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Reliability-Based Design Optimization with Equality Constraints

Dr. Xiaoping Du
Department of Mechanical and Aerospace Engineering
University of Missouri – Rolla

Corresponding author
Phone: 01-573-341-7249
Fax: 01-573-341-4115
E-mail: dux@umr.edu

Beiqing Huang
Graduate Research Assistant, PhD Candidate
Department of Mechanical and Aerospace Engineering
University of Missouri – Rolla
E-mail: beiqing.huang@gmail.com

SUMMARY

Equality constraints have been well studied and widely used in deterministic design optimization, but they have rarely been addressed in reliability-based design optimization (RBDO). The inclusion of an equality constraint in RBDO results in the dependency (correlation) among random variables. Theoretically, one random variable can be expressed in terms of the remaining random variables given an equality constraint; and the equality constraint can then be eliminated. However, in practice, eliminating an equality constraint may be difficult or impossible because of complexities such as coupling, recursion, high dimensionality, nonlinearity, implicit formats, and high computational costs. The objective of this work is to develop a methodology to model equality constraints and a numerical procedure to solve an RBDO problem. A sequential optimization and reliability analysis strategy is proposed to solve RBDO with equality constraints. The First Order Reliability Method (FORM) is employed for reliability analysis. The proposed method is illustrated by a mathematical example and a two-member frame design problem.

KEY WORDS: Optimization; reliability; equality constraints; probabilistic constraints

1. INTRODUCTION

With the advancements of computational technologies, design optimization has been increasingly used in engineering design. Combined with mathematical models and simulation tools such as finite element analysis, design optimization enables engineers to reach an inexpensive and optimal design solution in an automatic manner. In real-world problems, uncertainties such as variations in design variables and model parameters always exist. Deterministic optimization without considering uncertainties usually pushes the design to the limits of constraints, leaving little or no room for accommodating uncertainties in modeling and simulation and manufacturing imperfections. Consequently, deterministic optimization could lead to unreliable decisions.

To meet the need of higher product quality and safety, optimization under uncertainty has been increasingly applied as an alternative to deterministic optimization. Reliability-based design optimization (RBDO) [1-4] is one of the representative methods of optimization under uncertainty.

RBDO maintains design constraint satisfaction at expected probability (reliability) levels. There are two common RBDO formulations: reliability index approach (RIA) [4, 5] and performance measure approach (PMA) [1, 2]. In RIA, design feasibility is formulated as the probability of constraint satisfaction equal to or greater than the desired reliability. In PMA, design feasibility is formulated by a percentile constraint function value that corresponds to the desired reliability. PMA has a couple of numerical advantages over RIA [6]: (1) PMA is more robust in terms of convergence; and (2) PMA is more efficient in reliability analysis because it performs reliability assessment only up to a necessary level.

An RBDO problem is generally formulated from a deterministic optimization problem by converting deterministic constraints into probabilistic ones while the objective function is evaluated at the mean values of random variables. A general deterministic optimization problem involves both inequality and equality constraints. However, with the presence of uncertainty, formulating an equality constraint is much more complicated [7, 8]. Only few researchers have addressed the equality constraints with random variables in robust design optimization, which is another optimization methodology under uncertainty. In Das's work [9], equality constraints are eliminated by solving out the dependent random variables from equality constraints. Yu and Ishii [10] formulate equality constraints at the mean values of random variables. In Mattson and Messac's work [8], a comprehensive discussion on equality constraints with random variables is provided. They classify the treatments of equality constraints under uncertainty into three categories: (a) to relax equality constraints, (b) to satisfy equality constraints in a probabilistic sense, and (c) to remove the equality constraints through substitution.

All of the aforementioned approaches are conducted under the framework of robust design optimization. Handling equality constraints in RBDO is still a rarely touched area. In this work, a general method of modeling and handling equality constraints for RBDO is developed. A sequential optimization and reliability analysis (SORA) strategy and the First Order Reliability Method (FORM) are applied to solve an RBDO problem with equality constraints.

The rest of the paper is organized as follows. Section 2 reviews the general models of deterministic optimization and RBDO with equality constraints. Section 3 classifies equality constraints into two types: physics-based equality constraints and demand-based equality constraints. Section 4 formulates an RBDO problem for the two types of equality constraints. In Section 5, a computational algorithm is developed to solve RBDO problems involving physics-based equality constraints. A mathematical example and a two-member frame design problem are used to illustrate the effectiveness of the proposed method in Section 6. Section 7 concludes this research work.

2. DETERMINISTIC OPTIMIZATION AND RELIABILITY-BASED DESIGN OPTIMIZATION WITH EQUALITY CONSTRAINTS

In this section, we first briefly review the model of deterministic design optimization with an emphasis on equality constraints. Then, we discuss the RBDO model with equality constraints.

2.1 Deterministic design optimization model

A deterministic optimization model is given by

$$\begin{cases} \min\limits_{\mathbf{d}} \ f(\mathbf{d}) \\ s.t. \quad g_i(\mathbf{d}) \le 0, \ i = 1, 2, \ \cdots, n_g \\ \qquad h_j(\mathbf{d}) = 0, \ j = 1, 2, \ \cdots, n_h \end{cases} \tag{1}$$

In the above model, $\mathbf{d} = \left(d_1, d_2, \cdots, d_{n_d}\right)$ is the vector of design variables; $f(\mathbf{d})$ is the objective function; $g_i(\mathbf{d})$ are inequality constraint functions; $h_j(\mathbf{d})$ are equality constraint functions; n_d is the number of design variables; n_g is the number of inequality constraints; and n_h is the number of equality constraints.

An equality constraint imposes a functional relationship on design variables. Theoretically, given an equality constraint, one design variable can be solved out and be expressed in terms of the remaining design variables. In other words, eliminating one equality constraint means eliminating one design variable. If we can eliminate all the equality constraints by solving the simultaneous equality equations, $h_j(\mathbf{d}) = 0$ ($j = 1, 2, \cdots, n_h$), the number of independent design variables will be $n_d - n_h$. However, in practice, eliminating an equality constraint may be very difficult or impossible because of complexities such as coupling, high dimensionality, nonlinearity, implicit functional relationships (block-boxes), and high computational costs. Furthermore, in collaborative multidisciplinary design optimization [11], equality constraints are artificially added to maintain the consistency among disciplines.

2.2 Reliability-based design optimization model

In reliability-based design optimization, random variables are used to account for uncertainties from various sources, such as variations in material's properties, manufacturing processes, and operating environments. By converting deterministic model in Equation (1) into a RBDO model, we obtain

$$\begin{cases} \min\limits_{\mathbf{d}} \; f(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{P}}) \\ \text{s.t.} \quad \Pr\{g_i(\mathbf{d}, \mathbf{P}) \le 0\} \ge R_i, \; i = 1, 2, \; \cdots, n_g \\ \qquad h_j(\mathbf{d}, \mathbf{P}) = 0, \; j = 1, 2, \; \cdots, n_h \end{cases} \tag{2}$$

In the above model, $\mathbf{P} = \left(P_1, P_2, \cdots, P_{n_p}\right)$ is the vector of *random* variables and n_p is the number of random variables. $\mathbf{P}$ are treated as design parameters that are out of designers' control. The objective function $f(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{P}})$ is evaluated at the means, $\boldsymbol{\mu}_{\mathbf{P}}$, of the random variables $\mathbf{P}$. $\Pr\{\cdot\}$ denotes a probability, and R_i stands for the desired reliability for constraint *i*. Equality constraints $h_j(\mathbf{d},\mathbf{P})$ involve random parameters **P**.

At each design point $\mathbf{d}=\left(d_1,d_2,\cdots,d_{n_d}\right)$, reliability analysis is needed in order to calculate the reliability $\Pr\left\{g_k(\mathbf{d},\mathbf{P})\le 0\right\}$. Typical reliability analysis methods [12, 13] or Monte Carlo Simulation [15] can be used for the reliability analysis purpose.

It is worthwhile to explain and differentiate the concepts of the design space (deterministic space) and reliability analysis space (random space) used in RBDO. The *design space* consists of deterministic design variables $\mathbf{d}=\left(d_1,d_2,\cdots,d_{n_d}\right)$. It is an n_d-dimensional space. The *reliability analysis space* consists of random variables $\mathbf{P}=\left(P_1,P_2,\cdots,P_{n_p}\right)$. It is an n_p- dimensional space.

3. CLASSIFICATION OF PROBABILISTIC EQUALITY CONSTRAINTS

To properly formulate equality constraints in an RBDO problem, it is necessary to study their features and then classify them into different categories. Adopting Mattson and Messac's idea [8], we classify equality constraints into two categories: *physics-based* equality constraint and *demand-based* equality constraint. Statistically, these two types are fundamentally different.

3.1 Type 1: Physics-based equality constraint

A physical-based equality constraint function is determined by a physical principle; the equality condition always holds regardless of the variations of its constituting variables. For instance, Newton's second law states the equality relationship among the mass, $m(P_1)$, of a particle, external resultant force, $F(P_2)$, acting on the particle, and the acceleration, $a(P_3)$, of the particle. The equation is given by

$$h(\mathbf{P}) = h(P_1, P_2, P_3) = F - ma = 0 \tag{3}$$

If F and m are random, a is also a random variable dependent on F and m. The above equation should be *always* satisfied during a numerical implementation.

In a cantilever beam design problem, four random variables P_1, P_2, P_3, and P_4 are involved as shown in Figure 1.

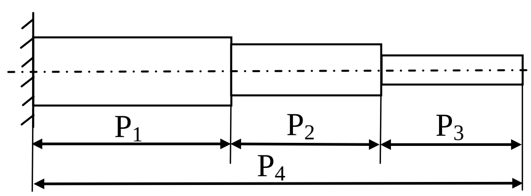

Figure 1. A cantilever beam.

The total length P_4 is the sum of the other variables.

$$P_4 = P_1 + P_2 + P_3 \tag{4}$$

Then, an equality constraint consisting of P_1, P_2, P_3, and P_4 is given by

$$h(\mathbf{P}) = P_1 + P_2 + P_3 - P_4 = 0 \tag{5}$$

This equality constraint is determined by the material continuity of the beam.

In the example of a cubic container design (see Figure 2), there are three random parameters P_1, P_2, and P_3. P_1 is the length and width, P_2 is the height, and P_3 is the volume, which is given by

$$P_3 = P_1^2 P_2 \tag{6}$$

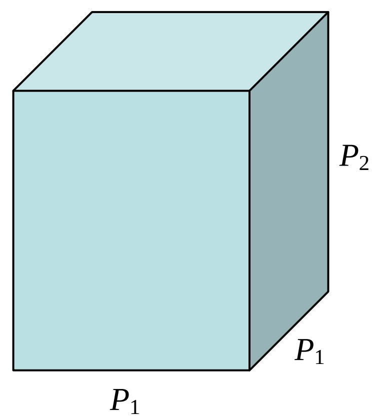

Figure 2. A cubic container.

Therefore, the equality constraint is expressed by

$$h(\mathbf{P}) = P_1^2 P_2 - P_3 = 0 \tag{7}$$

The above equality constraint is determined by a geometric relationship.

Features of a physical based equality constraint include:

(1) The equality condition should always hold.

Since an equality constraint represents a physical principle, the equality condition should always hold. In other words, the probability of the equality constraint satisfaction should always be 1.0, no mater how large uncertainties are. The violation of a physics-based constraint will result in an infeasibility design. For example, the violation of the equality constraint in Equation (5) implies the breakage of the beam.

(2) The equality condition holds in the random analysis space.

From Equations (3) and (7), we see that the equality condition holds in the random space, no matter how random variables vary.

(3) Satisfying an equality condition in deterministic design space may not be sufficient.

As discussed in Section 1, in some literature an equality constraint is formulated at the means of random variables in the deterministic design space as

$$h(\boldsymbol{\mu}_{\mathbf{p}}) = 0 \tag{8}$$

This equation may or may not guarantee the equality condition in the random space. For the cantilever beam problem in Figure 1, let P_1, P_2, and P_3 be independently normally distributed with their means, μ_1, μ_2, and μ_3, and standard deviations, σ_1, σ_2, and σ_3, respectively. Since P_4 is a linear combination of the other normally distributed variables, P_4 is also normally distributed with its mean

$$\mu_4 = \mu_1 + \mu_2 + \mu_3 \tag{9}$$

and its standard deviation

$$\sigma_4 = \sqrt{\sum_{i=1}^{3} \sigma_i^2} \tag{10}$$

With Equation (8), the equality constraint in the deterministic design space is given by

$$h(\boldsymbol{\mu}) = \mu_1 + \mu_2 + \mu_3 - \mu_4 = 0 \tag{11}$$

This equation guarantees the same equality condition in the random analysis space when the standard deviation of P_4 is determined by Equation (9).

However, for the cubic container design problem, let P_1 and P_2 be independently normally distributed with their means, μ_1 and μ_2, and their standard deviations, σ_1 and σ_2, respectively. The mean of the volume P_3 is given by

$$\mu_3 = (\mu_1^2 + \sigma_1^2)\mu_2 \tag{12}$$

The equality constraint in the deterministic design space is

$$\mu_3 = \mu_1^2 \mu_2 \tag{13}$$

Since the standard deviation σ_1 is not zero, Equation (11) conflicts with Equation (12). This indicates that the equality constraint specified in Equation (13) in the deterministic design space cannot guarantee that the equality constraint is satisfied in the random analysis space.

Generally, whether an equality constraint in deterministic design space ensures the equality condition in the random analysis space depends on the constraint function form and the distribution types of random variables.

(4) A physics-based equality constraint imposes correlations among random variables.

A physical-equality constraint inexplicitly reduces one degree of freedom of random variables. For example, for the cantilever problem, there are correlations between P_4 and P_1, P_4 and P_2, and P_4 and P_3. The correlations increase the complexity of computing the probabilistic characteristics of equality and inequality constraints.

3.2 Type 2: Demand-based equality constraint

A demand-based equality constraint is determined by designers' preferences or desires, not by any physical principle. For example, in the above cubic container design problem, the design variables are chosen as the length and width (P_1) and height (P_2), and they are mutually independent. A designer may wish the volume to be as close as possible to a target value of 1 m^3. This is a demand-based equality constraint.

Features of a demand based equality constraint include:

(1) The equality condition may not be required to be strictly satisfied.

Since a demand-based equality constraint is not determined by physical principles, a slight constraint violation does not necessarily mean an infeasible design. Instead, this type of constraint is somewhat "flexible" since it only reflects designers' preferences or desires.

(2) A demand-based equality constraint cannot always be satisfied in random analysis space.

In the above cubic container design problem, we wish the volume to be 1 m^3. Then, we could express our preferences in the random analysis space as

$$P_1^2 P_2 = 1 \tag{14}$$

or

$$h(\mathbf{P}) = P_1^2 P_2 - 1 = 0 \tag{15}$$

Since the volume $P_1^2 P_2$ is a continuous random variable and the probability of the volume being equal to a specific value is zero. Equations (14) or (15) will never be satisfied in the random analysis space. We must find other means to formulate this type of constraint (see the next section).

(3) A demand-based equality constraint may not impose a correlation on the random variables.

A demand-based equality constraint may not necessarily indicate the reduction of a degree of freedom of random variables that are involved in an equality constraint. For instance, in the above example, P_1 and P_2 can be changed independently even though we wish the volume to be a certain value. Therefore, there is no correlation between P_1 and P_2.

4. MODELING EQUALITY CONSTRAINTS IN RBDO

We have discussed the features of two fundamentally different types of equality constraints. Next, we discuss how to formulate both types of constraints.

4.1 Formulating a demand-based equality constraint

A demand-based equality constraint will *never* be satisfied. Therefore, we have to relax the *ideal* equality condition. The simplest treatment is to formulate it just at the mean values of random variables. The optimization model is therefore given by

$$\begin{cases} \min\limits_{\mathbf{d}} \; f(\mathbf{d}, \boldsymbol{\mu}_P) \\ s.t. \;\; \Pr\left\{ g_i(\mathbf{d}, \mathbf{P}) \le 0 \right\} \ge R_i, \; i = 1, 2, \; \cdots, n_g \\ \qquad h_j(\mathbf{d}, \boldsymbol{\mu}_P) = 0, \; j = 1, 2, \; \cdots, n_h \end{cases} \tag{16}$$

Another treatment is to relax and replace the equality constraints $h_j(\mathbf{d}, \mathbf{P}) = 0, \; j = 1, 2, \; \cdots, n_h$, as shown in Equation (2) by the following two equality constraints:

$$\Pr\left\{ h_j(\mathbf{d}, \mathbf{P}) \le \delta_j \right\} \ge R_j, \tag{17}$$

and

$$\Pr\left\{h_j(\mathbf{d},\mathbf{P}) \geq -\delta_j\right\} \geq R_j^{'}, \tag{18}$$

where δ_j, R_j and $R_j^{'}$ are a small tolerance and two desired reliabilities, respectively.

By use either of the above two approaches, there is no equality constraint. Therefore, the problem can be solved by any existing RBDO algorithm. Next, we thus focus on formulating and solving RBDO problems with physics-based equality constraints.

4.2 Formulating a physics-based equality constraint

As we discussed in Section 3, a physics-based equality constraint must be satisfied with a probability of 1.0 in the random analysis space. However, in the optimization model, we can only formulate an equality constraint in the deterministic design space, and an equality constraint formulated in the deterministic design space may not guarantee the equality condition in the random analysis space. To this end, it is nearly impossible to satisfy a physics-based probabilistic equality constraint in the deterministic design space. The only way to ensure the probability of 1.0 is to eliminate the equality constraints by eliminating dependent random variables. Since eliminating equality constraints may not be practical, we develop the following numerical procedure based on the principle of variable elimination.

Let equality constraint functions in the random analysis space be

$$h_i(\mathbf{d},\mathbf{P}) = 0 \quad i = 1, 2, \ \cdots, \ n_h \tag{19}$$

$\mathbf{d}$ are deterministic design variables and are treated as constants in the random analysis space. The inclusion of equality constraints in RBDO results in the dependency (correlations) among random variables. In this work we assume that there are $n_P - n_h$ independent random variables. We partition random variables $\mathbf{P}$ into independent variables $\mathbf{X}$ and dependent variables $\mathbf{Y}$ such that

$$\mathbf{P} = \left(\mathbf{X},\ \mathbf{Y}\right) \tag{20}$$

Solving the simultaneous equations in Equation (19) yields

$$\mathbf{Y} = \mathbf{G}\left(\mathbf{d}, \mathbf{X}\right) \tag{21}$$

where $\mathbf{G}$ represents the relationship between $\mathbf{X}$ and $\mathbf{Y}$.

Then, an RBDO problem is formulated as

$$\begin{cases} \min\limits_{\mathbf{d}}\ f(\mathbf{d}, \boldsymbol{\mu}_X) \\ \text{s.t. } \Pr\left\{g_i\left[\mathbf{d}, \mathbf{X},\ \mathbf{G}\left(\mathbf{X}\right)\right] \le 0\right\} \ge R_i \quad i = 1, 2,\ \cdots, n_i \end{cases} \tag{22}$$

where equality constraints are eliminated.

In practice, eliminating equality constraints is difficult. We propose to use the First Order Reliability Method [12] to formulate and solve the above RDBO model. As discussed in the introduction, performance measure approach (PMA) [1, 2] has advantages over the traditional reliability index approach (RIA). We therefore use PMA to formulate the reliability constraints in Equation (20). Suppose the Most Probable Point (MPP) [12] of the constraint function $g_i\left[\mathbf{d}, \mathbf{X},\ \mathbf{G}\left(\mathbf{X}\right)\right]$ is found at $\mathbf{x}_i^*$ given the desired reliability R_i, then according to RIA, the reliability constraint $\Pr\left\{g_i\left[\mathbf{d}, \mathbf{X},\ \mathbf{G}\left(\mathbf{X}\right)\right] \le 0\right\} \ge R_i$ is equivalent to $g_i\left[\mathbf{d}, \mathbf{x}_i^*,\ \mathbf{G}\left(\mathbf{x}_i^*\right)\right] \le 0$. Hence the RBDO model can be rewritten as

$$\begin{cases} \min\limits_{\mathbf{d}}\ f(\mathbf{d}, \boldsymbol{\mu}_X) \\ \text{s.t. } g_i\left[\mathbf{d}, \mathbf{x}_i^*,\ \mathbf{G}\left(\mathbf{x}_i^*\right)\right] \le 0 \quad i = 1, 2,\ \cdots, n_i \end{cases} \tag{23}$$

Using the concept of variable elimination but not actually eliminating dependent variables, we include the mean values $\boldsymbol{\mu}_Y$, of the dependent variables $\mathbf{Y}$ and their values $\mathbf{y}_i^*$ at the MPP $\mathbf{x}_i^*$ as design variables as well. $\mathbf{Y}_i^*$ can be calculated by

$$\mathbf{Y}_i^* = \mathbf{G}\left(\mathbf{d}, \mathbf{x}_i^*\right) \tag{24}$$

The RBDO model then becomes

$$\begin{cases} \min\limits_{\mathbf{d}, \boldsymbol{\mu}_Y, \mathbf{y}_i^*} f(\mathbf{d}, \boldsymbol{\mu}_X, \boldsymbol{\mu}_Y) \\ \text{s.t. } g_i\left[\mathbf{d}, \mathbf{x}_i^*, \mathbf{G}\left(\mathbf{x}_i^*\right)\right] \le 0 \quad i = 1, 2, \cdots, n_i \\ \qquad h_j(\mathbf{d}, \boldsymbol{\mu}_\mathbf{X}, \boldsymbol{\mu}_\mathbf{Y}) = 0,\ j = 1, 2, \cdots, n_h \\ \qquad h_j(\mathbf{d}, \mathbf{x}_i^*, \mathbf{y}_i^*) = 0,\ j = 1, 2, \cdots, n_h,\ i = 1, 2, \cdots, n_g \end{cases} \tag{25}$$

In the above model, since each inequality constraint has its own MPP, the totally number of equality constraints at MPPs is $n_h \times n_g$. Solving this model needs to search the MPP $\mathbf{x}_i^*$ for the reliability constraint $g_i \le 0$. The MPP search for reliability analysis itself is also an optimization problem [1, 2], and therefore, directly solving the above RBDO problem will involve an expensive double-loop procedure. Next, we develop a numerical procedure that decouples the optimization outer loop from the reliability analysis inner loop with the sequential single-loop strategy.

5. SOLVE RBDO WITH PHYSICS-BASED EQUALITY CONSTRAINTS

In this section, we first briefly review the recently developed sequential single-loop strategy for RBDO without equality constraints. Then, we develop the formulation and numerical procedure for RBDO with physics-based equality constraints.

5.1 Sequential single-loop strategy for RBDO

RBDO without equality constraints has been extensively studied. Traditional approaches to RBDO without equality constraints require a nested double-loop process. Under the optimization outer loop, the inner loop of reliability analysis calculates the reliability for each of probabilistic constraints. The optimization outer loop searches for the optimal solution by updating design variables and calling the reliability analysis repeatedly. Because of the nested framework, this process is computationally intensive. To improve computational efficiency, single loop methods have been developed. The original approaches can be found in Chen and Hasselman [16], Wu and Wang [17] and Wu, et al. [18]. The sequential optimization and reliability assessment (SORA) [2] method is one of later developed methods. SORA consists of a few cycles as shown in Figure 3. In each cycle, there are two decoupled parts: deterministic optimization (DO) and reliability analysis (RA); deterministic optimization is performed first, followed by reliability analysis.

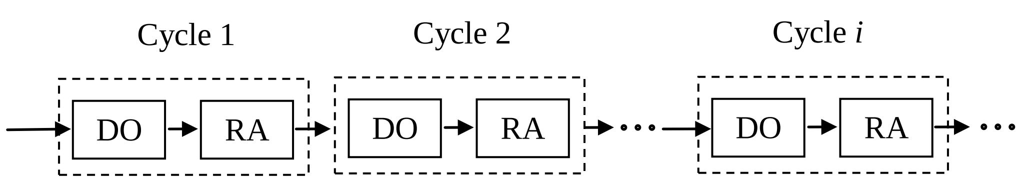

Figure 3. Sequential single-loop strategy for RBDO.

In each cycle, optimization and reliability analysis are decoupled from each other and run sequentially. The deterministic optimization is performed to achieve an optimal design solution; the reliability analysis is conducted after optimization to verify the satisfaction of reliability constraints and also provides improvement direction for updating design solution within optimization in the next cycle. If the process does converge, a new deterministic optimization model is formulated for the next cycle based on the reliability analysis results just obtained. The new optimization formulation shifts the constraint boundaries of unsatisfied reliability constraints toward the feasible region and therefore guarantees the reliability improvement.

5.2 Deterministic Optimization

The deterministic optimization in each cycle is given by Equation (25), where in addition to the original design variables **d**, two new groups of variables, $\boldsymbol{\mu}_Y$ and $\mathbf{y}^*$, are also added as design variables. The inequality constraints are evaluated at the MPPs $\left(\mathbf{x}^*, \mathbf{y}^*\right)$ of independent random variables **X** and dependent random variables **Y**. The MPPs $\mathbf{x}^*$ of independent variables are the results from the reliability analysis in the previous cycle. Equality constraints hold at both the means and the MPPs.

5.3 Reliability analysis

The First Order Reliability Method (FORM) is employed to calculate the MPPs corresponding to the desired reliability. Two main steps are involved.

(1) Transformation: Rosenblatt transformation [19] is used to convert the independent random variables **X** in each of the constraint functions g_i into standard normal variables **U**. The transformation is given by

$$u_j = \Phi^{-1}\left[F_{X_j}(x_j)\right], \quad j = 1, 2, \cdots, n_X \tag{26}$$

where $\Phi^{-1}[\cdot]$ is the inverse cumulative distribution function (CDF) of the standard normal distribution, $F_{X_j}(\cdot)$ is the CDF of X_j. After the transformation, the constraint function $g_i(\mathbf{d}, \mathbf{X}, \mathbf{y}_i^*)$ in the original space becomes another function $g_i(\mathbf{d}, \mathbf{U}, \mathbf{y}_i^*)$ in the transformed normal space with regarding to **X**. $\mathbf{d}$ and $\mathbf{y}_i^*$ are known because they are obtained in the deterministic optimization in the previous cycle.

(2) MPP search: For constraint i, a maximization (optimization) procedure is used to search the MPP $\mathbf{u}^*$ in the transformed normal space. The model is given by for all g_i, $i = 1, 2, \cdots, n_g$

$$\begin{cases} \max\limits_{\mathbf{u}, \mathbf{Y}} \ \mathrm{g}_i(\mathbf{d}, \mathbf{u}, \mathbf{y}_i^*) \\ st. \quad \|\mathbf{u}\| = \Phi^{-1}(R_i) \\ \qquad h_j(\mathbf{d}, \mathbf{u}, \mathbf{y}_i^*) = 0, \ j = 1, 2, \ \cdots, \ n_h \end{cases} \tag{27}$$

Different from the conventional MPP search without any equality constraint, the above model includes all the equality constraints to ensure their satisfaction at the MPP. The solution $\mathbf{u}_i^*$ to the above problem is the MPP for constraint i in the transformed space. The MPP $\mathbf{x}_i^*$ in the original space can be obtained by using Eq. (26).

The overall flowchart is depicted in Figure 4.

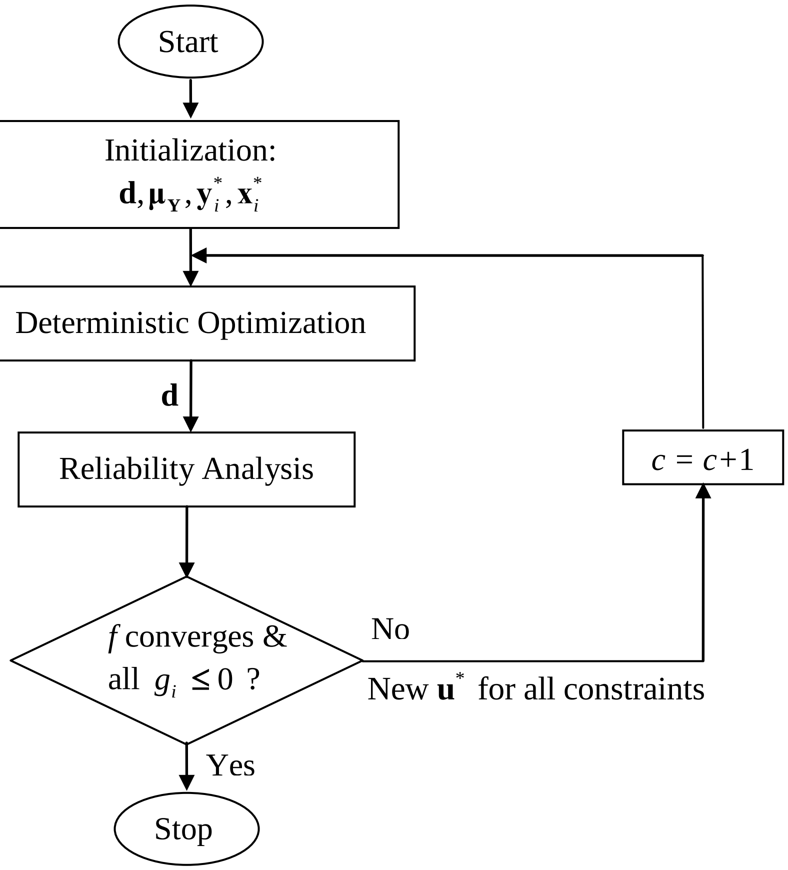

Figure 4. Flowchart of RBDO with Equality Constraints.

6. NUMERICAL EXAMPLES

In this section, we use two examples to demonstrate the proposed method, including a mathematical problem and a design of a two-member frame. For the two-member frame design, the result from the proposed method is also verified with that from the elimination method, in which equality constraints are eliminated.

6.1 A mathematical problem

A deterministic optimization problem is given by

The problem involves design variables $\mathbf{d} = (d_1, d_2, d_3, d_4, d_5, y_1, y_2)$ and two equality constraints. The last two design variables (y_1, y_2) can be eliminated because of the two equality constraints.

To demonstrate the effectiveness of the proposed method, this deterministic optimization problem is reconfigured as an RBDO problem, in which $d_i \quad (i = 1, 2, ..., 5)$ are consider as random design variables, which follow normal distributions $N(\mu_i, \sigma_i)$, $\sigma_i = 0.1$. To use the RBDO model as shown in Equality (25), a random design variable is split into two parts, namely, $\mu_i + X_i$, where $X_i \sim N(0, \sigma_i)$. X_i can be treated as independent random variables, and (y_1, y_2) become the dependent random variables (Y_1, Y_2) . Therefore, $\mathbf{d} = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$, $\mathbf{P} = (\mathbf{X}, \mathbf{Y})$, $\mathbf{X} = (X_1, X_2, X_3, X_4, X_5)$, $\mathbf{Y} = (Y_1, Y_2)$, and the RBDO model is given by

$$\begin{cases} \min\limits_{\mathbf{d}=(d_1,d_2,d_3,d_4,d_5)} \quad f(d_1, d_2, d_3) = d_1^2 + 2d_2 + d_3 - d_2\mu_{Y_2} \\ st:\ 12 \text{-} \left(\sqrt{d_1 + X_1} + (d_4 + X_4) + (d_5 + X_5)\sqrt{0.4Y_1}\right) \ \le 0 \\ \quad 11 \text{-} \left((d_1 + X_1)_1^2 + 2(d_2 + X_2) + (d_3 + X_3) + (d_2 + X_2)(-Y_2)\right) \le 0 \\ \quad Y_1 = (d_1 + X_1)^2 + 2(d_2 + X_2) - (d_3 + X_3) + 2\sqrt{Y_2} \\ \quad Y_2 = (d_1 + X_1)(d_4 + X_4) + (d_4 + X_4)^2 + (d_5 + X_5) + Y_1 \\ \quad 0 \le d_i \le 10, \quad i = 1, 2, ..., 5 \end{cases}$$

The desired reliability is 0.9 for the two inequality constraints. The results from the proposed method are given in Table I. The entire RBDO procedure converges with three cycles. The final optimal design solution is

$\mathbf{d} = (d_1, d_2, d_3, d_4, d_5) = (2.2494, 0.0000, 7.5075, 0.0000, 8.4180)$, and the objective value is 12.5671.

Table I. The results for the mathematical RBDO problem.

Cycle	Design solution $\mathbf{d} = (d_1, d_2, d_3, d_4, d_5)$	Objective
1	(2.3135, 0.1380,7.1118, -0.0000, 6.9902)	11.0000
2	(2.2489, 0.0000, 7.5052, 0.0000, 8.4167)	12.5627
3	(2.2494, 0.0000, 7.5075, 0.0000, 8.4180)	12.5671

6.2 RBDO for a two-member frame [20]

A two-member frame is subject to out-of-plane load as shown in Figure 5. Such frames are commonly encountered in automotive, aerospace, mechanical and structural engineering applications. The design is to minimize the volume of the frame with the stress constraints such that the maximization stresses should be less than or equal to the allowable material strengths. Three design variables are the width (*d*), the height (*h*), and the wall thickness (*t*) of the member, namely, $\mathbf{d} = (d, h, t)$.

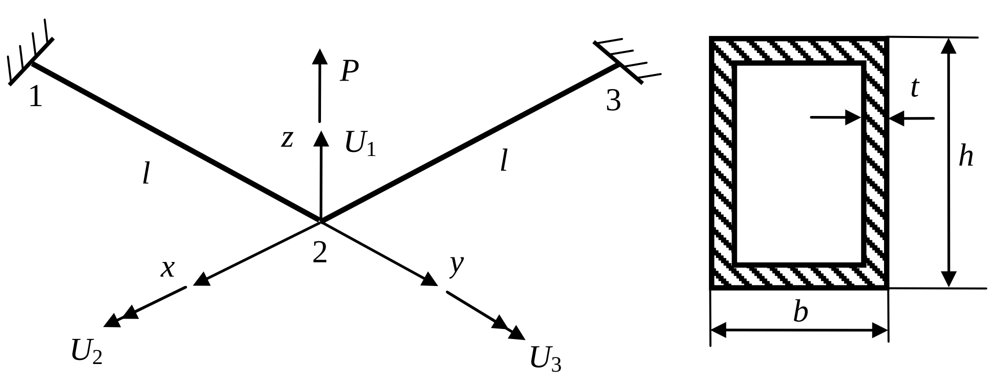

Figure 5. A two-member frame.

The volume of the structure is given by

$$f(\mathbf{d}) = 2l(2dt + 2ht - 4t^2)$$

The members are subjected to both bending and shear stresses, and the combined stress constraint needs to be imposed at points 1 and 2. According to von Mises yield condition, the two constraints are

$$g_1 = \frac{1}{\sigma_a^2}\left(\sigma_1^2 + 3\tau^2\right) - 1.0 \le 0$$

$$g_2 = \frac{1}{\sigma_a^2}\left(\sigma_2^2 + 3\tau^2\right) - 1.0 \le 0$$

where σ_1, and σ_2 are the maximum bending stresses at points 1 and 2, respectively; and τ is the shear stress in the members.

To calculate the stresses, the vertical displacement U_1 at Point 2, the rotation U_2 about line 3-2 and the rotation U_3 about line 1-2 need to be first computed using finite element analysis procedure, which involves solving the following three equality constraints,

$$\frac{EI}{L^3}\begin{bmatrix} 24 & -6L & 6L \\ -6L & 4L^2 + \frac{GL}{EI}L^2 & 0 \\ 6L & 0 & 4L^2 + \frac{GL}{EI}L^2 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix},$$

where

$$I \text{ (moment of inertia)} = \frac{1}{12}\left[dh^3 - \left(d - 2t\right)\left(h - 2t\right)^3\right],$$

$$G \text{ (polar moment of inertia)} = \frac{2t\left(d - t\right)^2\left(h - t\right)^2}{\left(d + h\right) - 2t},$$

and

$$\text{A (area for calculation of torsional shear stress)} = \left(d - t\right)\left(h - t\right).$$

Once the U_1, U_2 , U_3 have been solved out, the torque T, and bending moments M_1 and M_2 at points 1 and 2 for the member 1-2 can be calculated as

$$T = -\frac{GJ}{L}U_3,$$

$$M_1 = \frac{2EI}{L}\left(-3U_1 + U_2 L\right),$$

and

$$M_2 = \frac{2EI}{L}\left(-3U_1 + 2U_2 L\right).$$

Using the torque and moments, the torsional shear and bending stresses can be calculated as

$$\tau = \frac{T}{2At},$$

$$\sigma_1 = \frac{1}{2I}\left(M_1 h\right),$$

and

$$\sigma_2 = \frac{1}{2I}\left(M_2 h\right).$$

For more details about this example, refer to [20].

Five independent random variables are involved. They consist of the length L, the modulus of elasticity E, the shear modulus G, the allowable stress (material strength) σ_a , and the external force P. Their distribution parameters are given in Table II.

Table II. Distributions of random variables.

Variable	Distribution type	Mean (μ)	Standard deviation(σ)
L	Normal	100 in	1 in
E	Normal	30,000,000 psi	3,000,000 psi
P	Normal	-10, 000 lb	1,000 lb
G	Normal	11,540,000 psi	1,000,000 psi
σ_a	Normal	40,000 psi	4,000 psi

The formulation with both inequality and equality constraints is briefly shown as

$$\begin{cases} \min\limits_{\mathbf{d}} \ f(\mathbf{d}) = 2L\left(2dt + 2ht - 4t^4\right) \\ st: \ g_1(\mathbf{d},\mathbf{P}) = \dfrac{1}{\sigma_a^2}\left(\sigma_1^2 + 3\tau^2\right) - 1.0 \le 0 \\ \quad\ \ g_2(\mathbf{d},\mathbf{P}) = \dfrac{1}{\sigma_a^2}\left(\sigma_2^2 + 3\tau^2\right) - 1.0 \le 0, \\ \quad\ \ \dfrac{EI}{L^3}\begin{bmatrix} 24 & -6L & 6L \\ -6L & 4L^2 + \dfrac{GL}{EI}L^2 & 0 \\ 6L & 0 & 4L^2 + \dfrac{GL}{EI}L^2 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} \\ \quad\ \ 2.5 \le h \le 10;\ 2.5 \le d \le 10;\ 0.1 \le t \le 1.0 \end{cases} \tag{29}$$

in which $\mathbf{d} = \left(d, h, t\right)$ and $\mathbf{P} = \left(\mathbf{X}, \mathbf{Y}\right) = \left(E, L, G, P, \sigma_a, U_1, U_2, U_3\right)$. The independent random variables are $\mathbf{X} = \left(E, L, G, P, \sigma_a\right)$ and the dependent random variables are $\mathbf{Y} = \left(U_1, U_2, U_3\right)$.

To show the effectiveness of the proposed method, we first solve the RBDO problem with three equality constraints and two probabilistic inequality constraints, which is converted from the deterministic optimization model shown in Eq. (29). The desired reliability is 0.9 for both inequality constraints. The results are shown in Table II. The entire RBDO process converges with three serial cycles. The final design solution is **d** = (d,

h, t) = (9.7518, 10.0000, 0.1000) (in). The total volume of the two-member frame structure is 782.0697 in^3.

Table III. Results from the proposed RBDO with equality constraints.

Cycle	Design solution **d** = (d, h, t) (in)	Objective
1	(7.7987, 10.0000, 0.1000)	703.9467
2	(9.7517, 10.0000, 0.1000)	782.0695
3	(9.7518, 10.0000, 0.1000)	782.0697

This problem can also be formulated as an RBDO problem with only two inequality constraints. To verify the proposed method, we also use the RBDO with only two inequality constraints. The result from the RBDO model with only two inequality constraints is the same as the result from the proposed RBDO formulation with both inequality and equality constraints.

7. CONCLUSIONS

Equality constraints exist in many engineering problems and have been well-addressed in deterministic optimizations. With random variables, it is straightforward to deal with equality constraints in RBDO. Equality constraints are therefore overlooked in literature or are simply treated as deterministic ones at the mean values of random variables.

To appropriately handle equality constraints in RBDO, we classify them into two types. The first type consists of demand-based equality constraints, which are resulted from a designer's preferences and desires. A demand-based equality constraint may not be satisfied when random variables exist. They can be treated as deterministic equality constraints or as inequality reliability constraints that bound an equality constraint with a small tolerance.

The second type consists of physics-based equality constraints, which must be satisfied with a probability of 1.0. This type of constraints can be eliminated by solving and expressing the dependent random variables in terms of independent random variables. However, in practice, variable elimination may not be feasible due to coupling, high dimensionality, nonlinearity, implicit format, and high computational costs. A numerical procedure is proposed to handle and solve the RBDO problems with physics-based equality constraints. A sequential deterministic optimization and reliability analysis strategy is employed. Formulations for the deterministic optimization and reliability analysis are developed. To maintain the equality relationships without eliminating equality constraints, additional design variables are added, including the mean values and the MPPs of the dependent random variable. The two examples indicate the feasibility and effectiveness of the proposed method.

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