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Analytical Robustness Assessment for Robust Design

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Abstract

Robust design ensures the product performance be insensitive to various uncertainties and therefore results in high quality and productivity. Robustness assessment, which evaluates the variability of the performance, is an important and indispensable component of robust design. An accurate and efficient robustness assessment is essential for obtaining a real robust solution. The aim of this paper is to investigate features of the existing model-based methods for robustness assessment in terms of accuracy, efficiency, and reliability. Recommendations on the use of those methods are provided based on the comparison study through example problems.

Keywords Robust design · Robustness · Variance · Percentile

1. Introduction

Robust design, pioneered by Dr. Taguchi (1987, 1993), is a powerful design tool for high quality and productivity. By assessing the variations that a product experiences during design, manufacture, and operation, robust design ensures that a product perform its intended function regardless of these variations or uncertainties. Examples of uncertainties include variations in loading, material properties, and part dimensions. They are caused by various reasons such as imprecision in manufacturing processes, random operation conditions, and computational errors.

The fundamental principle of robust design is to improve product quality or stabilize a system performance by minimizing the effects of variations without eliminating their causes (Chen et al. 1996). As a result, robust design (Phadke 1989)

1) allows the use of low grade materials and components since the product performance is insensitive to the variations in material properties;
2) reduces labor and material cost since the effects of manufacturing variations are minimized;
3) improves the reliability and reduces operation cost since the product performance is inert to environmental uncertainties; and
4) ensures the high productivity because of the structured and systematic process of the development process.

Traditional robust design (Taguchi's method) is conducted through statistical experiments (Fowlkes and Creveling 1995, Taguchi et al. 1999, Montgomery 2005). With the availability of the CAE (Computer Aided Engineering) simulations and the introduction of the nonlinear programming to robust design, robust design has also been

performed through model-based optimization (Chen et al. 1996, Santner et al. 2003, Frey et al. 2005). To distinguish the experiment-based robust design from the model-based robust design, we use the term of *analytical robust design* to represent the latter. The analysis that assesses the robustness under the analytical robust design framework is therefore called *analytical robustness assessment* in this paper. We focus our discussions on analytical robust design and analytical robustness assessment herein. For an overview of robust design, interested readers may refer to Park et al. (2006).

In analytical robust design, both the robustness of design objectives and the robustness of design constraints (called design feasibility) are considered. Under the optimization framework, the objective of robust design is to optimize the mean performances and minimize the variations of the performances simultaneously. The design constraints should be also satisfied in the presence of uncertainty.

How to model robust design problems has been widely investigated in the literature. The modeling methods address four main issues:

(1) Mathematical models, which formulate a robust design problem as an optimization problem with stochastic nonlinear programming (probabilistic approach) (Sundaresan et al. 1995, Lee et al. 1996, Chen and Yuan 1999, Du and Chen 2000, Jung and Lee 2002, Huang and Du 2006a) or deterministic programming (Belgundu and Zhang 1992, Badhrinath and Rao 1994, Gunawan and Azarm 2004, 2005);

(2) Objectives handling, which converts multi-objectives into a single objective, such as the single-to-noise ratio (Taguchi 1987, Phade1989), the weighting-sum method (Du and Chen 2000, Lee and Park 2001), quality utility (Chen et al. 1999), and physical programming (Chen et al. 2000);

(3) Solving strategies, which achieve efficient and quick convergence in the optimal design, such as sequential procedures (Parkinson et al. 1993, Du and Chen 2004), re-weighting scheme (Fu and Sahin 2004), and metamodeling or response surface methods (Simpson et al. 2001, Jin et al. 2004); and

(4) Robustness assessment, which estimates the variations (dispersions) of objective and constraint functions under uncertainties (Du et al. 2004, Thaweepat et al. 2004, Montgomery 2005, and Frey et al. 2005).

The probabilistic formulation with the consideration of the variations in both objective and constraints is widely used. In real-world complex engineering problems, the challenge of using this formulation is the costly computation for robustness assessment. To accurately capture the statistical dispersion of the performance functions (objective and constraint functions), a number of deterministic analyses are required. A good balance between efficiency and accuracy is the major consideration of choosing a robustness assessment method.

Although various methods have been developed for robustness assessment, their characteristics in terms of accuracy, efficiency, and reliability are not yet fully investigated. The aim of this paper is to investigate the accuracy, efficiency, and limitations of existing robustness assessment methods and then provide recommendations for the use of those methods. For the investigation of the feasibility robustness of design constraints, one may also refer to Du and Chen (2000).

The rest of this paper is organized as follows. In Section 2, two models of robustness assessment, variance formulation and percentile difference formulation, are discussed. Methods for evaluating variance and percentile difference are reviewed and

discussed in Sections 3 and 4, respectively. Section 5 presents three examples to demonstrate the features of various methods. The comparison summary and recommendations are made in Section 6.

2. Robustness Assessment Modeling

Robustness of a performance reflects the dispersion of the performance from its mean. There are two ways to measure the robustness: the traditional variance and the recently proposed percentile difference. We will discuss how to model the robustness in this section and then discuss how to evaluate robustness in the next section.

2.1 Model Robustness by Variance

Traditionally, robustness has been measured by the variance (or standard deviation) of the performance. Let a random variable *Y* be a response variable that represents a performance in robust design. The performance could serve as either an objective or a constraint. Let *Y* be in the form of

$$Y = g(\mathbf{X}), \tag{1}$$

where $\mathbf{X}$ is the vector consisting of *n* input random variables of the model. The elements of $\mathbf{X}$ can be design variables that are controllable by a designer or the noise factors that are uncontrollable. In this paper, we consider the situation where the random variables $\mathbf{X}$ are mutually independent.

Theoretically, the variance σ_Y^2 of *Y* is evaluated by

$$\sigma_Y^2 = E\left[\left(Y-\mu_Y\right)^2\right] = \int_{-\infty}^{\infty}\left[g(\mathbf{x})-\mu_Y\right]^2 f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}, \tag{2}$$

where μ_Y is the mean of *Y*, which is calculated by

$$\mu_Y = E[Y] = \int_{-\infty}^{\infty} g(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}. \quad (3)$$

In the above equations, $f_{\mathbf{X}}$ is the joint probability density function (PDF) of the random variables **X**.

2.2 Model Robustness by Percentile Difference

Recently, Du et al. (2004) proposed a new robustness measure called *percentile difference*, which is the difference between the performances at both tails of the distribution. The percentile difference is defined as

$$\Delta y_{\alpha_1}^{\alpha_2} = y^{\alpha_2} - y^{\alpha_1}, \quad (4)$$

where y^{α_1} and y^{α_2} are two values of *Y* given by (Bain and Engelhardt 1991)

$$\text{Prob}\{Y \le y^{\alpha_i}\} = \alpha_i \quad (i = 1, 2). \quad (5)$$

α_1 is the cumulative distribution function (CDF) of *Y* at the left tail, and its value can be taken as 0.05 or 0.1. α_2 is the CDF of *Y* at the right tail, and its value can be taken as 0.95 or 0.99.

Figure 1 illustrates the concept of the percentile difference. As shown in the figure, reducing the percentile difference is equivalent to shrinking the dispersion of the distribution of *Y* when the distribution of *Y* is unimodal.

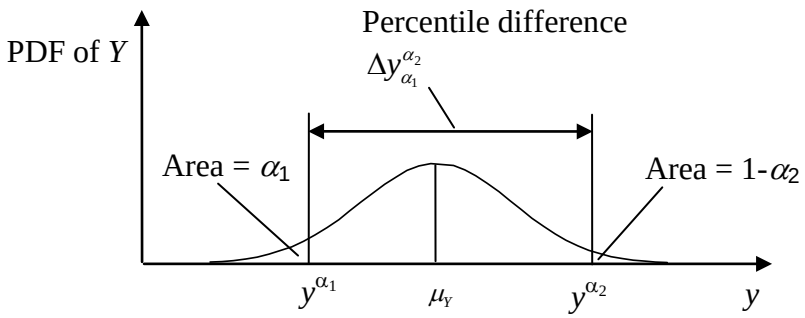

Fig. 1 Illustration of the percentile difference

2.3 Comparison of the Two Measures

Variance (or standard deviation) is the conventional measure for robustness and is widely used in engineering. It is easy to understand and is applicable to any distributions such as unimodal and multimodal distributions. Percentile difference is not as straightforward as the standard deviation, but it has several advantages. One major advantage is that a percentile is related to the probability at the tail areas of a performance distribution; it therefore carries more information than the standard deviation, such as the skewness of a distribution. The standard deviation only captures the dispersion around the mean value. From the percentile difference, we know to what extent or at what confidence level the design robustness is achieved. This confidence level is given by $\alpha_2 - \alpha_1$. The other major advantage is that the percentile difference can be obtained from the well-developed reliability analysis methods such as the First Order Reliability Method (Hasofer and Lind 1974, Breitung 1984, Ayyub and Haldar 1984, Hohenbichler et al. 1987, Tu and Choi 1999, Youn et al. 2003, Du et al. 2004, Du 2006). However, percentile difference may not be applicable to multimodal distributions which may be rare in design applications.

In general, if the performance follows a unimodal distribution, reducing the standard deviation is equivalent to reducing the percentile difference. Therefore, either measure could be used, depending upon the computational convenience. In the next sections, we discuss how to computationally evaluate the two measures.

3. Methods for Estimating Variance

In this section, we provide an extensive review on the existing methods for estimating variance. As shown in Eqs. (1) to (3), calculating variance is a multidimensional integration problem. For a real-world engineering system, it is very difficult or even impossible to obtain a closed-form solution to Eq. (2) because of the complicated integrand and the high dimensionality. Various simulation and approximation methods have been developed. Existing methods can be categorized into three types: Taylor series expansion methods, simulation methods, and point estimate methods.

3.1 Taylor Series Expansion Methods

Taylor series expansion methods (Robinson 1998, Putko et al. 2002, Du and Chen 2000, Jung and Lee 2002) are the simple and efficient approaches which employ Taylor series approximation of the performance function $Y = g(\mathbf{X})$ at the mean values, $\boldsymbol{\mu}_{\mathbf{X}}$, of $\mathbf{X}$. When the first order Taylor series expansion of Y is used, the variance of Y is given by

$$\sigma_Y^2 \cong \sum_{i=1}^{n} \left(\left. \frac{\partial g}{\partial X_i} \right|_{\boldsymbol{\mu}_{\mathbf{X}}} \right)^2 \sigma_{X_i}^2 . \tag{6}$$

When the second order Taylor series expansion is applied, the variance of Y can be calculated by

$$\sigma_Y^2 \cong \sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}_\mathbf{X}}\right)^2 \sigma_{X_i}^2 + \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left.\frac{\partial^2 g}{\partial X_i \partial X_j}\right|_{\boldsymbol{\mu}_\mathbf{X}} \sigma_{X_i}\sigma_{X_j}\right)^2 . \tag{7}$$

The second order Taylor series expansion method is generally more accurate than the first order Taylor series expansion method, but the former is more computationally expensive than the latter. If the derivatives are computed with the forward or backward finite difference technique (Dennis and Schnabel 1983), the total number of function evaluations is n+1 and $\frac{1}{2}n(n+1)+1$ for the first and second order Taylor series expansion methods, respectively. Both methods require the differentiability of the performance function. In addition, it is should be pointed out that if the mean-value point $\boldsymbol{\mu}_\mathbf{X}$ is a stationary point of the performance function (e.g. all first-order partial derivatives equal zero at $\boldsymbol{\mu}_\mathbf{X}$) , the first order Taylor series expansion method gives inaccurate results. For instance, for the one-dimensional performance function $Y = g(\mathbf{X})$, where $\mathbf{X} = [x]$, in the three cases as shown in Fig. 2, as x varies with the distribution shown, the performance function $Y = g(\mathbf{X})$ has a zero variance while the true variance is obviously nonzero. The result provided by the second order Taylor series expansion method is also not accurate for Case (c).

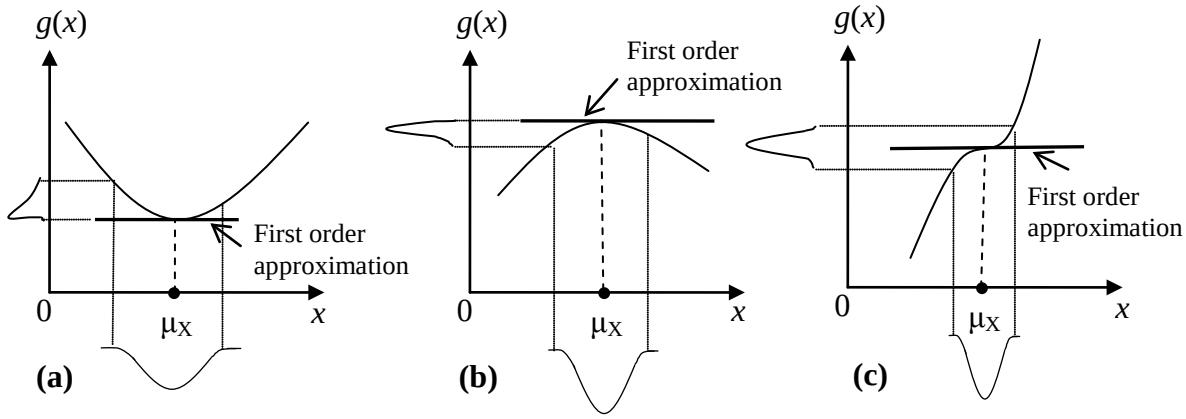

Fig. 2 Mean-value points at a stationary point of the performance function

3.2 Simulation Methods

Simulation methods (Rubinstein 1981, Law and Kelton 1982) give a very straight formulation as

$$\sigma_Y^2 \cong \frac{1}{N-1}\sum_{j=1}^{N}\left[g(\mathbf{x}^j) - \mu_Y \right]^2, \tag{8}$$

where

$$\mu_Y \cong \frac{1}{N}\sum_{j=1}^{N} g(\mathbf{x}^j) \quad (j = 1, 2, ..., N), \tag{9}$$

and $\mathbf{x}^j$ are the samples of random vector **X**, which are drawn from the distributions of **X**. The samples can be obtained by different sampling techniques. There are three commonly used sampling techniques: random sampling, Latin hypercube sampling, and Hamersley sequence sampling.

3.2.1 Random Sampling (RS)

Random sampling (RS) (Rubinstein 1981), also called Monte Carlo sampling, is the simplest sampling procedure. Random variable sampling requires the generation of *N*

samples of a random vector consisting of all of the n random variables, $\mathbf{X} = \left[X_1,\ X_2, ...,\ X_n \right]^{\mathrm{T}}$. With random sampling for uncorrelated variables, if each variable X_i follows a CDF F_{X_i}, the component N samples x_i^j are *independently* generated using Quantile-Quantile transformation from the N samples v_i^j, which are *uniformly distributed* on [0, 1), $U\left[0,1\right)$,

$$x_i^j \ = \ F_{X_i}^{-1}\left(v_i^j\right) \quad \left(i = 1, 2, ..., n;\ j = 1, 2, ..., N\right), \tag{10}$$

where $F_{X_i}^{-1}\left(\cdot\right)$ is the inverse CDF of X_i.

Then,

$$\mathbf{x}^j = \left[x_1^j, x_2^j, ..., x_n^j \right] \quad \left(j = 1, 2, ..., N \right). \tag{11}$$

3.2.2 Latin Hypercube Sampling (LHS)

LHS, first introduced by McKay et al. (1979), is a stratified sampling technique in which the uniformly distributed samples $v_i^j \ \left(j = 1, 2, ..., N \right)$ are drawn by

$$v_i^j \ = \ \frac{\pi_i\left(j - 1 \right) \ + \ w_i^j}{N} \quad \left(i = 1, 2, ..., n;\ j = 1, 2, ..., N\right), \tag{12}$$

where π_i is a uniform permutation of 0,…, N-1, and w_i^j is a random observation from $U\left[0,1\right)$. v_i^j can also be obtained by the widely used median version of Eq. (12)

$$v_i^j \ = \ \frac{\pi_i\left(j - 1 \right) \ + \ 0.5}{N} \quad \left(i = 1, 2, ..., n;\ j = 1, 2, ..., N\right). \tag{13}$$

Then,

$$x_i^j \ = \ F_{X_i}^{-1}\left(v_i^j\right) \quad \left(i = 1, 2, ..., n;\ j = 1, 2, ..., N\right). \tag{14}$$

The N values thus obtained for X_1 are paired at random (equally likely combination) without replacement with the N values of X_2. These N pairs are combined in a random manner without replacement with the N values of X_3 to form N triplets. This process continues until N n-tuplets are formed. The N n-tuplets may be contained in an $N \times n$ sample matrix, in which each row corresponds to one sample of $\mathbf{X}$. Totally there are N samples of $\mathbf{X}$ in a matrix.

Because LHS exhaustively *stratifies* across the whole range of each sampled variable, it mitigates the problem that important intervals with low probability but high consequences are likely to be missed (Owen 1997). Compared to random sampling, LHS requires fewer samples for a similar accuracy. It is more efficient for estimating statistical moments (mean, variance, etc.) and produces more stable results than random sampling (Helton and Davis 2003). However, the main shortcoming of LHS stratification scheme is one-dimensional and does not provide good uniform properties on a multidimensional unit hypercube (Diwekar 2003). LHS provides no significant practical advantage over random sampling if the performance function is highly nonlinear (Manteufel 2000, Guiunta 2003).

3.2.3 Hamersley Sequence Sampling (HSS)

HSS employs a quasi-random sequence that is uniformly distributed in the *unit hypercube* $[0, 1)^n$. The sequence is generated using the following expansion of a nonnegative integer j (Hamersley 1960, Kalagnanam and Diwekar 1997)

$$j = a_0 + a_1 p + a_2 p^2 + \cdots + a_r p^r, \tag{15}$$

where p is a prime number and a_i (i = 0,1,…, r) are nonnegative integer coefficients smaller than p. p and a_i are then used to generate a number between 0 and 1 by

$$u_{j,p} = \frac{a_0}{p} + \frac{a_1}{p^2} + \frac{a_2}{p^3} + \cdots + \frac{a_r}{p^{r+1}}. \tag{16}$$

When j takes values from $1, 2, ..., N$, any sequence of n-1 different prime numbers, $p_1, p_2, ..., p_{n-1}$, leads to N Hamersley points in the *unit hypercube* [0, 1)^n as follows:

$$\mathbf{v}^j = \left[v_1^j, v_2^j, ..., v_n^j \right] = \left[\frac{j - 0.5}{N}, u_{j,p_1}, u_{j,p_2}, ..., u_{j,p_{n-1}} \right], \quad (j = 1, 2, ..., N). \tag{17}$$

After the N Hamersley points are acquired, the N samples of **X** can then be obtained through Eq. (10).

A variant of HSS is the Halton sequence sampling (Halton 1960, Reichert et al. 2002), in which N Halton points in the *unit hypercube* [0, 1)^n are given by

$$\mathbf{v}^j = \left[v_1^j, v_2^j, ..., v_n^j \right] = \left[u_{j,p_1}, u_{j,p_2}, ..., u_{j,p_n} \right], \quad (j = 1, 2, ..., N), \tag{18}$$

where $p_1, p_2, ..., p_n$ are a sequence of n different prime number.

Compared with LHS and HSS, Halton sequence sampling has the advantage that the sample size can be easily increased by adding some new sampling points without modifying the old sampling points. Using some test problems, Kalgnanam and Diwekar (1997) demonstrated that HSS has low-discrepancy and good space filling properties and converges at 3-100 times faster than LHS and random sampling. Figs. 3(a) – (c) illustrate the space filling properties of random sampling, LHS, and HSS. It is evident that HSS are much more uniformly distributed than random sampling and LHS. However, when the dimensionality is high, the uniform property of HSS deteriorates, such as the aliasing problem (Crow 1977, Lau 1995). For instance, the points tend to line up in slanting lines

on a 2D plane when the prime numbers 13 and 17 are used, which is illustrated in Fig. 3 (d).

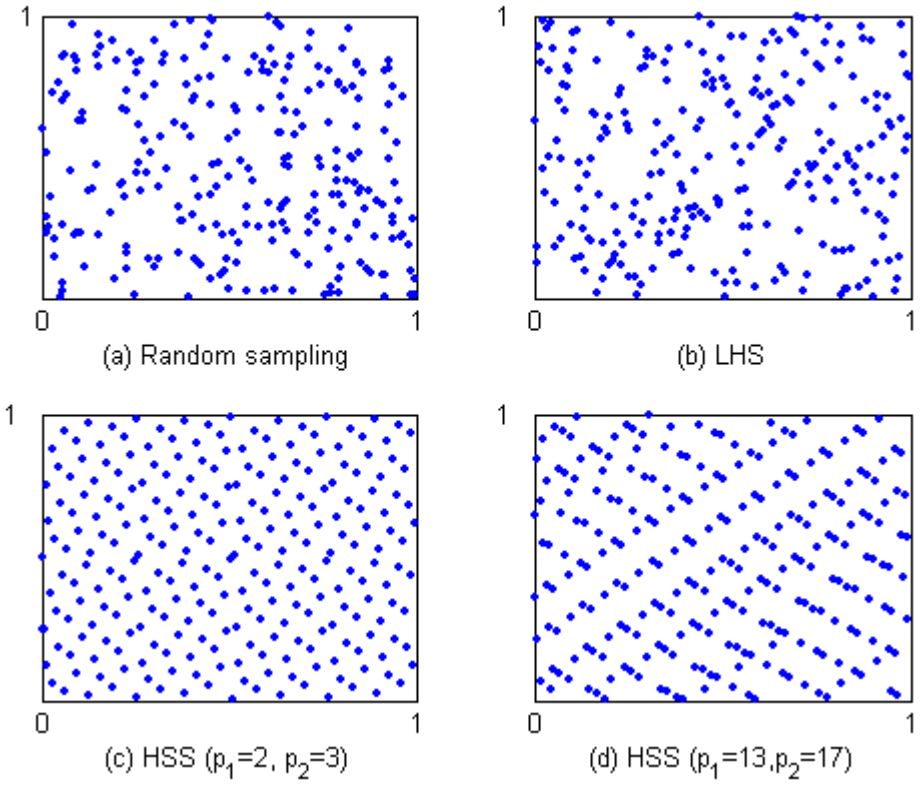

Fig. 3 Sample points (250) on a unit square using different sampling techniques

3.3 Point Estimate Method

Point estimate methods, originally proposed by Rosenblueth (1975), are another type of methods for estimating statistical moments and have been used in engineering applications (Zhao and Hon 2000, 2003; Wang 2004). The point estimate methods use the first few moments (mean, standard deviation, etc) of each random variable to obtain the points and weights for each random variable to compute the moments of the response variable (performance function). The points and weights are different for different random variables.

3.3.1 Taguchi's Method

The experimental design method for variance estimation in robust design is generally attributed to Taguchi (1978). He chose $\mu_{X_i} \pm \sigma_{X_i}$ for a two-level design, and $\mu_{X_i} - \sqrt{\frac{3}{2}}\sigma_{X_i}, \mu_{X_i}, \text{and} \, \mu_{X_i} + \sqrt{\frac{3}{2}}\sigma_{X_i}$ for a three-level design (Yu and Ishii 1998). To apply the three-level method, the performance function *Y* is evaluated at all $N = 3^n$ combinations. And then the variance is computed with the *N* values of *Y* by using Eqs. (8) and (9). Taguchi's method is very straightforward and easily understandable to engineers and scientists.

3.3.2 Quadrature-Based Methods

Quadrature-based methods (D'Errico and Zaino 1988, Seo and Kwak 2002, Frey et al. 2005) are another kind of estimate for statistical moments. In these methods, a few points and corresponding weights for every random variable are chosen for calculating the variance of the performance function. Different methods use different ways to choose the points and weights. D'Errico and Zaino (1988) modify Taguchi's method through using three different points $\left\{x_i^1, x_i^2, x_i^3\right\} = \left\{\mu_{X_i} - \sqrt{3}\sigma_{X_i}, \mu_{X_i}, \mu_{X_i} + \sqrt{3}\sigma_{X_i}\right\}$ and the three corresponding weights $\left\{w_i^1, w_i^2, w_i^3\right\} = \left\{\frac{1}{6}, \frac{4}{6}, \frac{1}{6}\right\}$ for every variable X_i. Seo and Kwak (2002) extend the work of D'Errico and Zaino for non-normal distributions and propose a three-point method which has the three points $\left\{x_i^1, x_i^2, x_i^3\right\}$ and three weights $\left\{w_i^1, w_i^2, w_i^3\right\}$ for the random variables X_i. The points and weights are determined by

$$\left\{x_i^1, x_i^2, x_i^3\right\} = \begin{bmatrix} \mu_{X_i} + \dfrac{\sqrt{\beta_{X_i}}\sigma_{X_i}}{2} - \dfrac{\sigma_{X_i}}{2}\sqrt{4\gamma_{X_i} - 3\beta_{X_i}} \\ \mu_{X_i} \\ \mu_{X_i} + \dfrac{\sqrt{\beta_{X_i}}\sigma_{X_i}}{2} + \dfrac{\sigma_{X_i}}{2}\sqrt{4\gamma_{X_i} - 3\beta_{X_i}} \end{bmatrix}^{\mathrm{T}}, \tag{19}$$

$$\left\{w_i^1, w_i^2, w_i^3\right\} = \begin{bmatrix} \dfrac{\left(4\gamma_{X_i} - 3\beta_{X_i}\right) + \sqrt{\beta_{X_i}\left(4\gamma_{X_i} - 3\beta_{X_i}\right)}}{2\left(4\gamma_{X_i} - 3\beta_{X_i}\right)\left(\gamma_{X_i} - \beta_{X_i}\right)} \\ \dfrac{\gamma_{X_i} - \beta_{X_i} - 1}{\gamma_{X_i} - \beta_{X_i}} \\ \dfrac{\left(4\gamma_{X_i} - 3\beta_{X_i}\right) - \sqrt{\beta_{X_i}\left(4\gamma_{X_i} - 3\beta_{X_i}\right)}}{2\left(4\gamma_{X_i} - 3\beta_{X_i}\right)\left(\gamma_{X_i} - \beta_{X_i}\right)} \end{bmatrix}^{\mathrm{T}}, \tag{20}$$

where $\mu_{X_i}, \sigma_{X_i}, \sqrt{\beta_{X_i}}$, and γ_{X_i} are the mean, standard deviation, skewness, and kurtosis of X_i, respectively.

The formulation for variance is then given by

$$\sigma_Y^2 \cong \sum_{I_1=1}^{3} \cdots \sum_{I_n=1}^{3} w_1^{I_1} \cdots w_n^{I_n} \left[g\left(x_1^{I_1}, ..., x_n^{I_n}\right)\right]^2 - \mu_Y^2, \tag{21}$$

where

$$\mu_Y = \sum_{I_1=1}^{3} \cdots \sum_{I_n=1}^{3} w_1^{I_1} \cdots w_n^{I_n} g\left(x_1^{I_1}, ..., x_n^{I_n}\right). \tag{22}$$

Recently, Huang and Du (2006a) propose a method in which the Gauss points and Gauss weights in Gauss-Hermite quadrature (Davis and Rabinowitz 1983) are directly used. In this method, all the random variables **X** are first converted to the normal variables **Z** that follow a distribution of $N\left(0, \dfrac{1}{2}\right)$ (with a mean of 0 and a variance of $\dfrac{1}{2}$) as follows:

$$z_i = t_i(x_i) = \frac{1}{\sqrt{2}}\Phi^{-1}\left[F_{X_i}(x_i)\right] \quad (i = 1, 2, ..., n), \tag{23}$$

where $\Phi^{-1}(\cdot)$ is the inverse CDF of the standard normal distribution and $F_{X_i}(\cdot)$ are the CDFs of X_i.

Then the variance is given by

$$\sigma_Y^2 \cong \left(\frac{1}{\sqrt{\pi}}\right)^n \sum_{I_1=1}^{r_1} \cdots \sum_{I_n=1}^{r_n} w_1^{I_1} \cdots w_n^{I_n} \left\{g[t_1^{-1}(z_1^{I_1}), ..., t_n^{-1}(z_n^{I_n})]\right\}^2 - \mu_Y^2, \tag{24}$$

in which

$$\mu_Y = \left(\frac{1}{\sqrt{\pi}}\right)^n \sum_{I_1=1}^{r_1} \cdots \sum_{I_n=1}^{r_n} w_1^{I_1} \cdots w_n^{I_n} g[t_1^{-1}(z_1^{I_1}), ..., t_n^{-1}(z_n^{I_n})], \tag{25}$$

where $r_1, ..., r_n$ denote the quadrature order (the number of abscissas) used in the $z_1, ..., z_n$ directions; $(z_1^{I_1}, ..., z_n^{I_n})$ are Gauss points; and $w_1^{I_1}, ..., w_n^{I_n}$ are the corresponding Gauss weights. The points and weights are provided in Beyer (1991). $t_i^{-1}(\cdot)$ is the inverse function of that given in Eq. (23). Generally, the accuracy of the direct Gauss-Hermite quadrature approximation improves as the number of the abscissas for each integration variable increases. However, increasing the number of the points also results in the increase of the total number of performance function evaluations, because the number of function evaluations is r^n where n is the total number of random variables, and r is the number of abscissas used for each random variable. GHI provides an exact result for the variance only if $y(\cdot)$ is a polynomial of degree r-1 or less (Davis and Rabinowitz 1983, Frey et al. 2005). For most problems in structural or mechanical engineering, it may be good enough to take r = 2, 3 or 4 (Huang and Du 2006a).

In sum, D'Errico and Zaino's method is only applicable to the situation that all random variables are normally distributed. Seo and Kwak's method is suitable to normal as well as non-normal distributions, and D'Errico and Zaino's method is a special case of Seo and Kwak's method. Huang and Du's method is also applicable to any distributions. D'Errico and Zaino's method and Seo and Kwak's method both need 3^n function evaluations where n is the number of random variables. The computational cost of Huang and Du's method is r^n, where r is the number of abscissas used for each random variable and n is the number of random input variables. Therefore, Huang and Du's method has a variable computational cost, depending on the number of abscissas used for each random variable.

As indicated above, the aforementioned quadrature-based methods are expensive when the number of random variables is large since their computational cost grows exponentially with the number of random variables. Very recently, a bivariate dimension reduction integration (BDRI) method (Xu and Rahman 2004, Huang and Du 2006b) is proposed for statistical moments (including variance) estimation with less computational efforts. In BDRI, the $\left[g(\mathbf{z})\right]^2$ and $g(\mathbf{z})$ in Eqs. (24) and (25), respectively, are approximated by a summation of a series of two- and one-dimensional functions as follows:

$$g(\mathbf{z}) \cong \sum_{k_1<k_2} g(0,...,0,z_{k_1},0,...,0,z_{k_2},0,...,0) + (-1)\binom{n-2}{1}\sum_{k=1}^{n} g(0,...,0,z_k,0,...,0) + \binom{n-1}{2} g(0,...,0), \tag{26}$$

$$\left[g(\mathbf{z})\right]^2 \cong \sum_{k_1<k_2}\left[g(0,...,0,z_{k_1},0,...,0,z_{k_2},0,...,0)\right]^2 - \binom{n-2}{1}\sum_{k=1}^{n}\left[g(0,...,0,z_k,0,...,0)\right]^2 + \binom{n-1}{2}\left[g(0,...,0)\right]^2. \tag{27}$$

The variance is then calculated based on the above two approximation functions and the use of Gauss-Hermite integration. The procedure of Gauss-Hermite integration (GHI) is not repeated here. For more details, refer to Huang and Du (2006a). The computational effort in computing the integration of all the one- and two-dimensional in Eqs. (26) and (27) is generally much less than that in evaluating the original *n*-dimensional integration in Eqs. (21), (22), (24) and (25). The total computational effort (*CE*) can be measured by the number of function evaluations, which is given by

$$CE = \binom{n}{2}(r)^2 + \binom{n}{1}(r) + 1, \tag{28}$$

where *r* is the number of Gauss points used for each random variable, *n* is the number of random variables. For a performance function with 10 random variables and three points are used for each random variable, the total number of function evaluations is 436. It is significantly less than that of the direct GHI approach, which is 3^{10} = 59049. It should be mentioned that BDRI can achieve high efficiency when the number of random variables is high. However, when the number of random variables is low, say four or less, BDRI may not be more computationally efficient than other quadrature-based methods without applying bivariate dimension reduction.

4. Percentile Difference Approach

As described in Eqs. (4) and (5), the percentile difference requires calculating two percentiles of the performance function. Percentiles can be solved by various methods

such as Monte Carlo simulation. The inverse first order reliability method (FORM) (Tu and Choi 1999) is an efficient method to calculate the percentiles of the performance function. Three steps are involved in inverse FORM to calculate an α percentile of Y, y^{α}.

(1)Transformation: Rosenblatt transformation is used to convert all the random variables **X** in the performance function $g(\mathbf{X})$ into independent and standard normal variables **Z**. If all random variables **X** are mutually independent, the transformation is given by

$$z_i = \Phi^{-1}\left[F_{X_i}(x_i) \right] \quad (i = 1, 2, ..., n), \tag{29}$$

where $\Phi^{-1}(\cdot)$ is the inverse CDF of the standard normal distribution, $F_{X_i}(\cdot)$ are the CDF of X_i. After the transformation, the performance function $g(\mathbf{X})$ in the original space becomes another function $g(\mathbf{Z})$ in the transformed normal space.

(2) Inverse Most Probable Point (MPP) search: MPP is formally defined in the standard normal space as the point that is on the boundary $\mathrm{g}(\mathbf{Z}) = 0$ and has the minimum distance to the origin (Breitung 1984, Du and Chen 2001). The distance between the MPP and the origin represents the reliability or probability of $\mathrm{g}(\mathbf{Z}) \le 0$. The inverse MPP search problem is to find the minimum of $\mathrm{g}(\mathbf{Z})$ and the corresponding **z**, satisfying the given the reliability requirement. For more details about the inverse MPP, refer to Tu and Choi (1999) and Du et al. (2004). Therefore, the inverse MPP search is a minimization problem. The model for the inverse MPP, $\mathbf{z}_{MPP}$, is given by

$$\begin{cases} \underset{\mathbf{z}}{\text{Minimize:}}\ \mathrm{g}(\mathbf{z}) \\ \text{Subject to:}\ \|\mathbf{z}\| = \Phi^{-1}(\alpha), \end{cases} \tag{30}$$

where $\|\cdot\|$ stands for the norm operator, $\Phi^{-1}(\cdot)$ is the inverse CDF of the standard normal distribution.

(3) Percentile evaluation: The percentile value can be evaluated at $\mathbf{z}_{MPP}$ in the transformed normal space,

$$y^{\alpha} = g^{\alpha} = g(\mathbf{z}_{MPP}). \tag{31}$$

5. Examples and Comparison Study

In this section, three examples are used to investigate the effectiveness of different methods of robustness assessment. The computational cost is measured by the number of performance function evaluations. For the Taylor series expansion methods, derivatives are computed with the forward finite difference approach. The results of the percentile difference method are also given. The computational effort of the percentile difference method also includes the number of performance function evaluations used for the MPP search.

5.1 Example 1: A Beam Structure

A statically indeterminate beam (Seo and Kwak 2002) is depicted in Fig. 4.

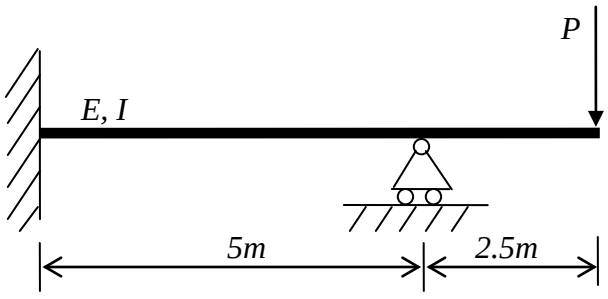

Fig. 4 Statically indeterminate beam and concentrated force

The performance function represents the failure mode of the deflection of the beam and is given by

$$Y = g\left(P, E, I\right) = EI - 78.125P, \tag{32}$$

where E is the Young's Modulus of the beam material, I is the moment of the inertia, and P is the external vertical concentrated force. Three random variables are involved in this problem. E and I are normally distributed. P follows a Gumbel distribution. The distribution information of all random variables is listed in Table 1.

Table 1 Distribution information of random variables for the indeterminate beam

Random variable	Distribution	Mean (μ)	Standard deviation (σ)
P	Gumbel	4 kN	1 kN
E	Normal	2×10^7 kN/m^2	0.5×10^7 kN/m^2
I	Normal	1.0×10^{-4} m^4	0.2×10^{-4} m^4

Figure 5 shows the standard deviations of Y calculated by random sampling (RS), Latin hypercube sampling (LHS) and Hamersley sequence sampling (HSS) techniques using ten different sampling sizes (100, 250, 500, 750, 1000, 2500, 5000, 10000, 100000, 1000000). It is noted that LHS and HSS is more stable than RS. When sample sizes are 100 and 250, LHS provides better results than HSS and RS. HSS results approach to the accurate solution asymptotically and earlier than the other two methods.

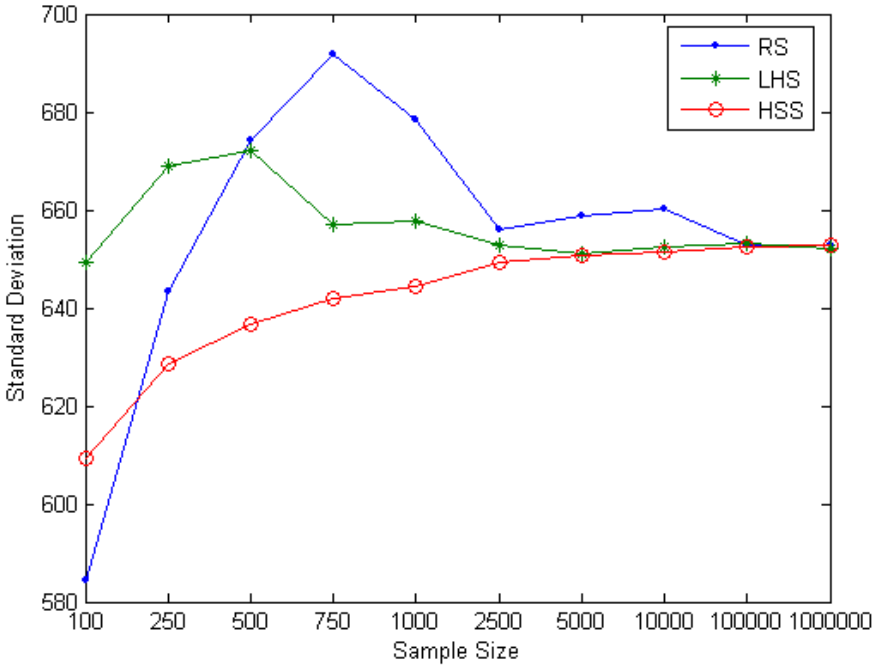

Fig. 5 Standard deviations from different sampling techniques (beam)

Table 2 gives the estimated standard deviations by the first and second order Taylor series expansion methods, the result (3-Point) from Seo and Kwak (2002), bivariate dimension reduction integration (BDRI) method, and HSS simulation method. The result of HSS with 1,000,000 samples is considered the accurate result. It is seen that the Taylor series expansion methods are the most efficient methods since the first order Taylor series expansion method only uses 4 function evaluations and the second order Taylor series expansion method uses 11 function evaluations. The second order Taylor series expansion method is more accurate than the first order Taylor series expansion method. Both BDRI and 3-Point methods provide almost the same results as the accurate result from HSS with 1,000,000 samples. BDRI is more efficient than 3-Point method

since BDRI only uses 19 function evaluations while the 3-Point method uses 27 function evaluations.

Table 2 Standard deviation estimated and computational effort (beam)

	Taylor		Point estimate		HSS		
	first	second	3-Point	BDRI	1,000	10,000	1,000,000
σ_Y	645.06	652.76	652.77	652.75	644.35	651.47	652.74
CE	4	11	27	19	1,000	10,000	1,000,000

Note: *CE* – computational effort in terms of the number of function evaluations; the 3-Point result is from Seo and Kwak (2002).

The results from the percentile difference method, with $\alpha_1 = 0.05$ and $\alpha_2 = 0.95$, are given in Table 3. The total number of performance function evaluations is 132, which is higher than Taylor series expansion methods and the point estimate methods. The result of percentile difference provides the robustness confidence level, $95\% - 5\% = 90\%$. In other words, the probability of the performance being between 812.81 and 2936.60 is 90%.

Table 3 Results of percentile difference (beam)

$y^{0.05}$	$y^{0.95}$	$\Delta y_{0.05}^{0.95}$	Total CE
812.81	2936.60	2123.79	132

5.2 Example 2: A Two-Bar Structure

A two-bar structure system (Jin et al. 2003) is illustrated in Fig. 6.

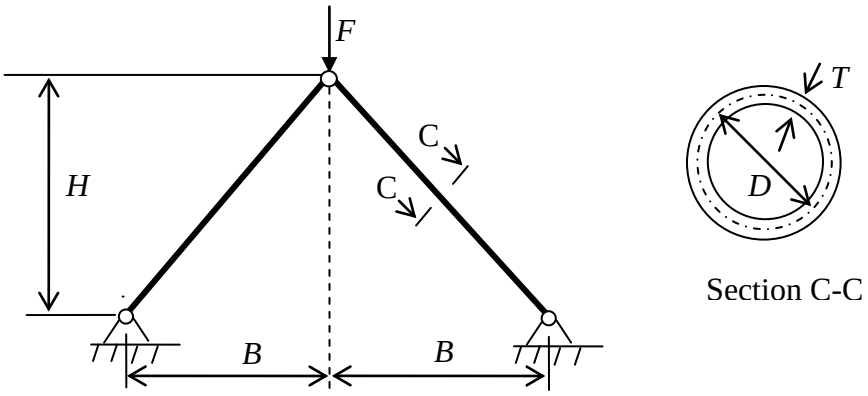

Fig. 6 A two-bar structure

The performance function is the buckling stress in the two bars, which is given by

$$Y = g\left(F, E, T, B, H, D\right) = \frac{F\sqrt{B^2 + H^2}}{2\pi TDH} - \frac{\pi^2 E\left(T^2 + D^2\right)}{8\left(B^2 + H^2\right)}. \tag{33}$$

The problem involves six random variables. F is the external force, E is the Young's modulus, T is the thickness of the cross section, H is the height of the two bar structure, and D is the inner nominal diameter of the cross section. F, E, T, H, and D are normally distributed, the distribution parameters are provided in Table 4.

Table 4 Distributions of random variables for the two-bar structure

Variable	Distribution	Mean (μ)	Standard deviation (σ)
F	Normal	150,000 N	30,000 N
E	Normal	210,000 N/mm^2	21,000 N/mm^2
T	Normal	2.5 mm	0.1 mm
B	Normal	750 mm	5 mm
H	Normal	720 mm	5 mm
D	Normal	40 mm	1 mm

The standard deviations of *Y* estimated by three sampling techniques with ten different sample sizes are depicted in Fig. 7. It is noted that LHS and HSS is more stable than RS and that LHS provides better results than HSS and RS when sample size are less than or equal to 500. HSS results asymptotically approach to the accurate solution with a quicker rate.

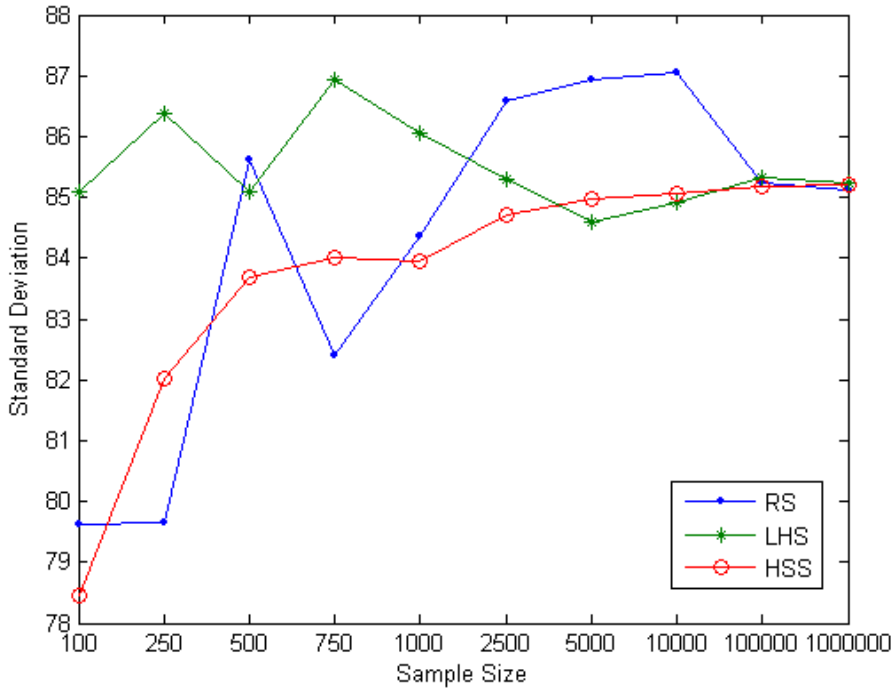

Fig. 7 Standard deviations from different sampling techniques (two-bar)

Table 5 gives the estimated standard deviations by the first and second order Taylor series expansion methods, BDRI, the result (3-Point) of Seo and Kwak's method (2002), and HSS simulation method. The result of HSS with 1,000,000 is considered the accurate result. It is seen that the Taylor series expansion methods are the most efficient methods since it uses the fewest function evaluations (7 and 22 function evaluations for the first and second order Taylor series expansion methods, respectively), but its results

are less accurate than those of BDRI and 3-Point methods. BDRI and 3-Point methods provide almost the same results as HSS with 1,000,000 samples. However, BDRI is more efficient than the 3-Point method since BDRI only uses 73 function evaluations while the 3-Point method uses 729 function evaluations.

Table 5 Standard deviation estimated and computational effort (two-bar)

	Taylor		Point estimate		HSS		
	first	second	3-Point	BDRI	1,000	10,000	1,000,000
σ_Y	84.95	86.87	85.21	85.19	83.94	85.05	85.20
CE	7	22	729	73	1,000	10,000	1,000,000

Note: *CE* – computational effort in terms of the number of function evaluations.

Table 6 shows the results from the percentile difference method with $\alpha_1 = 0.05$ and $\alpha_2 = 0.95$. The total number of function evaluations is 349, which is higher than Taylor series expansion methods and DBRI but is less than the 3-Point method and HSS.

Table 6 Results of percentile difference (two-bar)

$y^{0.05}$	$y^{0.95}$	$\Delta y_{0.05}^{0.95}$	Total CE
-187.71	101.25	288.96	349

5.3 Example 3: An Overrunning Clutch

An overrunning clutch, also called Fortini's clutch, (Greenwood and Chase 1990, Seo and Kwak 2002) , shown in Fig. 8, is used to further investigate the effectiveness of different methods. The performance function is the contact angle *Y*, which is given by

$$Y = \arccos\left[\frac{X_1 + 0.5\left(X_2 + X_3\right)}{X_4 - 0.5\left(X_2 + X_3\right)}\right], \tag{34}$$

where X_1, X_2, X_3 and X_4 are four dimensions of the clutch depicted in Fig. 8.

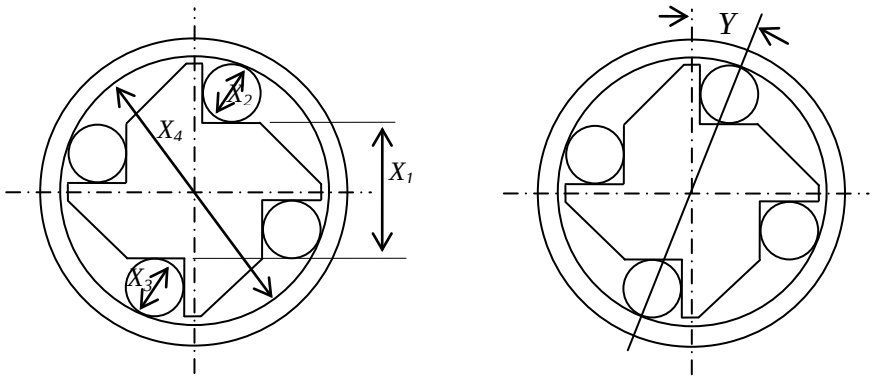

Fig. 8 Overrunning clutch assembly

The distribution information of the four random variables X_1, X_2, X_3 and X_4 is provided in Table 7.

Table 7 Distributions of random variables for the overrunning clutch

Random Variable	Distribution	Mean (mm)	Standard deviation (mm)	Parameters
X_1	Beta	55.29	0.0793	$A = B = 5.0$
X_2	Normal	22.86	0.0043	$\mu = 22.86,\ \sigma = 0.0043$
X_3	Normal	22.86	0.0043	$\mu = 22.86,\ \sigma = 0.0043$
X_4	Rayleigh	101.60	0.0793	$A = 0.1211,\ \left(X_4 \geq 101.45\right)$

Similar to Examples 1 and 2, the standard deviations of *Y* are also estimated by the RS, LHS and HSS with ten different sample sizes, and the results are shown in Fig. 9. Same conclusions are obtained in this problem as those in the previous two examples.

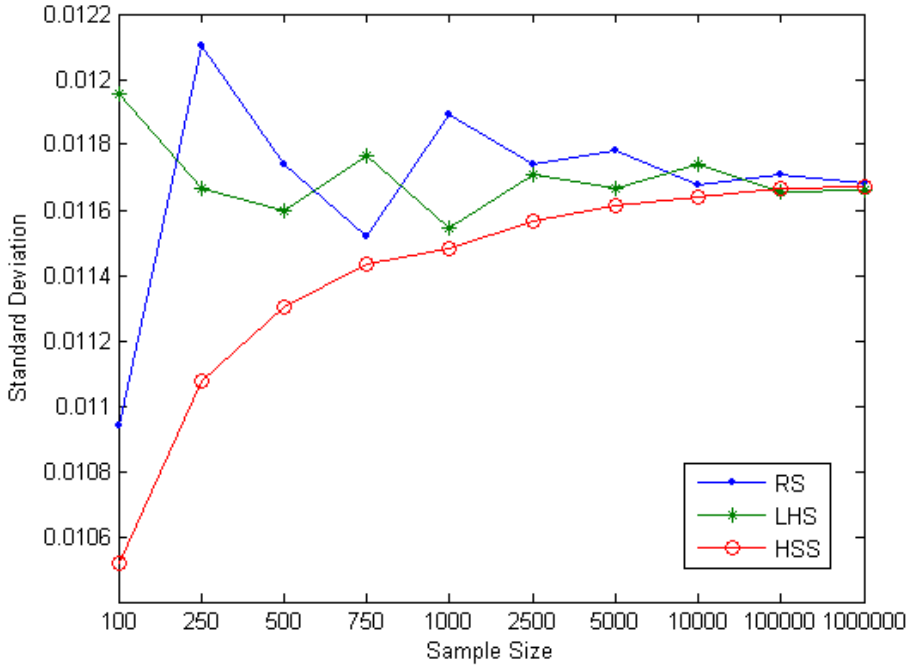

Fig. 9 Standard deviations from different sampling techniques (clutch)

Table 8 gives the estimated standard deviations by the first and second order Taylor series expansion methods, BDRI, the result (3-Point) from Seo and Kwak (2002), and HSS simulation. The result of HSS with 1,000,000 sample points is considered the accurate result. It is seen that all methods provide good results and that the Taylor series expansion methods are the most efficient methods. HSS is the least efficient method and its result from 1,000 sample points is less accurate than Taylor series expansion methods, the 3-Point method, and BDRI methods. For this problem with only four random input variables, BDRI uses 91 function evaluations while the 3-Point method uses 81 function evaluations. This indicates that BDRI may not be as efficient as the other quadrature-based methods when the number of random variables is small. Table 9 provides the results from the percentile difference method with $\alpha_1 = 0.05$ and $\alpha_2 = 0.95$. The total

number of performance function evaluations is 162, which is higher than Taylor series expansion methods and the point estimate methods.

Table 8 Standard deviation estimated and computational effort (clutch)

	Taylor		Point estimate		HSS		
	first	second	3-Point	BDRI	1,000	10,000	1,000,000
σ_Y	0.01162	0.01165	0.01169	0.01162	0.01147	0.01163	0.01166
CE	5	11	81	91	1,000	10,000	1,000,000

Note: *CE* – computational effort in terms of the number of function evaluations, the 3-Point result is from Seo and Kwak (2002).

Table 9 Results of percentile difference (clutch)

$y^{0.05}$	$y^{0.95}$	$\Delta y_{0.05}^{0.95}$	Total *CE*
0.10143	0.14127	0.03984	162

6. Comparisons and Conclusions

Based on the test examples and the formulation of the methods, the robustness assessment methods are summarized and compared in Table 10. Efficiency, accuracy, the requirement of performance function differentiability, and other features are considered for the comparison.

Table 10 Comparisons of different methods for robustness assessment

	Variance			Percentile difference
	First Order Taylor	**Point Estimate**	**Simulation**	
Efficiency	Highest $n+1$	Moderate $O(n^2)$ * or $O(3^n)$ Costly for a large scale problem	Lowest when dimensionality is small. May be more efficient than other methods for a large scale problem	$O(n)$ when using FORM
Function differentiability	Yes	No	No	Yes when using FORM. No when using simulation approach
Accuracy	Good when the variances of input random variables are small and the nonlinearity of the performance function is not high, otherwise the accuracy is low.	More accurate than Taylor series expansion methods in general	Accurate when the number of sample size is sufficient	Accurate if a sampling method is used with a sufficient sample size. Accurate when FORM is used with a less nonlinear performance function.
Other features	• Widely used • Easy to use • Fails at a stationary point	• Easy to use	• Easy and flexible to use	• A little hard to understand • Contains more distribution information • Provides confidence levels

$O(\cdot)$ means "order…" which represents the rates of growth (Weiss 1999).

The major concern of selecting the methods is the trade-off between efficiency and accuracy. In general, higher accuracy requires more function evaluations and therefore implies low efficiency. The conflict exists between the methods, as well as within a specific method. For example, one can select a sampling method instead of a Taylor series expansion method to ensure a higher accuracy. Even though one decides to

use a specific method, he or she still must make decision on the trade-off between accuracy and efficiency. For instance, after a sampling method is chosen, one needs to specify the number of samples he or she could afford according to the problem on hand. There is no such a method that is universally superior to other method. The selection of method should be determined by the specific problem. However, some guidelines are provided as below.

When computational effort is a major concern and the derivatives of the performance function exist, the first order Taylor series expansion method is a good choice. However, the first order Taylor series expansion method may not give accurate results. The second order series expansion method provides better results but it needs more computational cost. If the performance function is very expensive to evaluate and are not differentiable, the point estimate methods are good alternatives. Among the point estimate methods, BDRI significantly reduces the computational cost when the number of random input variables is large. Simulation methods are flexible to use and easy to implement, but usually they are the most computationally expensive methods. However, when a very large number of random variables (for example, 1000) are involved, they will be more efficient than other methods that are dimensionality-dependent. They are applicable to the performance functions that are not time-consuming to evaluate. HSS and LHS are more efficient and achieve more stable results than random sampling. HSS exhibits a non-decreasingly asymptotic property.

The percentile difference approach is a promising alternative for robustness assessment, it can provide more distribution information about the performance, and it may use less computation effort when the number of random input variables is large.

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References

Ayyub BM, Haldar A (1984) Practical structural reliability techniques. Journal of Structural Engineering 110(8):1707-1724.

Badhrinath K, Rao JR (1994) Bi-level models for optimization designs which are insensitive to perturbations in variables and parameters. Advanced in Design Automation 69(2):15-23.

Bain LJ, Engelhardt M (1991) Introduction to probability and mathematical statistics (Duxbury classic series), 2nd Edition. Thomson Learning.

Belgundu AD, Zhang S (1992) Robustness of design through minimum sensitivity. ASME Journal of Mechanical Design 114:213-217.

Beyer WH (1991) CRC standard mathematical tables and formulae, 29th Edition. CRC Press 449.

Breitung K (1984) Asymptotic approximations for multinormal integrals. Journal of Engineering Mechanics 110(3):357-367.

Chen W, Allen JK, Tsui, K-L, Mistress F (1996) A procedure for robust design: minimizing variations caused by noise factors and control factors. ASME Journal of Mechanical Design 118:478-485.

Chen W, Sahai A, Messac A, Sundararaj GJ (2000) Exploration of the effectiveness of physical programming for robust design. ASME Journal of Mechanical Design, 122(2):155-163.

Chen W, Wiecek MM, Zhang J (1999) Quality utility-a compromise programming approach to robust design. ASME Journal of Mechanical Design 121(2):179-187.

Chen W, Yuan C (1999) A probabilistic-based design model for achieving flexibility in design. ASME Journal of Mechanical Design 121:77-83.

Crow F (1977) The aliasing problem in computer-generated shaded images. Communication of the ACM 20(11):799-805.

D'Errico JR, Zaino NA (1988) Statistical tolerancing using a modification of Taguchi's method. Technometrics 30(4):397-405.

Davis PJ, Rabinowitz P (1983) Methods of numerical integration, 2nd Edition. Academic Press Inc. (London).

Dennis JE, Schnabel RB (1983) Numerical methods for unconstrained optimization and nonlinear equations. Prentice-Hall, Englewood Cliffs.

Diwekar UM (2003) A novel sampling approach to combinatorial optimization under uncertainty. Computational Optimization and Applications 24:335-371.

Du X (2006) Reliability-based design using saddlepoint approximation. Submitted to ASME Journal of Mechanical Design.

Du X, Chen W (2000a) Methodology for managing the effect of uncertainty in simulation –based deign. AIAA Journal 38(8):1471-1478.

Du X, Chen W (2000b) Towards a better understanding of modeling feasibility robustness in engineering design. ASME Journal of Mechanical Design 122(4): 385-394.

Du X, Chen W (2001) A most probable point based method for uncertainty analysis. Journal of Design and Manufacturing Automation 4(1): 47-66.

Du X, Chen W (2004) Sequential optimization and reliability assessment for probabilistic design. ASME Journal of Mechanical Design 126(2):225-233.

Du X, Sudjianto A, Chen, W. (2004) An Integrated Framework for Optimization under Uncertainty Using Inverse Reliability Strategy. ASME Journal of Mechanical Design 126(4):561-764.

Fowlkes WY, Creveling CM (1995) Engineering methods for robust product design: using Taguchi methods in technology and product development. PTR Prentice Hall.

Frey DD, Reber G, Lin Y (2005) A quadrature-based sampling technique for robust design with computer models. Proceedings of IDETC/CIE ASME 2005 Design Engineering Technical Conference and Computers and Information in Engineering Conference, Long Beach, California, USA.

Fu Y, Sahin KH (2004) Better optimization of nonlinear uncertainty systems (BONUS) for vehicle structural design. Annals of Operations Research 132:69-84.

Giunta AA, Wojtkiewicz SF Jr., Eldred MS (2003) Overview of modern design of experiments methods for computational simulations. AIAA 2003-0649, 41st AIAA Aerospace Sciences Meeting and Exhibit, Reno, NV, USA.

Greenwood WH, Chase KW (1990) Root sum squares tolerance analysis with nonlinear problems. ASME Journal of Engineering for Industry 112:382-394.

Gunawan S, Azarm S (2004) Non-gradient based parameter sensitivity estimation for single objective robust design optimization. ASME Journal of Mechanical Design 126(3):395-402.

Gunawan S, Azarm S (2005) A feasibility robust optimization method using sensitivity region concept. ASME Journal of Mechanical Design 127(5):858-865.

Halton JH (1960) On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals. Numerische Mathematik 2:84-90.

Hamersley JM (1960) Monte Carlo methods for solving multivariate problems. Annals of the New York Academy of Science 86:844-874.

Hasofer AM, Lind NC (1974) Exact and invariant second-moment code format. ASCE Journal of the Engineering Mechanics Division 100(EM1):111-121.

Helton JC, Davis FJ (2003) Latin hypercube sampling and the propagation of uncertainty in analysis of complex systems. Reliability Engineering & System Safety 81:23-69.

Hohenbichler M, Gollwitzer S, Kruse W, Rackwitz R (1987) New light on first- and second-order reliability methods. Structural Safety 4:267-284.

Huang B, Du X (2006a) A robust design method using variable transformation and Gauss-Hermite integration. International Journal for Numerical Methods in Engineering, in press.

Huang B, Du X (2006b) Uncertainty analysis by dimension reduction and saddlepoint approximations. ASME Journal of Mechanical Design 128(1):26-33.

Jin R, Chen W, Sudjianto A (2004) Analytical metamodel-based global sensitivity analysis and uncertainty propagation for robust design. Paper 2004-01-0429, SAE Congress, March 8-11, Detroit, MI. USA.

Jin R, Du X, Chen W (2003) The use of metamodeling techniques for design under uncertainty. Structural and Multidisciplinary Optimization 25(2):99-116.

Jung DH, Lee BC (2002) Development of a simple and efficient method for robust optimization. International Journal for Numerical Methods in Engineering 53:2201-2215.

Kalagnanam JR, Diwekar UM (1997) An efficient sampling technique for off-line quality control. Technometrics 39:308-319.

Lau W (1995) An adaptive supersampling method. Image Analysis Applications and Computer Graphics, Chin et al. (Eds), LNCS 1024, Spring-Verlag 205-214.

Law AM, Kelton WD (1982) Simulation modeling and analysis. McGraw-Hill Company, New York, USA.

Lee K-H, Eom I-S, Park G-J, Lee W-I (1996) Robust design for unconstrained optimization problems using the Taguchi's method. AIAA Journal 34(5):1059-1063.

Lee KH, Park GJ (2001) Robust optimization considering tolerances of design variables. Computers and Structures 79:77-86.

Mateufel RD (2000) Evaluating the convergence of Latin hypercube sampling. AIAA-2000-1636. 41st AIAA structures, Structural Dynamics, and Materials Conference, Atlanta, GA, USA.

Mckay MD, Conover WJ, Beckman RJ (1979) A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. Technometrics 21(2):39-245.

Montgomery DC (2005) Design and analysis of experiments, 6th Edition. John Wiley and Sons, Inc.

Owen A (1997) Monte Carlo variance of scrambled equidistribution quadrature. SIAM Journal of Numerical Analysis 34(5):1884-1910.

Park GJ, Lee TH, Lee KH, Hwang KH (2006) Robust design: an overview. AIAA Journal 44 (1):181-191.

Parkinson A, Sorensen C, Pourhassan N (1993) A general approach for robust optimal design. ASME Journal of Mechanical Design 115:74-80.

Phade MS (1989) Quality engineering using robust design. Prentice-Hall, Englewood Cliffs, New Jersey.

Phadke MS (1989) Quality engineering using robust design. PTR Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Putko MM, Taylor III AC, Newman PA, Green LL (2002) Approach for input uncertainty propagation and robust design in CFD using sensitivity derivatives. ASME Journal of Fluids Engineering 124:60-69.

Reichert P, Schervish M, Small MJ (2002) An efficient sampling technique for Bayesian inference with computationally demanding models. Technometrics 44(4): 318-327.

Robinson DG (1998) A survey of probabilistic methods using in reliability, risk, and uncertainty analysis: analytical techniques I. U.S. Sandia Report SAND98-118.

Rosenblueth E (1975) Point estimates for probability moment. Proceedings of the National Academy Sciences 72(10):3812-3814.

Rubinstein RY (1981) Simulation and the Monte Carlo method. New York: Wiley.

Santner TJ, Williams BJ, Notz WI (2003) The design and analysis of computer experiments. Springer Series in Statistics, Springer.

Seo HS, Kwak BM (2002) Efficient statistical tolerance analysis for general distribution using three-point information. International Journal of Production Research 40(4):931-944.

Simpson TW, Peplinskim J, Koch PN, Allen JK (2001) Metamodels for computer-based engineering design: survey and recommendations 17(2):129-150.

Sundaresan S, Ishii K, Hourser DR (1995) A robust optimization procedure with variations on design variables and constraints. Engineering Optimization 24:101-117.

Taguchi G (1978) Performance analysis design. International Journal of Production Research 16:521-530.

Taguchi G (1987) System of experimental design, Vols 1 & 2, Kraus International, New York.

Taguchi G (1993) Taguchi on robust technology development: bringing quality engineering upstream. ASME press, New York.

Taguchi G, Chowdhury S (1999) Robust engineering: learn how to boost quality while reducing costs & time to market. McGraw-Hill Professional.

Thaweepat B, Cao J, Chen W (2004) A weighted three-point-based strategy for variance estimation. Proceedings of ASME 2004 Design Engineering Technical

Conference and Computers and Information in Engineering Conference, Salt Lake, Utah, USA.

Tu J, Choi KK (1999) A new study on reliability based design optimization. ASME Journal of Mechanical Design 121(4):557-564.

Wang L, Beeson D, Wiggs G (2004) Efficient and accurate point estimate method for moments and probability distribution. 10th AIAA/ISSMO Multidisciplinary Analysis and Optimization conference, 30 August - 1 September 2004. Albany, New York.

Weiss MR (1999) Data structures & algorithm analysis in C++, second Edition. Addison Wesley Longman, Inc.

Xu H, Rahman S (2004) A generalized dimension-reduction method for multidimensional integration in stochastic mechanics. International Journal for Numerical Methods in Engineering 60(12):1992-2019.

Youn BD, Choi KK, Park YH (2003) Hybrid analysis method for reliability-based design optimization. ASME Journal of Mechanical Design 125(2):221-232.

Yu J, Ishii K (1998) Design optimization for robustness using quadrature factorial models. Engineering Optimization 30 (3/4):203-225.

Zhao Y-G, Hon A H-S A (2003) System reliability assessment by method of Moments. ASCE Journal of Structural Engineering 129(10):1341-1349.

Zhao Y-G, Ono T (2000) New point estimates for probability moments. ASCE Journal of Engineering Mechanics 126(4):433-436.