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Probabilistic Uncertainty Analysis by Mean-Value First Order Saddlepoint Approximation

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Abstract

Probabilistic uncertainty analysis quantifies the effect of input random variables on responses (outputs). It is an integral part of decision making under uncertainty and risk, reliability-based design, robust design, and design for Six Sigma. The efficiency and accuracy of probabilistic uncertainty analysis is a trade-off issue in engineering applications. In this paper, an efficient and accurate Mean-Value First Order Saddlepoint Approximation (MVFOSA) method is proposed. Similar to the Mean-Value First Order Second Moment (MVFOSM) approach, a response function is approximated with the first order Taylor expansion at the mean values of all the random input variables. Instead of simply using the first two moments of the random variables as in MVFOSM, MVFOSA estimates the probability density function and cumulative distribution function of the response by the accurate Saddlepoint Approximation. Because of the use of complete distribution information, MVFOSA is generally more accurate than MVFOSM with the same computational effort. Without the nonlinear transformation from non-normal variables to normal variables as required by the First Order Reliability Method (FORM), MVFOSA is more accurate than FORM in certain circumstances, especially when the transformation significantly increases the nonlinearity of the response function. It is also more efficient than FORM due to no need of an iterative search process for the so-called Most Probable Point in FORM. The features of the proposed method are demonstrated with four numerical examples.

Keywords: Uncertainty analysis, Cumulant generating function, Saddlepoint Approximation

1. Introduction

The performance of an engineered system or product is often affected by unavoidable uncertainties [1]. Accommodating and managing the effects of uncertainties on the performance (response), such as the consideration of reliability, safety, and robustness, is rapidly spreading in industry [2-7]. Quantitative uncertainty analysis has become an essential part of design and decision making under uncertainty and risk [8-13].

Uncertainty can be viewed as the difference between the present state of knowledge and the complete knowledge. Uncertainty is usually classified into aleatory and epistemic types [14]. Aleatory uncertainty, also termed as objective or stochastic uncertainty, describes the inherent randomness (variation) associated with a physical system or environment. This type of uncertainty is not reducible since it is a property of the system itself. Aleatory uncertainty is dealt with by probability theory. Epistemic uncertainty, on the other hand, results from some level of ignorance or incomplete information about a system [1, 14-22]. Epistemic uncertainty is reducible if more information is collected. Because of this reason, it is also termed as subjective or reducible uncertainty. Epistemic uncertainty can be modeled by probability theory, or other theories such as evidence theory, possibility theory, and fuzzy set.

Even though the incorporation of both types of uncertainty is important, the focus of this paper is uncertainty analysis that involves only aleatory uncertainty. We will call this type of analysis *probabilistic uncertainty analysis* and will use probability distributions to describe input variables with aleatory uncertainty. The major task of probabilistic uncertainty analysis is to obtain the distribution of a response (performance) Y or the probability of $Y \le y$ given the distributions of random inputs $\mathbf{X} = \left[X_1, X_2, ..., X_n\right]^{\mathrm{T}}$. The functional relationship between Y and $\mathbf{X}$ is called a performance function and is expressed by $Y = g(\mathbf{X})$. In many engineering applications, the performance function $Y = g(\mathbf{X})$ is usually a black-box and is computationally expensive to evaluate. For example, the vehicle crash simulation modeled by finite element analysis requires hours or even days of computational time for one single analysis run. Since probabilistic uncertainty analysis needs to evaluate the performance function $Y = g(\mathbf{X})$ (e.g. the vehicle crash simulation) a number of times, the challenge issue is the intensive computational demand.

In this paper, the following symbol convention is used. An *uppercase* letter denotes a random variable, a *lowercase* letter denotes an observation (or a realization) of a random variable, and a *bold* letter denotes a vector. For instance, X stands for a random variable; x denotes a realization of X; **X** represents a vector of random variables $\left[X_1, X_2, ..., X_n\right]^{\mathrm{T}}$, and **x** is an observation of **X**.

Theoretically, the cumulative density function (CDF) of Y can be calculated by a multi-dimensional integral,

$$F_Y(y) = P\{Y \le y\} = \int_{g(\mathbf{X}) \le y} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}, \tag{1}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of random variables $\mathbf{X} = \left[X_1, X_2, ..., X_n\right]^{\mathrm{T}}$. Herein $X_1, X_2, ..., X_n$ are assumed *mutually independent*. In practice, the nonlinear integration boundary $g(\mathbf{X}) = y$ and the high dimensionality make it difficult or even impossible to obtain an analytical solution to the probability integration in Eq. (1) [3]. Simulation and approximation methods are therefore used for probabilistic uncertainty analysis.

Commonly used simulation or approximation methods can be roughly categorized into three types: (1) sampling-based methods, (2) moment matching methods, and (3) Most Probable Point (MPP) based methods. Sampling-based methods [23-27], such as direct Monte Carlo simulation, Latin hypercube sampling and importance sampling, are accurate and feasible to use. If sufficient simulations are used, an accurate result can be obtained with high confidence. However, sampling-based methods are inefficient for many engineering design problems where high reliability and computationally expensive performance function $g(\mathbf{X})$ are involved [28]. For example, if the probability of failure of a structural system is 10^{-6}, theoretically and statistically, only one failure can be observed from 10^6 simulations. To ensure an accurate estimation of the probability of failure, at least 10×10^6 simulations should be conducted. In other words, the structural analysis should be performed 10×10^6 times. If the structural analysis is expensive, the task will be extremely time-consuming. To ease the computational difficulties for design under uncertainty, moment matching methods [5,29-31] are usually employed. This type of methods approximates the distribution of Y by selecting a distribution from a class of assumed distributions through fitting the first few moments. Various approaches, such as numerical integrations, point estimate methods [5, 31] and Taylor series approximations [6], have been developed to estimate the moments. Typically, the first two statistical moments are used. This type of methods is highly efficient since the performance function is approximated by the first or second order Taylor expansion. The commonly used method is the Mean-Value First Order Second Moment (MVFOSM) approach, which employs the first order Taylor expansion at the mean values of random variables [2]. MVFOSM is used in robust design optimization and rough reliability analysis. Despite its high efficiency, the accuracy of a moment matching method is generally lower than that of sampling-based methods. To obtain a good balance between efficiency and accuracy, MPP-based methods have been developed. They are widely used in reliability-based design optimization and structural reliability analysis. Compared with sampling-based methods, MPP-based methods have the advantages of satisfactory accuracy and moderate computational cost. They approximate the performance function with Taylor Series expansion at a special point, the Most Probable Point (MPP) to ensure minimal accuracy loss. Typical MPP-based methods include the First Order Reliability Method (FORM), the Second Order Reliability Method (SORM) [32, 33], and the recently developed First Order Saddlepoint Approximation (FOSA) [34]. In most cases, the MPP has to be located numerically, and the number of the function evaluations is approximately equal to the number of random variables times the number of iterations for FORM and FOSA. The number is much higher for SORM. Hence MPP-based methods may still be computationally expensive for large scale problems.

This work is aimed to address the computational issue of probabilistic uncertainty analysis. We will focus on the situation where the performance function is expensive to evaluate and only the MVFOSM method is feasible to use. The research question is: *How can we improve the accuracy of MVFOSM without sacrificing efficiency?* To answer the question, we propose to apply the accurate Saddlepoint Approximation [35] after the performance function is linearized at the mean values of random variables. The method is therefore termed as Mean-Value First Order Saddlepoint Approximation (MVFOSA) method.

The rest of the paper is organized as follows. In Section 2, three existing methods, MVFOSM, FORM and FOSA, are briefly reviewed. The proposed method is then introduced in detail in Section 3. The comparison between the proposed method and existing methods are performed through four examples in Section 4. Section 5 presents the conclusions.

2. Existing methods – MVFOSM, FORM and FOSA

In this section, we briefly review the existing methods, MVFOSM, FORM and FOSA. The review provides the background information and serves as the basis for the comparison study between the existing methods and our proposed method.

2.1 Mean-value first order second moment (MVFOSM) approach

MVFOSM is based on the first order Taylor expansion of the performance function $Y = g(\mathbf{X})$ at the mean values of random variables. The performance function $g(\mathbf{X})$ is linearized as

$$Y \cong g(\boldsymbol{\mu}) + \sum_{i=1}^{n} \left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} \left(X_i - \mu_{X_i} \right), \tag{2}$$

where $\boldsymbol{\mu} = \left[\mu_{X_1}, \mu_{X_2}, ..., \mu_{X_n} \right]^{\mathrm{T}}$ is the vector of the mean values of $\mathbf{X} = \left[X_1, X_2, ..., X_n \right]^{\mathrm{T}}$.

Based on Eq. (2), the mean value and standard deviation of *Y* are calculated by

$$\mu_Y \cong g(\boldsymbol{\mu}), \tag{3}$$

and

$$\sigma_Y \cong \sqrt{\sum_{i=1}^{n} \left(\left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} \sigma_{X_i} \right)^2}, \tag{4}$$

respectively, where σ_{X_i} is the standard deviation of random variable X_i. The probability of $Y \leq y$ is calculated by [2]

$$F_Y(y) = P\{Y \leq y\} \cong \Phi\left(\frac{y - \mu_Y}{\sigma_Y} \right). \tag{5}$$

Equation (5) implies an assumption that the response *Y* is normally distributed. Since after the linearization, *Y* is a linear combination of X_i $(i = 1, 2, ..., n)$ in Eq. (2), the assumption is also equivalent to assuming that X_i are normally distributed. For black-box and explicit performance functions, the *n* partial derivatives $\frac{\partial g}{\partial X_i}$ at $\boldsymbol{\mu} = \left[\mu_{X_1}, \mu_{X_2}, ..., \mu_{X_n} \right]^{\mathrm{T}}$ in Eq. (2) can be numerically calculated by finite difference methods [36]. When forward finite difference approach is used, the derivatives at the mean values of random variables are calculated as

$$\left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} = \frac{g\left(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_i} + \Delta x_i, \ldots, \mu_{X_n}\right) - g\left(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_i}, \ldots, \mu_{X_n}\right)}{\Delta x_i}, \quad i = 1, 2, \ldots, n, \qquad (6)$$

where Δx_i are the difference intervals. It is noted from Eq. (6) that the total number of function evaluations (g function evaluations) is equal to $n + 1$. After the derivatives are obtained, no more function evaluation is needed for probability calculation. The backward and central difference approaches can also be employed for the partial derivatives calculation. The central difference approach is generally more accurate than the forward and backward difference approaches, but it is more computationally expensive (it needs $2n$ function evaluations). For more details on finite difference methods, refer to [36].

Compared with MPP-based methods and sampling-based methods, MVFOSM is much more efficient. However, it has obvious deficiencies: it uses only the first two moments of the random variables instead of the complete distribution information [2,28], and it assumes that the response is normally distributed.

2.2 First order reliability method (FORM)

FORM solves the probability integral in Eq. (1) by simplifying the performance function $g(\mathbf{X})$ using the first order Taylor series approximation at the Most Probable Point (MPP). The classical FORM involves the following steps:

(1) Transform the original random variables $\mathbf{X} = \left[X_1, X_2, \ldots, X_n\right]^{\mathrm{T}}$ (generally non-normal) in $\mathbf{X}$-space into standard normal variables $\mathbf{U} = \left[U_1, U_2, \ldots, U_n\right]^{\mathrm{T}}$ in $\mathbf{U}$-space (the standard normal space) by Rosenball transformation [37]

$$u_i = \Phi^{-1}\left[F_{X_i}(x_i)\right], \quad i = 1, 2, ..., n, \tag{7}$$

where $\Phi^{-1}[\cdot]$ is the inverse CDF of the standard normal distribution, and $F_{X_i}(x_i)$ is the CDF of X_i. After the transformation, the performance function $Y = g(\mathbf{X})$ in **X**-space is expressed as $Y = g(\mathbf{U})$ in **U**-space.

(2) Search the MPP – the point on the integration boundary $g(\mathbf{U}) = y$ with *the minimum distance* to the origin in **U**-space. This step needs an iterative optimization process, i.e.,

$$\begin{cases} \min_{\mathbf{u}} \beta = \|\mathbf{u}\| \\ \text{s.t.} \quad g(\mathbf{u}) = y. \end{cases} \tag{8}$$

(3) Calculate the probability. At the MPP, the joint probability density function (JPDF) of **U** has the highest value on the limit state $Y = y$ in **U**-space. The performance function is linearized at the MPP in **U**-space to ensure the minimum accuracy loss. The probability in Eq. (1) is then expressed analytically by the following equation [29],

$$F_Y(y) = P\{Y \le y\} \cong \Phi(-\beta), \tag{9}$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution, β is the distance between the MPP and the origin in **U**-space defined in Eq. (8) and is commonly called *reliability index*.

Generally, FORM produces more accurate solutions than MVFOSM. However, locating the MPP, as shown in Eq. (8), is an optimization (minimization) problem and needs more function evaluations than MVFOSM. FORM may not generate accurate results when the transformation, given in Eq. (7), increases the nonlinearity of the performance function [8,34]. It is not as robust as MVFOSM, since it may fail when there exists more than one MPP or the MPP search process does not converge.

2.3 First order Saddlepoint Approximation (FOSA)

To overcome the drawback of increasing nonlinearity in the performance function because of the nonlinear transformation in Eq. (7), the recently developed First Order Saddlepoint Approximation (FOSA) [34] linearizes the performance function in the *original* random **X**-space without any random variable transformation. The expansion point is termed as the Most Likelihood Point (MLP) which has the highest probability density on the limit state $Y = y$. It is shown that FOSA is more accurate and efficient than FORM and in some cases even more accurate than the Second Order Reliability Method (SORM), and it is also proved that FORM is a special case of FOSA [34]. However, locating the MLP is also *an optimization process*. Therefore, FOSA has the same drawbacks as FORM, even though it is more accurate than FORM. For more details about FOSA, refer to [34].

It should be noted that MVFOSM, FORM and FOSA all use the first order Taylor series approximation to the performance function. Such approximation may not accurately capture the nonlinearity of the performance function, and thus may not be suitable for the situations where highly nonlinear performance functions are involved. However, they are widely used in structural reliability analysis [2,38], reliability-based design [8,10,12,28,33], and robust design [3,6], because in many situations, they provide the only practical alternatives for engineering design under uncertainty.

The aim of this work is to develop an efficient and accurate method, which is expected to have the same efficiency and robustness as MVFOSM, but with much higher accuracy. The new method is detailed in the following section.

3. Mean-value first order Saddlepoint Approximation (MVFOSA)

The central idea of MVFOSA is to use the accurate Saddlepoint Approximation to evaluate the cumulative distribution function (CDF) and probability density function (PDF) of a response (performance function) $Y = g(\mathbf{X})$. The Saddlepoint Approximation requires the cumulant generating function (CGF) of the response *Y*. The CGF of *Y* can be readily obtained if the performance function $Y = g(\mathbf{X})$ is linearized. Therefore, the procedure of the proposed method consists of three steps: (1) linearize the performance function at the mean values of random variables, 2) calculate the CGF of the performance function, and (3) estimate the CDF and PDF of the performance function. In the following subsections, the definition of CGF and its two useful properties are introduced, followed by the calculation of the CGF of the linearized response. The Saddlepoint Approximation for CDF and PDF estimations is then discussed.

3.1 Cumulant generating function (CGF)

The moment generating function of a random variable X, $M_X(t)$, is defined as [39]

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx. \tag{10}$$

The CGF of *X* is defined as the *natural logarithm* of $M_X(t)$, i.e.

$$K_X(t) = \ln\left[M_X(t)\right]. \tag{11}$$

The CGF of some commonly used distributions are listed in Table 1. Interested readers can refer to References [39] and [40] for other CGFs.

The two useful properties of CGF are as follows.

Property I: If $X_1, X_2, ..., X_n$ are independent random variables and their CGFs are $K_{X_i}(t)$ $(i = 1, 2, ..., n)$, then the CGF of $Y = \sum_{i=1}^{n} X_i$ is

$$K_Y(t) = \sum_{i=1}^{n} K_{X_i}(t). \tag{12}$$

Property II: If X is a random variable with a CGF, $K_X(t)$, the CGF of $Y = aX + b$ is given by

$$K_Y(t) = K_X(at) + bt, \tag{13}$$

where both a and b are constants.

For example, if X is normally distributed with mean value of μ_X and variance of σ_X^2, then the CGF of $Y = aX + b$ is

$$K_Y(t) = (a\mu_X + b)t + \frac{1}{2}\sigma_X^2 a^2 t^2. \tag{14}$$

Based on Properties I and II, if a linear performance function is given by

$$Y = a_0 + \sum_{i=1}^{n} a_i X_i, \tag{15}$$

where a_i $(i = 1, 2, ..., n)$ are constants, the CGF of Y reads

$$K_Y(t) = a_0 t + \sum_{i=1}^{n} K_{X_i}(a_i t). \tag{16}$$

After a general performance function is linearized at the mean values of random variables with a form of Eq. (15), we can use Eq. (16) to calculate the CGF of the linearized performance function.

3.2 The CGF of a response by the first order Taylor expansion

The first order Taylor expansion of $Y = g(\mathbf{X})$ at the mean values $\boldsymbol{\mu} = \left[\mu_{X_1}, \mu_{X_2}, ..., \mu_{X_n}\right]^{\mathrm{T}}$ of random variables $\mathbf{X}$ is given by

$$Y \cong g(\boldsymbol{\mu}) + \sum_{i=1}^{n} \left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} \left(X_i - \mu_{X_i}\right). \tag{17}$$

With Eq. (16), the CGF of Y is then evaluated by

$$K_Y(t) \cong \left(g(\boldsymbol{\mu}) - \sum_{i=1}^{n} \frac{\partial g}{\partial X_i}\bigg|_{\boldsymbol{\mu}} \mu_{X_i} \right) t + \sum_{i=1}^{n} K_{X_i}\left(\frac{\partial g}{\partial X_i}\bigg|_{\boldsymbol{\mu}} t \right). \quad (18)$$

3.3 Saddlepoint Approximation for the CDF and PDF of a response

The Saddlepoint Approximation has become a powerful tool to estimate CDF and PDF since it was first introduced by Daniels [35,41-47]. It has several excellent features. First, it yields extremely accurate probability estimation, especially in the tail areas of a distribution [41]; second, it requires only the process of finding one saddlepoint without any integration; and third, it provides estimations of both CDF and PDF, thereby there is no need of taking numerical derivative of CDF to obtain PDF or taking integration of PDF to obtain CDF. The theory of Saddlepoint Approximation is quite complex, but its use is fairly easy with simple formulas [45] as shown next.

Once the CGF of Y is obtained as shown in Eq. (18), it is straightforward to apply the Saddlepoint Approximation to CDF and PDF estimations. A simple formula for computing the PDF of Y is expressed as [35]

$$f_Y(y) \cong \left\{ \frac{1}{2\pi K_Y''(t_s)} \right\}^{\frac{1}{2}} e^{[K_Y(t_s) - t_s y]}, \quad (19)$$

where $K_Y''(\cdot)$ is the second order derivative of the CGF of Y, and t_s is the *saddlepoint*, which is the solution to the equation,

$$K_Y'(t) = y, \quad (20)$$

where $K_Y'(\cdot)$ is the first order derivative of the CGF of Y.

Lugannani and Rice [48] gave a concise formula for calculating the CDF of Y,

$$F_Y(y) = P\{Y \le y\} \cong \Phi(w) + \phi(w)\left(\frac{1}{w} - \frac{1}{v}\right), \tag{21}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the CDF and PDF of the standard normal distribution, respectively;

$$w = sgn(t_s)\left\{2\left[t_s y - K_Y(t_s)\right]\right\}^{1/2} \tag{22}$$

and

$$v = t_s\left[K_Y^{''}(t_s)\right]^{1/2}, \tag{23}$$

where $sgn(t_s) = +1,\ 0,\ \text{or}\ -1$, depending on whether the saddlepoint t_s is positive, negative or zero.

From the CGF of Y given in Eq. (18), the first order derivative of the CGF of Y is

$$K_Y^{'}(t) = \left(g(\boldsymbol{\mu}) - \sum_{i=1}^{n}\left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} \mu_{X_i}\right) + \sum_{i=1}^{n}\left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} K_{X_i}^{'}\left(\left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} t\right). \tag{24}$$

Solving the following equation,

$$K_Y^{'}(t) = \left(g(\boldsymbol{\mu}) - \sum_{i=1}^{n}\left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} \mu_{X_i}\right) + \sum_{i=1}^{n}\left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} K_{X_i}^{'}\left(\left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} t\right) = y, \tag{25}$$

yields the *saddlepoint* t_s.

Since Eq. (25) is generally a nonlinear function of t, a numerical method is required to solve it. However, the solution process does not need to evaluate the performance function any more. Once the saddlepoint t_s is obtained, w and v will be easily calculated form Eqs. (22) and (23), respectively; and then the PDF and CDF of Y will be calculated from Eqs. (19) and (21), respectively.

In the appendix, the above process is applied to a performance function with *only normally distributed random variables*. The derivation shows that the result from MVFOSA is the same as that from MVFSOM. Therefore, MVFOSM is a special case of MVFOSA. Since

MVFOSA uses the full distribution information while MVFOSM does not, the former is generally more accurate than the latter. We will demonstrate this with the following example problems.

4. Examples and discussions

In this section, four examples are used to illustrate the effectiveness of the proposed method (MVFOSA); comparisons are made with FORM, FOSA, MVFOSM, and Monte Carlo simulation (MCS) to appraise accuracy and efficiency. The efficiency is measured by the number of function evaluations. MCS is used as a reference for the accuracy comparison. The function evaluations used by MVFOSM, MVFOSA, FORM and FOSA include those for finite difference derivative calculation. The function evaluations used by FORM and by FOSA also include those for MPP or MLP search.

Example 1: A mathematical problem

Consider a performance function with two independent random variables given by [34]

$$Y = g(\mathbf{X}) = \frac{X_1 + X_2 - 2}{\sqrt{2}}, \tag{26}$$

where X_1 and X_2 follow a standard exponential distribution with the PDF,

$$f_{X_i}(x_i) = e^{-x_i},\ i = 1, 2. \tag{27}$$

The CDF of Y is

$$F_Y(y) = P\{Y \le y\} = P\left\{\frac{X_1 + X_2 - 2}{\sqrt{2}} \le y\right\} = P\left\{X_1 + X_2 \le \sqrt{2}y + 2\right\}. \tag{28}$$

A theoretical solution exists for $P\left\{X_1 + X_2 \le \sqrt{2}y + 2\right\}$ since $X_1 + X_2$ follows a gamma distribution. FORM, MVFOSM, FOSA, and MVFOSA are employed to estimate the CDF of the performance function over a range of $[-1.414,\ 4.0]$, and the results are given in Table 2.

The results are also visualized in Fig. 1, which shows that the result of MVFOSA is almost identical to the theoretical solution while those of FORM and MVFOSM have large errors. For this *linear* problem, FOSA produces the same result as MVFOSA. Theoretically, the error of MVFOSA comes from the linearization of the performance function and Saddlepoint Approximation. But for this *linear* performance function, the error is only from the Saddlepoint Approximation, the results indeed verify the high accuracy of the Saddlepoint Approximation. The reason that MVFOSM is inaccurate is that it only uses the first two moments of X_1 and X_2 instead of the full distribution information [28], and assumes that Y is normally distributed. The error of FORM comes from the linearization of the performance function in transformed **U**-space. Even though the performance function is linear, after the nonlinear transformation from exponential variables to standard normal variables, the performance function in Eq. (26) becomes a highly nonlinear function in **U**-space [34]. The increase of the nonlinearity of the performance function due to the transformation deteriorates the accuracy of FORM.

Example 2: A cantilever beam

A cantilever beam made of isotropic material as shown in Fig. 2 is subjected to a distributed transverse load [49].

The performance function is the tip displacement, which is expressed as

$$Y = \delta = g(\mathbf{X}) = \frac{QL^4}{8EI}, \tag{29}$$

where $\mathbf{X} = [Q, L, E, I]^{\mathrm{T}}$, in which Q is the constant distributed transverse load acting on the beam, L is the length of the beam, E is the Young's modulus of the beam material, and I is the moment of the cross-section. Q, L, E and I are normally distributed and their distribution parameters are provided in Table 3.

The first case for this problem is to calculate the probability of $g(\mathbf{X}) < 4.0$, and the results from different methods are given in Table 4. Since a sufficiently large number of simulations (10^6) is used, the result of MCS is considered an accurate reference. It is noted that all the methods produce accurate solutions. However, both MVFOSA and MVFOSM use only 5 function evaluations while FORM and FOSA use 26 and 21 function evaluations, respectively. Therefore, MVFOSA and MVFOSM are more efficient than FORM and FOSA.

Table 4 only gives the CDF estimation at the left tail of the distribution of the tip displacement. To investigate the accuracy and efficiency at the right tail and near the median region, other two cases are run, and their results are shown in Table 5 (near the median) and Table 6 (right tail), respectively. The results also indicate that all the methods provide accurate solutions and that MVFOSA and MVFOSM are the most efficient.

As discussed previously, for this special problem where all the random variables are normally distributed, both MVFOSA and MVFOSM should produce the same resolution. This is verified by the results from these three cases. The small difference between MVFOSM and MVFOSA in the first case (see Table 4) is just caused by numerical error. As indicated in [34], FORM is a special case of FOSA, and when all random variables are normally distributed, both methods generate the same solution. This is also confirmed by the results from the three cases.

The estimated CDF and PDF curves of the performance function from MVFOSA with only 5 function evaluations and from MCS with 10^6 function evaluations are shown in Figs. 3

and 4, respectively. The two curves almost overlap each other over the entire range of the distribution on both Figs. 3 and 4, respectively.

If FORM or FOSA were used to generate the full distribution as shown in Fig. 3 or 4, their computational cost would be much higher since, at each point on *y* axis in Figs 3 or 4, the MPP or MLP is different and must be located numerically.

Example 3: A speed reducer

The performance function of a shaft in a speed reducer is defined as [34]

$$Y = g(\mathbf{X}) = S - \frac{32}{\pi D^3}\sqrt{\frac{F^2 L^2}{16} + T^2}, \tag{30}$$

where $\mathbf{X} = \left[S, D, F, L, T\right]^{\mathrm{T}}$, S is the material strength, D is the diameter of the shaft, F is the external force, T is the external torque, and L is the length of the shaft. The performance function represents the difference between the strength and the maximum stress.

The distribution details of all the random variables are given in Table 7.

In order to compare the performance of different methods over the whole distribution range, three realizations (limit states) of Y covering two tails and the median region are selected. Tables 8 through 10 depict the results at the three limit states. It is noted that FOSA is the most accurate method, and MVFOSA is more accurate than FORM and MVFOSM. Even though MVFOSA is slightly less accurate than FOSA, it is much more efficient than FOSA. FORM has a large error for this problem.

From this example and the previous one, readers may notice that the number of function evaluations used by FORM and by FOSA varies from one case to another. This is because both of FORM and FOSA involve an optimization process to search for the MPP or MLP. This indicates that the computational cost is not controllable when FORM or FOSA is used for uncertainty analysis for decision making.

Figure 5 shows the CDF curves obtained from the proposed method requiring only 6 function evaluations and from MCS with 10^6 simulations. It is noted that the two CDF curves are almost identical to each other over the entire distribution range. The two PDF curves are also almost indistinguishable as depicted in Fig. 6.

This example demonstrates again that MVFOSA is very economical and accurate to obtain the complete distribution information of a performance function. For the same reason given in the previous example, if FORM or FOSA were used to generate the CDF and PDF curves, much more function evaluations would be needed.

Example 4: A composite beam

Consider a composite beam with 20 independent random variables (see Fig. 7) [10]. The beam with Young's modulus E_w and *A mm* wide by *B mm* high by *L mm* long, has an aluminum plate with Young's modulus E_a and a net section of *C mm* wide by *D mm* high, securely fastened to its bottom face. Six external vertical forces, P_1, P_2, P_3, P_4, P_5 and P_6, are applied at six different locations along the beam, L_1, L_2, L_3, L_4, L_5, and L_6. The allowable tensile stress is S.

In this problem, the twenty random variables are

$$\mathbf{X} = \left[X_1, X_2, ..., X_{20} \right]^{\mathrm{T}} = \left[A, B, C, D, L_1, L_2, L_3, L_4, L_5, L_6, L, P_1, P_2, P_3, P_4, P_5, P_6, E_a, E_w, S \right]^{\mathrm{T}}.$$

Details of these random variables are given in Table 11.

The maximum stress occurs in the middle cross-section M-M and is given by

$$\sigma = \frac{\left[\frac{\sum_{i=1}^{6} P_i(L-L_i)}{L} L_3 - P_1(L_2-L_1) - P_2(L_3-L_2)\right]\left[\frac{0.5AB^2 + \frac{E_a}{E_w}DC(B+D)}{AB + \frac{E_a}{E_w}DC}\right]}{\frac{1}{12}AB^3 + AB\left\{\left[\frac{0.5AB^2 + \frac{E_a}{E_w}DC(B+D)}{AB + \frac{E_a}{E_w}DC}\right] - 0.5B\right\}^2 + \frac{1}{12}\frac{E_a}{E_w}CD^3 + \frac{E_a}{E_w}CD\left\{0.5D + B - \left[\frac{0.5AB^2 + \frac{E_a}{E_w}DC(B+D)}{AB + \frac{E_a}{E_w}DC}\right]\right\}^2}. \tag{31}$$

The maximum stress σ should be less than the allowable stress (strength) *S*. The performance function of the beam is

$$Y = g(\mathbf{X}) = \sigma - S. \tag{32}$$

Table 12 gives the probabilities of $\sigma - S < 0$ (the reliability of the beam). MVFOSA generates accurate result even though FOSA is slightly more accurate. However, MVFOSA is much more efficient than FOSA.

Referenced to MCS with 10^6 simulations, MVFOSA with only 21 function evaluations provides very good CDF and PDF curves, which are shown in Figs. 8 and 9, respectively.

5. Conclusions and discussions

The purpose of this work is to improve the accuracy of the traditional Mean-Value First Order Second Moment (MVFOSM) method. The Mean-Value First Order Saddlepoint Approximation (MVFOSA) is developed for this purpose. The proposed method uses the first order Taylor expansion of a performance function at the mean values of random input variable. Then it employs the accurate Saddlepoint Approximation to estimate the CDF and PDF of the linearized performance function. The advantage of MVFOSA over MVFOSM is the improvement of accuracy without deteriorating efficiency. Same as MVFOSM, the new method requires only $n+1$ performance function evaluations (*n* is the number of random variables) when the finite difference method is used for derivative estimation. The new method is generally more accurate than MVFOSM since it uses the complete distribution information. MVFOSM is a special case of MVFOSA in the sense that both of them produce an identical solution when all the random variables are normally distributed. Therefore, it is preferable to use MVFOSA instead of MVFOSM, when computation cost is a major concern. As demonstrated by the four examples, MVFOSA may also be an attractive alternative to existing methods, such as FORM, FOSA and MCS. Compared with FORM and FOSA, the proposed method does not involve an optimization process to search the MPP or MLP, it is therefore much more efficient and robust.

There are two sources that contribute to the error of MVFOSA, the linearization of the performance function at the mean values of random variables and the Saddlepoint Approximation for the CDF and PDF of the linearized performance function. Since Saddlepoint Approximation is extremely accurate for a linear function (see Example 1), the linearization of the performance function is the dominant factor that causes errors. If the performance function is highly nonlinear and the uncertainty (variances) of random variables is large, the linearization at the mean values can not approximate the performance function well, and MVFOSA may consequently results in a large error. In this case, one may select to use FOSA, FORM, or SORM, or even Monte Carlo simulation. However, one should be aware that the error of FOSA FORM, and SORM may also be large if the MPP or MLP is not the global solution and if multiple MPPs or MLPs exist. Moreover, FORM, SORM, and FOSA are only suitable for reliability analysis but are not appropriate for the complete distribution generation because of their high computational cost. The accuracy of the proposed method can be further improved by the Advanced Mean Value (AMV) method [50,51]. Our future work will target to integrate the AMV method with MVFOSA.

It should be noted that the proposed probabilistic uncertainty analysis is only suitable for aleatory uncertainty, which is described by probability distributions. Since a performance function that specifies the relationship between the performance and input random variables is required, the proposed method is applicable to the physics-based uncertainty analysis where the system states can be modeled by the performance function. It is not designed for uncertainty analysis for time-dependent problems based on statistical data. Another limitation of the proposed method is that the analytical cumulant generating function (CGF) of a random variable should exist. A few distributions do not have an analytical CGF, for example, Weibull distribution. This problem can be solved by transforming the distributions into any other distributions that have an analytical CGF, or by using sampling method to obtain an empirical CGF [34].

Acknowledges

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Appendix: Relationship between MVFOSA and MVFOSM

MVFOSM implies that the performance function $Y = g(\mathbf{X})$ is normally distributed after it is linearized at the mean values of random variables. With a finite number of random variables, the implication is equivalent to the assumption that all the random variables are normally distributed. Next, we will use MVFOSA to derive the CDF of Y for the situation where only normal random variables are involved and demonstrate that MVFOSM is a special case of MVFOSA.

If $\mathbf{X} = \left[X_1, X_2, ..., X_n\right]^{\mathrm{T}}$ are *normally distributed* with mean of μ_{X_i} and variance of $\sigma_{X_i}^2$ $(i = 1, 2, ..., n)$, the mean-value first order Taylor series approximation of $Y = g(\mathbf{X})$ is

$$Y \cong g(\boldsymbol{\mu}) + \sum_{i=1}^{n} \left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} \left(X_i - \mu_{X_i}\right). \tag{A1}$$

According to Eq. (18) and Table 1, the CGF of *Y* is

$$K_Y(t) = g(\boldsymbol{\mu})t + \frac{1}{2}\sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} \sigma_{X_i}\right)^2 t^2. \tag{A2}$$

Solving Eq. (20),

$$K_Y^{'}(t) = y, \tag{A3}$$

the *saddlepoint* t_s is obtained as

$$t_s = \frac{y - g(\boldsymbol{\mu})}{\sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} \sigma_{X_i}\right)^2}. \tag{A4}$$

Substituting Eq. (A4) into Eq. (A2), the CGF at the saddlepoint t_s is obtained as

$$K_Y(t_s) = \frac{y^2 - \left[g(\boldsymbol{\mu})\right]^2}{2\sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} \sigma_{X_i}\right)^2}. \tag{A5}$$

The second order derivative of CGF at t_s is given as

$$K_Y^{''}(t_s) = \sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial X_i}\right|_{\boldsymbol{\mu}} \sigma_{X_i}\right)^2. \tag{A6}$$

Substituting Eqs (A4) through (A6) into Eqs. (22) and (23) yields

$$w = v = \frac{y - g(\boldsymbol{\mu})}{\sqrt{\sum_{i=1}^{n} \left(\left. \frac{\partial g}{\partial X_i} \right|_{\boldsymbol{\mu}} \sigma_{X_i} \right)^2}}, \tag{A7}$$

and according to Eq. (21), the CDF of Y becomes

$$F_Y(y) = P\{Y \le y\} \cong \Phi\left(\frac{y - g(\boldsymbol{\mu})}{\sqrt{\sum_{i=1}^{n} \left(\left. \frac{\partial g}{\partial X_i} \right|_{\boldsymbol{\mu}} \sigma_{X_i} \right)^2}} \right) = \Phi\left(\frac{y - \mu_Y}{\sigma_Y} \right), \tag{A8}$$

which is the identical result of MVFOSM given in Eq. (5). Therefore, MVFOSM is a special case of MVFOSA, when all random variables *are normally distributed*.

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List of figures

Tables

Table 1

The CGF of some common distributions

Distribution	PDF	CGF
Uniform	$f_X(x) = \frac{1}{b-a}$	$K_X(t) = \ln\left(e^{bt} - e^{at}\right) - \ln\left(b-a\right) - \ln\left(t\right)$
Normal	$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left[\frac{(x-\mu)}{\sigma}\right]^2}$	$K_X(t) = \mu t + \frac{1}{2}\sigma^2 t^2$
Exponential	$f_X(x) = \frac{1}{\beta} e^{-\frac{1}{\beta}x}$	$K_X(t) = -\ln\left(1-\beta t\right)$
Type I Extreme Value (Gumbel)	$f_X(x) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}} \exp\left(-e^{-\frac{x-\mu}{\sigma}}\right)$	$K_X(t) = \mu t + \ln\Gamma(1-\sigma t)$
Gamma	$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$	$K_X(t) = \alpha\left\{\ln\left(\beta\right) - \ln\left(\beta - t\right)\right\}$
χ^2	$f_X(x) = \frac{1}{\Gamma(n/2)2^{n/2}} x^{n/2-1} e^{-\frac{1}{2}x}$	$K_X(t) = -\frac{1}{2} n \ln\left(1-2t\right)$

Table 2
CDF $F_Y(y) = P\{Y \le y\}$

y	FORM	MVFOSM	MVFOSA FOSA	Exact
-1.414	0.0000	0.0787	0.0000	0.0000
-1.014	0.1661	0.1553	0.1108	0.1108
-0.614	0.4045	0.2696	0.3121	0.3125
-0.214	0.6013	0.4153	0.5052	0.5059
0.186	0.7426	0.5738	0.6597	0.6605
0.586	0.8377	0.7211	0.7730	0.7738
0.986	0.8993	0.8379	0.8519	0.8525
1.386	0.9382	0.9171	0.9050	0.9055
1.786	0.9625	0.9630	0.9398	0.9402
2.186	0.9774	0.9856	0.9623	0.9625
2.586	0.9864	0.9951	0.9766	0.9768
2.986	0.9919	0.9986	0.9856	0.9857
3.386	0.9952	0.9996	0.9911	0.9912
3.786	0.9972	0.9999	0.9946	0.9947
4.000	0.9979	1.0000	0.9959	0.9959

Table 3
Parameters of random variables in the cantilever beam problem

Variable	Mean	Standard deviation	Distribution
Distributed load *Q* (*N/mm*)	10	3	Normal
Length *L* (*mm*)	5,000	2	Normal
Young Modulus *E* (N/mm^2)	73,000	1,000	Normal
Cross-section moment *I* (mm^4)	1.067×10^9	100,000	Normal

Table 4
Probability $P\{g(\mathbf{X}) < 4.0\}$

	FORM	FOSA	MVFOSM	MVFOSA	MCS
$P\{g(\mathbf{X}) < 4.0\}$	0.02255	0.02255	0.02264	0.02258	0.02253
Function evaluations	26	21	5	5	10^6

Table 5
Probability $P\{g(\mathbf{X})<10.03\}$

	FORM	FOSA	MVFOSM	MVFOSA	MCS
$P\{g(\mathbf{X})<10.03\}$	0.5	0.5	0.4999	0.4999	0.5011
Function evaluations	14	11	5	5	10^6

Table 6
Probability $P\{g(\mathbf{X}) < 20.0\}$

	FORM	FOSA	MVFOSM	MVFOSA	MCS
$P\{g(\mathbf{X}) < 20.0\}$	0.9995	0.9995	0.9995	0.9995	0.9995
Function evaluations	26	21	5	5	10^6

Table 7
Distribution details of random variables in the speed reducer problem

Variable	Parameter 1	Parameter 2	Distribution
Strength *S* (*MPa*)	70	80	Uniform [a]
Diameter *D* (*mm*)	39	0.1	Normal [b]
External force *F* (*N*)	1500	150	Gumbel [c]
Span *L* (*mm*)	400	0.1	Normal
Torque *T* (*Nm*)	250	35	Normal

[a] For a uniform distribution, Parameters 1 and 2 are lower and upper bounds, respectively.
[b] For normal distribution, Parameters 1 and 2 are mean and standard deviation, respectively.
[c] For Gumbel distribution, Parameters 1 and 2 are mean and standard deviation, respectively.

Table 8

Probability $P\{g(\mathbf{X})<1.0\times10^{7}\}$

	FORM	FOSA	MVFOSM	MVFOSA	MCS
$P\{g(\mathbf{X})<1.0\times10^{7}\}$	1.492×10^{-3}	7.437×10^{-3}	4.645×10^{-3}	6.682×10^{-3}	8.020×10^{-3}
Function evaluations	1003	37	6	6	10^{6}

Table 9
Probability $P\{g(\mathbf{X}) < 2.48\times10^7\}$

	FORM	FOSA	MVFOSM	MVFOSA	MCS
$P\{g(\mathbf{X}) < 2.48\times10^7\}$	0.2433	0.4897	0.4897	0.4902	0.4963
Function evaluations	72	49	6	6	10^6

Table 10
Probability $P\{g(\mathbf{X}) < 3.5\times10^7\}$

	FORM	FOSA	MVFOSM	MVFOSA	MCS
$P\{g(\mathbf{X}) < 3.5\times10^7\}$	0.7839	0.9535	0.9704	0.9519	0.9563
Function evaluations	79	55	6	6	10^6

Table 11
The distribution information of random variables in the composite beam problem

Variable No.	Variable	Mean	Standard deviation	Distribution
1	A (mm)	100	0.2	Normal
2	B (mm)	200	0.2	Normal
3	C (mm)	80	0.2	Normal
4	D (mm)	20	0.2	Normal
5	L_1 (mm)	200	1	Normal
6	L_2 (mm)	400	1	Normal
7	L_3 (mm)	600	1	Normal
8	L_4 (mm)	800	1	Normal
9	L_5 (mm)	1000	1	Normal
10	L_6 (mm)	1200	1	Normal
11	L (mm)	1400	2	Normal
12	P_1 (kN)	15	1.5	Gumbel
13	P_2 (kN)	15	1.5	Gumbel
14	P_3 (kN)	15	1.5	Gumbel
15	P_4 (kN)	15	1.5	Gumbel
16	P_5 (kN)	15	1.5	Gumbel
17	P_6 (kN)	15	1.5	Gumbel
18	E_a (GPa)	70	7	Normal
19	E_w (GPa)	8.75	0.875	Normal
20	S (MPa)	16	1.6	Gumbel

Table 12
Probability $P\{\sigma - S < 0\}$

	FORM	FOSA	MVFOSM	MVFOSA	MCS
$P\{\sigma - S < 0\}$	0.99999	0.96282	0.93732	0.96359	0.96262
Function evaluations	244	169	21	21	10^6

Figures

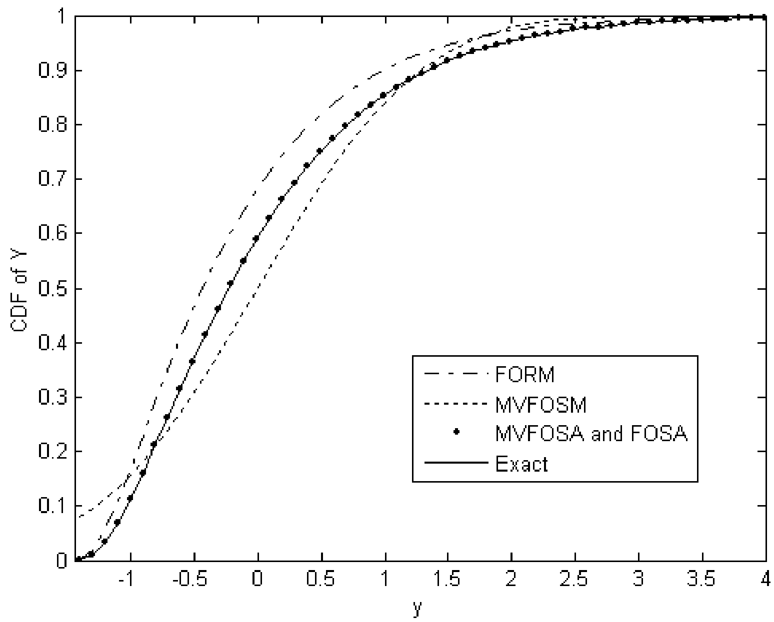

Fig. 1. Comparison of the CDF of *Y* from different methods.

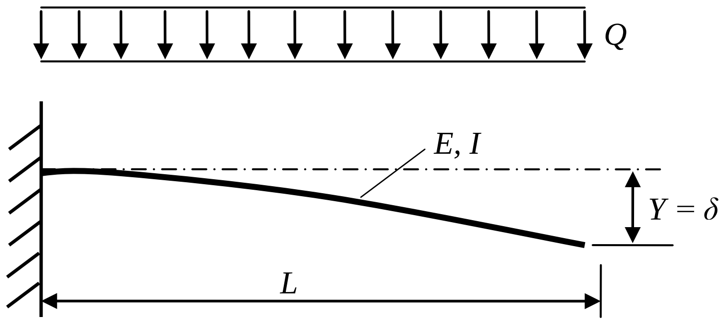

Fig. 2. A cantilever beam.

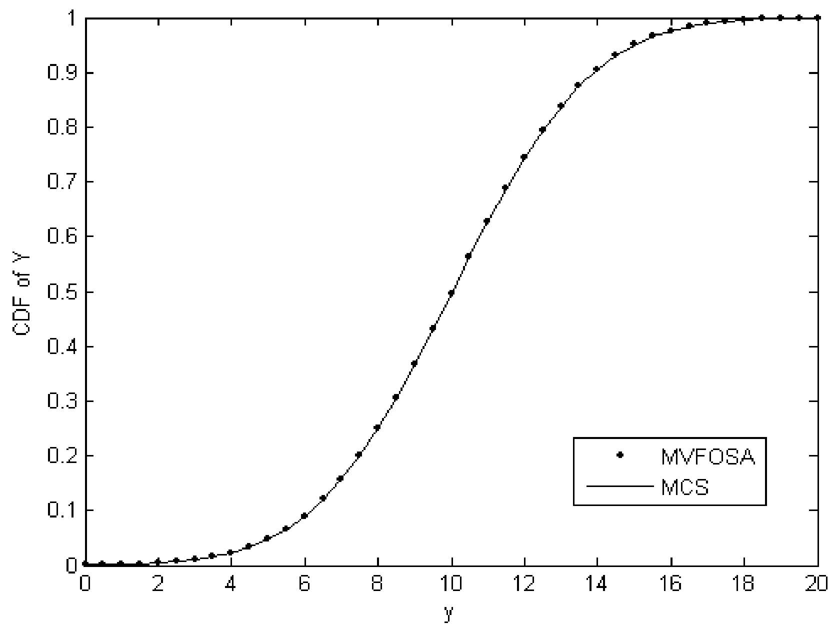

Fig. 3. CDF of *Y* (Example 2).

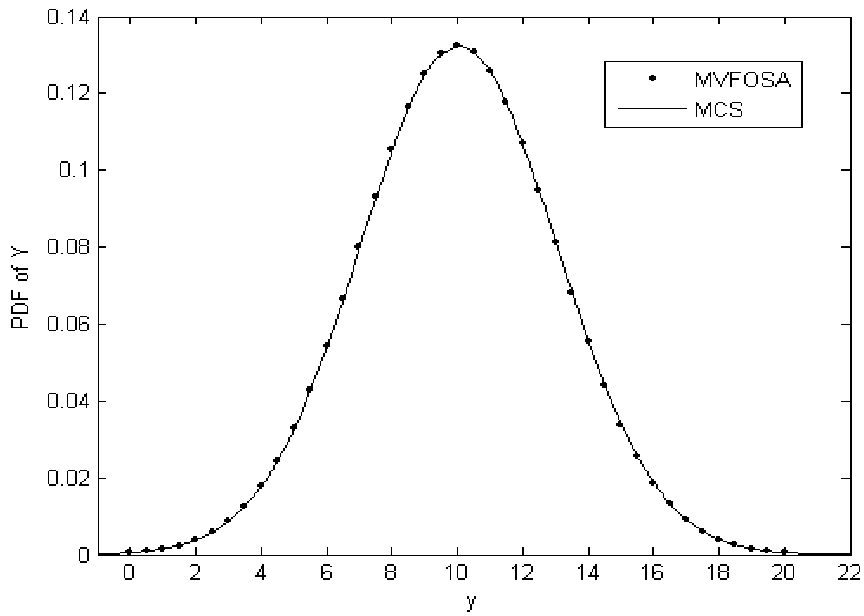

Fig. 4. PDF of *Y* (Example 2).

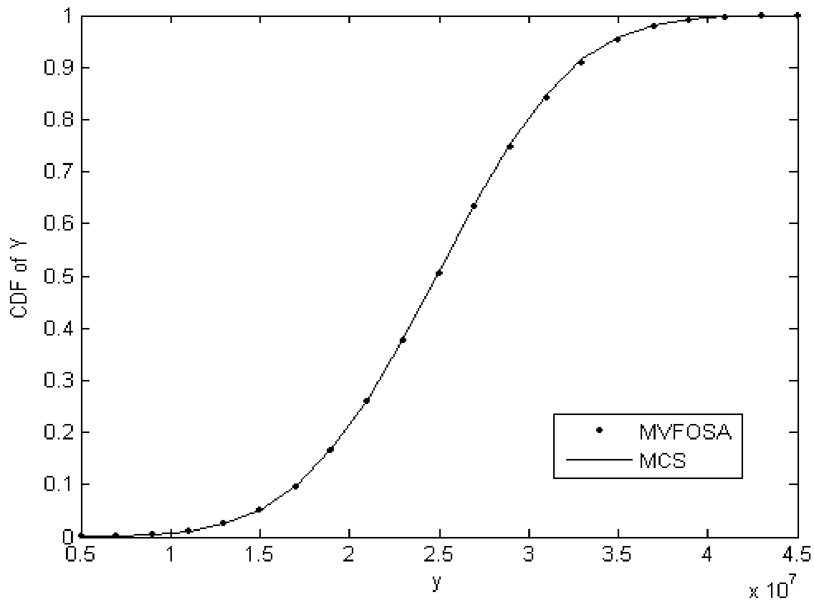

Fig. 5. CDF of *Y* (Example 3).

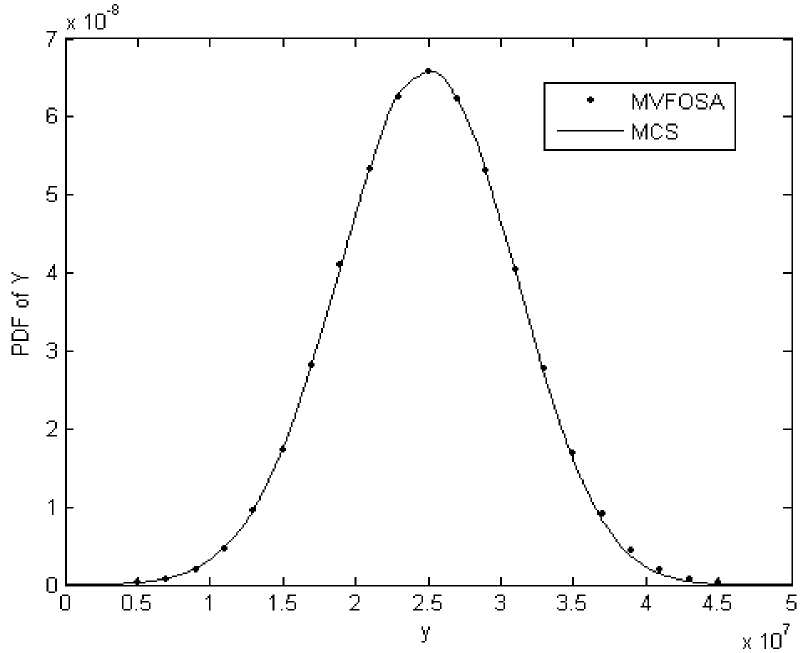

Fig. 6. PDF of *Y* (Example 3).

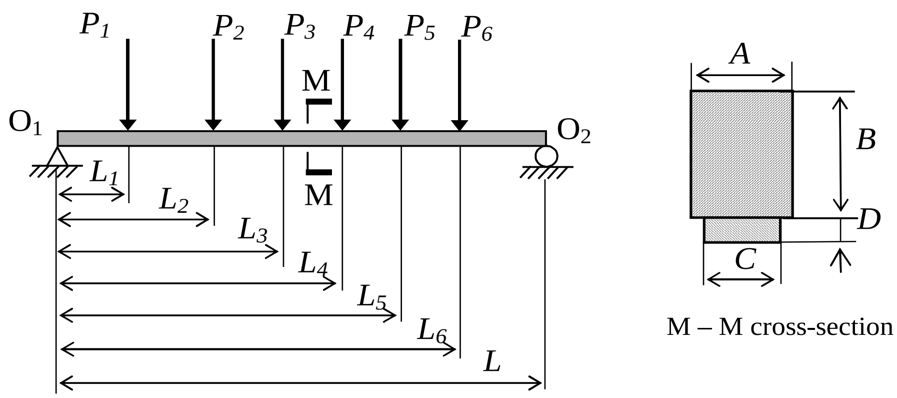

Fig. 7. A composite beam.

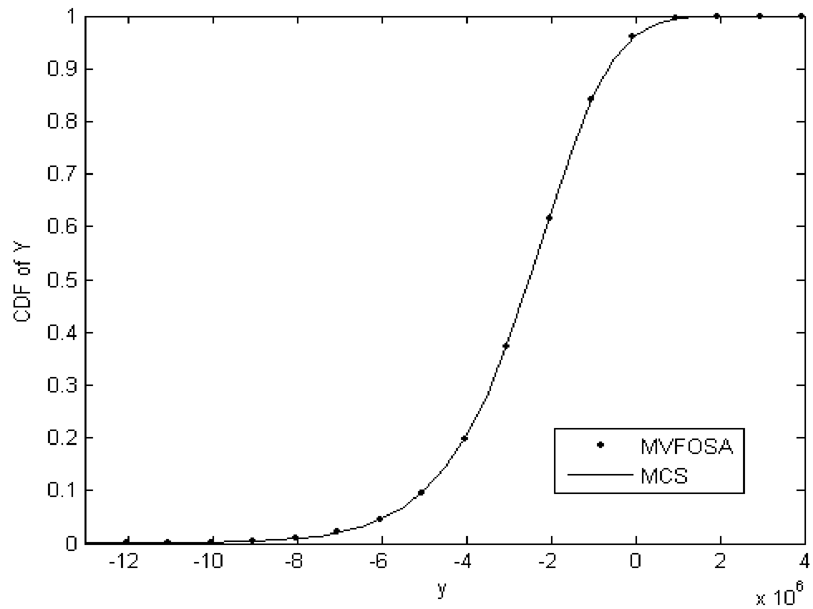

Fig. 8. CDF of Y (Example 4).

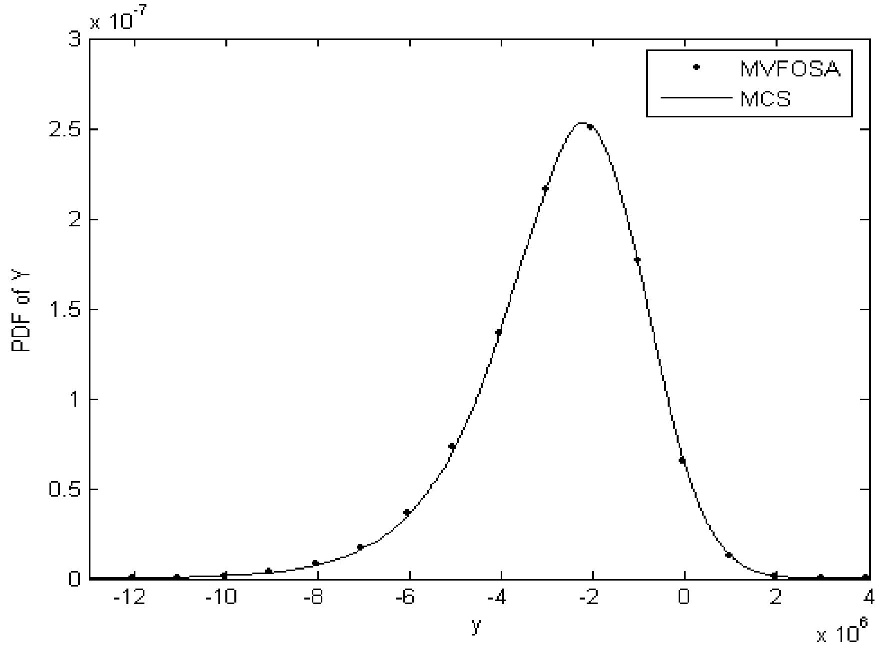

Fig. 9. PDF of *Y* (Example 4).