-1}(u_i)$ is the *inverse* function of the transformation function which is given by Equation (11); $f_{\mathbf{U}}(\mathbf{u})$ is the joint probability density function of **U** and is given by

$$f_{\mathbf{U}}(\mathbf{u}) = \prod_{i=1}^{m+n} f_{U_i}(u_i) \tag{19}$$

3.3 Estimating the mean and variance of a function by GHI

(a) *Y is a function of one random variable Z*

As indicated in Equation (16), after transformation, the mean of *Y* can be calculated as

$$\mu_Y = \int_{-\infty}^{\infty} y\left[t^{-1}(u)\right] f_U(u)du \tag{20}$$

Using Equation (9), Equation (20) becomes

$$\mu_Y \cong \sum_{I=1}^{r} w_I e^{u_I} \left\{ y\left[t^{-1}(u_I)\right] f_U(u_I)\right\} \tag{21}$$

Substituting Equation (12) into Equation (21) yields

$$\mu_Y = \frac{1}{\sqrt{\pi}} \sum_{I=1}^{r} w_I y\left[t^{-1}(u_I)\right] \tag{22}$$

As shown in Equation (17), the variance of *Y* reads

$$\sigma_Y^2 = \int_{-\infty}^{\infty} \left\{ y\left[t^{-1}(u)\right]\right\}^2 f_U(u)du - \mu_Y^2 \tag{23}$$

Applying Equation (9) to the integral term in Equation (23) obtains

$$\sigma_Y^2 \cong \sum_{I=1}^{r} w_I e^{u_I} \left\{ y\left[t^{-1}(u_I)\right]\right\}^2 f_U(u_I) - \mu_Y^2 \tag{24}$$

Plugging Equation (12) into Equation (24) yields

$$\sigma_Y^2 = \frac{1}{\sqrt{\pi}} \sum_{I=1}^{r} w_I \left\{ y\left[t^{-1}(u_I)\right]\right\}^2 - \mu_Y^2 \tag{25}$$

where u_I and w_I are the abscissas and weights of Gauss-Hermite integration, respectively, and *r* is the number of Gauss points (or the quadrature order) used.

(b) *Y is a function of multiple random variables* Z_i $(i = 1, 2, \cdots, n)$

Using the same procedure as in (a), the mean and variance of *Y* can be computed as

$$\mu_Y \cong \left(\frac{1}{\sqrt{\pi}}\right)^n \sum_{I_1=1}^{r_1} \cdots \sum_{I_n=1}^{r_n} w_{I_1} \cdots w_{I_n} y[t^{-1}(u_{I_1}), \cdots, t^{-1}(u_{I_n})] \tag{26}$$

and

$$\sigma_Y^2 \cong \left(\frac{1}{\sqrt{\pi}}\right)^n \sum_{I_1=1}^{r_1} \cdots \sum_{I_n=1}^{r_n} w_{I_1} \cdots w_{I_n} \left\{y[t^{-1}(u_{I_1}), \cdots, t^{-1}(u_{I_n})]\right\}^2 - \mu_Y^2 \tag{27}$$

respectively, where $r_1, \cdots, r_n$ denote the quadrature order (the number of abscissas) used in the $u_1, \cdots, u_n$ directions; $(u_{I_1}, \cdots, u_{I_n})$ are Gauss points; and $w_{I_1}, \cdots, w_{I_n}$ are the corresponding weights.

Generally, the accuracy of GHI approximation improves as the quadrature order *r* of each integration variable (the number of the abscissas for each integration variable) increases. However, increasing the quadrature order *r* also results in the increase of the total number of function (integrand) evaluations, because the number of function evaluations is $(n+m)^r$, where $n+m$ is the total number of random variables involved in the function, and *r* is the number of abscissas used for each random variable or parameter. Therefore, there is a trade-off between accuracy and efficiency. GHI gives an exact result for the mean if the function $y[\cdot]$ in Equation (22) or (26) is a polynomial of degree 2*r* -1 or less, where *r* is the quadrature order [14]. GHI provides an exact result for the variance only if $y[\cdot]$ is a polynomial of degree *r*-1 or less, due to the involvement of the square of $y[\cdot]$ in Equation (25) or (27). For most problems in structural or mechanical engineering, it is good enough to take *r* = 3 or 4.

The following example shows the accuracy and efficiency of the proposed method. The response variable is given by $Y = Z_1 Z_2$. Both Z_1 and Z_2 are normal random variables, $Z_1 \sim N(40, 4)$ and $Z_2 \sim N(70, 4)$. The quadrature order 2 (two abscissas) is used for both Z_1 and Z_2. Table II gives the results of different methods. The result of Monte Carlo simulation (MCS) is considered as an accurate reference since a large number of simulations (10^8) are performed.

Table II. Comparison of mean and variance from three methods.

Mean (μ_Y)			Variance (σ_Y^2)		
Taylor	GHI	MCS	Taylor	GHI	MCS
2800	2686	2686	26000	155583	155561

The above results indicate that the GHI method provides good approximations to both mean and variance and is much more accurate than the Taylor expansion method. In the Taylor expansion method, if the evaluations of two partial derivatives of the function Y are evaluated numerically by forward finite difference method, totally 3 function evaluations are taken. Since the quadrature order for both X_1 and X_2 is 2, hence the number of function evaluations used in GHI is 4. In this mathematical problem, the Taylor expansion method is slightly better than the GHI method in terms of efficiency, but the latter is much more accurate than the former.

3.4 Robust design optimization using Gauss-Hermite integration

Considering the desirable features of the proposed method, it is very promising to incorporate the proposed method into robust design optimization to achieve accurate and efficient solutions. The procedure of the proposed robust design optimization is depicted in Figure 2. The optimizer calls the proposed method repeatedly to evaluate the mean and standard deviation of the objective and constraint functions when updating the design variables.

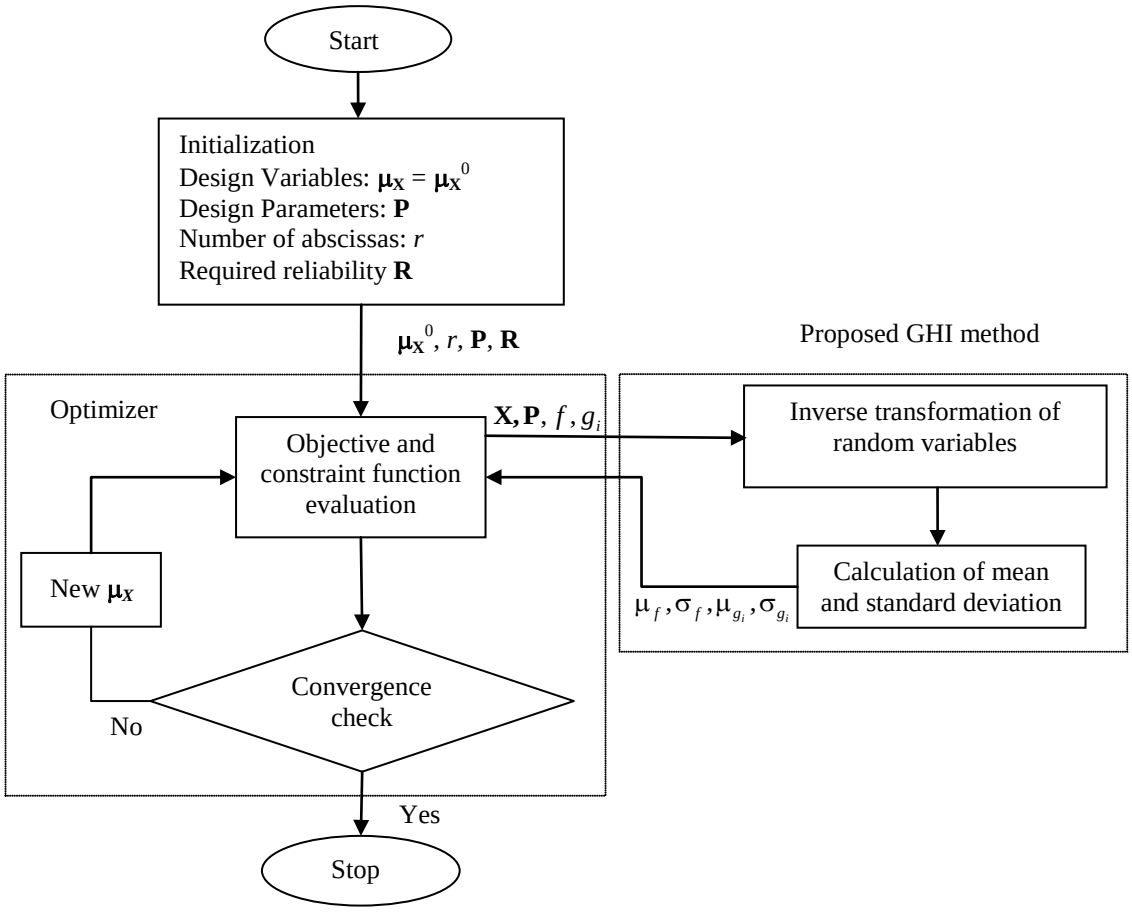

Figure 2. Flowchart of the robust design optimization using GHI.

4. NUMERICAL EXAMPLES

In this section, two engineering problems are used to demonstrate the proposed method compared with the Taylor expansion method and Monte Carlo simulation (MCS). Both design variables and design parameters are considered as random variables that follow normal distributions. For the Taylor expansion method, the forward finite difference approach is used to calculate the first order derivatives of objective and constraint functions. An identical number of abscissas are used for every random variable in each case for the proposed GHI method.

4.1. A two-bar truss

Figure 3 shows a structure of a two-bar truss [6]. Design variables $\mathbf{X} = \left[X_1, X_2 \right]^{\mathrm{T}}$ are the cross-sectional area of the truss and the half distance between the two bottom rollers. Design parameters $\mathbf{P} = \left[\rho, S, Q \right]^{\mathrm{T}}$ consist of the density of the bar material ρ , the bar material's maximum allowable tensile strength S , and the magnitude of the external force Q applied on the top of the truss. The design is to minimize the weight of the two-bar truss subjected to the two strength constraints about the axial stress in each bar.

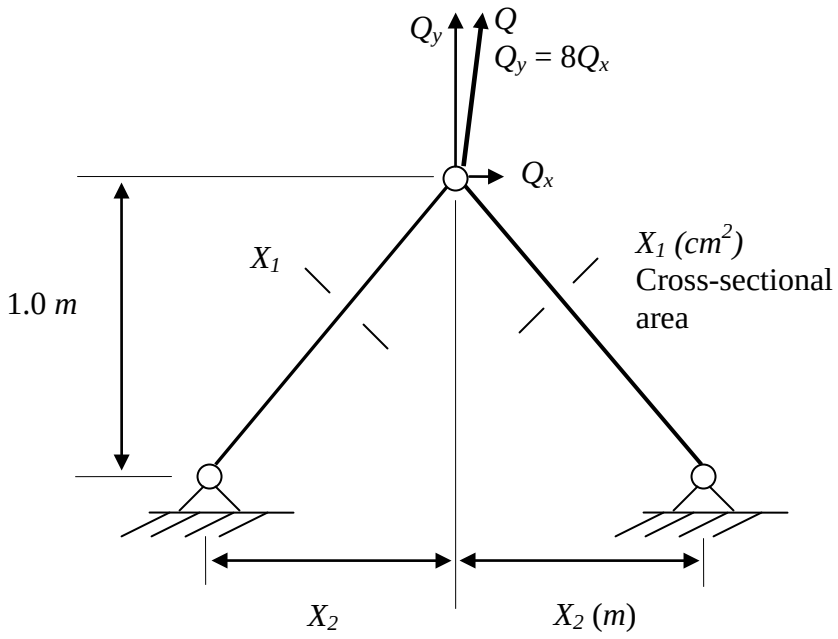

Figure 3. Two-bar truss.

The deterministic optimization design model of this problem is given by

$$\begin{aligned}
&\underset{X_1, X_2}{\text{minimize}}\ f(X_1, X_2) = \rho X_1 \sqrt{1 + X_2^2} \\
&\text{subject to}\quad g_1(X_1, X_2) = 1.0 - \frac{5Q}{\sqrt{65}S}\sqrt{1 + X_2^2}\left(\frac{8}{X_1} + \frac{1}{X_1 X_2}\right) \geq 0 \\
&\qquad\qquad g_2(X_1, X_2) = 1.0 - \frac{5Q}{\sqrt{65}S}\sqrt{1 + X_2^2}\left(\frac{8}{X_1} - \frac{1}{X_1 X_2}\right) \geq 0 \\
&\qquad\qquad 0.2 \leq X_1 \leq 20,\ \ 0.1 \leq X_2 \leq 1.6
\end{aligned} \tag{28}$$

where $\mathbf{P} = \left[\rho, S, Q\right]^{\mathrm{T}} = [10^4\ kg/m^3, 1050\ MPa, 800\ kN]^{\mathrm{T}}$ for the deterministic case.

In the robust design optimization, X_1, X_2, ρ, S, and Q are considered as random variables. A normal distribution is associated with each of the random variables. Distributions of design parameters are given in Table III. The means and standard deviations of design variables are unknowns. The means of design variables are to be

determined, and the standard deviations of the design variables are set to be 2% of the mean values.

Table III. Distribution information of design parameters.

Design parameter	Mean	Standard deviation
ρ	10,000 kg/m^3	2,000 kg/m^3
Q	800 kN	200 kN
S	1050 MPa	250 MPa

The robust design optimization model is given by

$$\begin{aligned} \underset{\mu_{X_1},\,\mu_{X_2}}{\text{minimize}} \quad & w_1 \frac{\mu_f}{\mu_f^*} + w_2 \frac{\sigma_f}{\sigma_f^*} \\ \text{subject to} \quad & G_1(X_1, X_2, Q, S) = \mu_{g_1} - k\sigma_{g_1} \geq 0 \\ & G_2(X_1, X_2, Q, S) = \mu_{g_2} - k\sigma_{g_2} \geq 0 \\ & 0.2 \leq \mu_{X_1} \leq 20 \\ & 0.1 \leq \mu_{X_2} \leq 1.6 \end{aligned} \tag{29}$$

where the constant k is chosen to be 3, which is equivalent to a reliability of 0.9987; and $w_1 = w_2 = 0.5$ is used.

Table IV. Results from different methods for the two-bar truss.

Method		Design variables (μ_{X_1}, μ_{X_2})
Deterministic		(5.3791, 0.3770)
Taylor		(10.9602, 0.3769)
GHI	2-abscissa	(11.7127, 0.3770)
	3-abscissa	(12.3913, 0.3770)
	4-abscissa	(12.6579, 0.3771)
MCS	$N=10^3$	(11.7566, 0.3772)
	$N=10^4$	(12.8132, 0.3770)
	$N=10^5$	(13.1514, 0.3772)

N – The number of Monte Carlo simulations

Table IV shows the optimal results from the deterministic optimization, the Taylor expansion method, MCS, and the proposed method (GHI). The result of MCS is considered as an accurate reference when a large number of simulations (10^5) are used for calculating the means and standard deviations. It is noted that the solutions of the proposed method are very close to those of MCS. The deterministic optimal solution is far from the results of MCS. The result from the Taylor expansion method has a considerable error compared with the accurate result. The proposed method produces more accurate solutions than the Taylor expansion method, and the approximate results get better as the number of the GHI abscissas increases from 2 to 4, which confirms the fact that the accuracy of GHI increases as the number of the abscissas increases. It is seen that the result of 2-abscissa case is close to that of MCS with 1000 simulations and the result of 3-abscissa case is better than the solution of MCS with 1000 simulations. The high accuracy of the proposed method relies in the accurate estimation of means and standard deviations from the GHI technique.

To demonstrate the accuracy of the proposed method further, the mean and standard deviation of the objective and constraint functions calculated at the respective optimal points (10.9602, 0.3769) from the Taylor expansion method and (12.3913, 0.3770) from the proposed method are also listed in Table V. Under the heading of (10.9602, 0.3769) from the Taylor expansion method, the first column displays the means and standard deviations calculated by the Taylor expansion method, and the second column displays those calculated by MCS with 10^5 simulations. The means and standard deviations from both the proposed method (3-absissa) and MCS with 10^5 simulations are also displayed under the heading of (12.3913, 0.3770) from GHI. The results calculated by Monte Carlo simulation are considered as accurate solutions. The table shows that the proposed method is more

accurate than the Taylor expansion method. Because of the large error, a feasible constraint in the Taylor expansion method may not be actually feasible. For example, at the optimal point from the Taylor expansion method, $G_1(X_1, X_2, Q, S) = \mu_{g_1} - k\sigma_{g_1} = 0.5091 - 3 \times 0.1697 = 0$ is active and feasible when the robustness assessment is based on the Taylor expansion method. However, if the accurate mean and standard deviation from MCS are used to calculate the constraint again, the constraint function will be $G_1(X_1, X_2, Q, S) = \mu_{g_1} - k\sigma_{g_1} = 0.4744 - 3 \times 0.2248 = -0.2$, which is no longer feasible.

Table V. Means and standard deviations at the two optima in Table IV.

Optimal solution (μ_{X_1}, μ_{X_2})	(10.9602, 0.3769)		(12.3913, 0.3770)	
	Taylor	MCS	GHI (3-abscissa)	MCS
Mean μ_f (kg)	11.7131	11.7153	13.2432	13.2454
Standard deviation σ_f (kg)	2.3544	2.3511	2.6626	2.6582
Mean μ_{g_1}	0.5091	0.4744	0.5360	0.5351
Standard deviation σ_{g_1}	0.1697	0.2248	0.1786	0.1798
Mean μ_{g_2}	0.7536	0.7363	0.7671	0.7667
Standard deviation σ_{g_2}	0.0852	0.1127	0.0897	0.0997

In the 2-abscissa case, 8 function evaluations are used to calculate the mean and standard deviation of the objective function, and 16 function evaluations are used to obtain the mean and standard deviation of each constraint function. In the 3-abscissa case, 27 and 81 function evaluations are used to get the mean and standard deviation of the objective function and each constraint function, respectively. The Taylor expansion method needs 4 function evaluations to calculate the mean and standard deviation of the objective function,

5 function evaluations to calculate the mean and standard deviation of each constraint function. Even though Taylor expansion method is more efficient than the proposed method, its accuracy is much lower. It is worthwhile to mention that more than four abscissas for each variable may be used to achieve higher accuracy in GHI. The disadvantage of more abscissas used is that it will result in a higher computational cost (more function evaluations) for estimating mean and variance, because the total number of function evaluations increases exponentially with the number of abscissas, as previously discussed.

4.2. An automobile torque arm

An automobile torque arm [19] is shown in Figure 4. This design problem has four design variables, $\mathbf{X} = \left[X_1, X_2, \mathrm{X}_3, X_4\right]^{\mathrm{T}} = \left[a, d_1,\ d_2,\ l\right]^{\mathrm{T}}$. The design parameters include the yield stress S_y, Young's modulus E, and the external force Q. The distributions of the random variables are listed in Table VI. The volume is to be minimized with strength and buckling constraints.

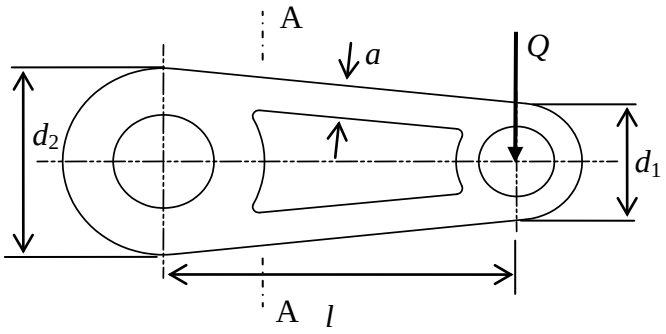

Figure 4. Automobile torque arm.

Table VI. Distribution information of design variables and parameters.

Variable	Mean	Standard deviation
X_1 (a)	-	0.05 mm
X_2 (d_1)	-	1.5 mm
X_3 (d_2)	-	2.5 mm
X_4 (l)	-	6.0 mm
Q	5500 N	900 N
S_y	170 N/mm^2	30 N/mm^2
E	2.1×10^{11} N/mm^2	2.0×10^{10} N/mm^2

The volume is given by

$$f(a,d_1,d_2,l)=\frac{\pi d_2^2}{4}a+2(l-\frac{d_1}{2}-\frac{d_2}{2})a^2 \tag{30}$$

The first constraint is related to the yield failure at section A-A and is given by

$$g_1(a,d_1,d_2,l)=1-\frac{Q(2l-d_2)d_2}{4IS_y}\geq 0 \tag{31}$$

where

$$I=\frac{a^2\left(d_2-a\right)^2}{2}+\frac{a^4}{6} \tag{32}$$

The second constraint is related to the buckling failure at the two connecting rods and is given by

$$g_2(a,d_1,d_2,l)=\frac{\pi^2Ea^4}{3(2l-d_1-d_2)^2}\frac{d_2-d_1}{Ql}-1\geq 0 \tag{33}$$

In robust design optimization, a,d_1,d_2,l,Q,S_y and E are considered as random variables. The robust design optimization model is

$$
\begin{aligned}
\underset{\mu_{X_1},\mu_{X_2},\mu_{X_3},\mu_{X_4}}{\text{minimize}} \quad & w_1 \frac{\mu_f}{\mu_f^*} + w_2 \frac{\sigma_f}{\sigma_f^*} \\
\text{subject to} \quad & G_1(X_1, X_2, X_3, X_4, Q, S_y) = \mu_{g_1} - k\sigma_{g_1} \geq 0 \\
& G_2(X_1, X_2, X_3, X_4, Q, E) = \mu_{g_2} - k\sigma_{g_2} \geq 0 \\
& 5 \leq \mu_{X_1} \leq 15 \\
& 45 \leq \mu_{X_2} \leq 55 \\
& 55 \leq \mu_{X_3} \leq 65 \\
& 110 \leq \mu_{X_4} \leq 210
\end{aligned}
\tag{34}
$$

where the constant k is chosen to be 3, which corresponds a reliability of 0.9987; and the weighting factors are set to $w_1 = w_2 = 0.5$.

Table VII gives the results from four different methods. The result of MCS with a large number of simulations (10^5) is again considered as an accurate solution for the comparison. According to the results, the proposed method provides the closest solution to the accurate solution when 4 abscissas are used for each random variable. Also the accuracy of the proposed method improves as the number of abscissas increases from 2 to 4. This phenomenon confirms again the fact that the accuracy of GHI improves as the number of the abscissas increases. The result of the Taylor expansion method is better than deterministic case in terms of robustness, but it is far away from the accurate robust solution.

Table VII. Comparisons of results from different methods.

Method		Design variables $(\mu_{X_1}, \mu_{X_2}\ \mu_{X_3}, \mu_{X_4})$ (mm)
Deterministic		(8.135, 55.000, 55.000, 110.000)
Taylor		(14.999, 47.204, 64.999, 134.367)
Proposed	2-abscissa	(12.510, 49.156, 65.000, 146.276)
	3-abscissa	(13.518, 49.295, 65.000, 157.986)
	4-abscissa	(13.674, 49.321, 65.000, 159.686)
MCS	N = 1000	(12.693, 45.000, 65.000, 141.897)
	N = 5000	(14.259, 49.274, 65.000, 167.396)
	N = 10^5	(13.708, 49.330, 65.000, 160.342)

Similar to Example 1, to demonstrate the accuracy of the proposed method further, the mean and standard deviation of the objective and constraint functions calculated by both the Taylor expansion method and the proposed method (3-abscissa) at their respective optimal points are listed in Table VIII. The results are compared with those from the MCS with 10^5 simulations. The results indicate again that the proposed method is more accurate than Taylor expansion method in estimating the mean and standard deviation.

Table VIII. Means and standard deviations at the two optima in Table VII

Optimal solution $(\mu_{X_1}, \mu_{X_2}, \mu_{X_3}, \mu_{X_4})$	(14.999, 47.204 , 64.999, 134.367) Taylor	MCS (10^5)	(13.518, 49.295, 65.000, 157.986) GHI (3-abscissa)	MCS (10^5)
Mean μ_f (mm^3)	8.4980×10^4	8.5064×10^4	8.1784×10^4	81782×10^4
Standard deviation σ_f (mm^3)	4.2700×10^3	3.9889×10^3	3.5465×10^3	3.5352×10^3
Mean μ_{g_1}	0.6303	0.6164	0.4471	0.4478
Standard deviation σ_{g_1}	0.0952	0.1049	0.1490	0.1495
Mean μ_{g_2}	3.4408×10^7	1.4411×10^8	4.2839×10^7	4.2691×10^7
Standard deviation σ_{g_2}	1.1372×10^7	4.7866×10^7	1.3753×10^7	1.3821×10^7

5. DISCUSSION AND SUMMURY

In order to achieve an accurate and efficient robust design solution, a new robustness assessment technique is proposed. In this method, the random variables are transformed to the normal variable that follows a distribution $N\left(0,\frac{1}{2}\right)$. The numerical Gauss-Hermite integration technique is then employed to calculate the mean and standard deviation of the response functions. A concise formula that only sums up a sequence of products of weights and response values is constructed for estimating the mean and standard deviation of response functions. As demonstrated in the examples, this method is more accurate than the widely used Taylor expansion approach and is more efficient than Monte Carlo simulation. Since there is no need for derivatives, the proposed method is suitable to the situations where the derivatives of a response function are difficult to obtain, or the derivatives do not exist. This method can be easily implemented in the framework of robust design.

It should be noted that, similar to other numerical integration techniques, the efficiency of the proposed method will decrease when the number of random variables increases. When the number of random variables is large, before the robust design, sensitivity analysis [17, 20, 21] may be performed to screen out the random variables which are not important to the objective and constraint functions. Design of Experiments [22], and response surface method [5, 23] can also be used for this purpose. During the robust design process, those random variables that are not of significance will be fixed at their

mean values. Another strategy is to use different number of abscissas for different random variables [12] to reduce the number of function evaluations.

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