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Sequential Optimization and Reliability Assessment for Multidisciplinary Systems Design

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Abstract

With higher reliability and safety requirements, reliability-based design has been increasingly applied in multidisciplinary design optimization (MDO). A direct integration of reliability-based design and MDO may present tremendous implementation and numerical difficulties. In this work, a methodology of Sequential Optimization and Reliability Assessment for MDO is proposed to improve the efficiency of reliability-based MDO. The central idea is to decouple the reliability analysis from MDO with sequential cycles of reliability analysis and deterministic MDO. The reliability analysis is based on the First Order Reliability Method (FORM). In the proposed method, the reliability analysis and deterministic MDO use two MDO strategies, Multidisciplinary Feasible Approach and Individual Disciplinary Feasible Approach. The effectiveness of the proposed method is illustrated with two example problems.

Key words: reliability-based design, multidisciplinary design optimization, reliability analysis, Most Probable Point

1 Introduction

Combined with optimization, the model based design enables engineers to identify design options effectively and automatically. However, the traditional deterministic optimization design ignores the fact that in real life there are many sources of uncertainty, such as manufacturing variations and various customer usages (Batill, et al., 2000; Du and Chen, 2000a; Du and Chen, 2000b) . Consequently, deterministic optimization designs may be too sensitive to the variation of system input (leading to quality loss), risky (high likelihood of undesired extreme events and low reliability), or uneconomic. For this reason, incorporating uncertainty in design has received increasing attention and applications, such as those found in automotive, civil, mechanical, and aerospace engineering.

The other reason of uncertainty consideration is that engineering systems have become increasingly sophisticated and that the occurrence of failure events may lead to higher catastrophic consequences. To this end, the expectation of higher reliability and lower environmental impact has become imperative. Reliability-based design (RBD) is a design method to meet this expectation. RBD seeks a design that has a probability of failure less than some acceptable (invariably small) value and therefore ensures that failure events be extremely unlikely. RBD has been used in engineering fields for several decades (Zang, et al., 2002). Most of RBD applications are for relatively simple systems where only one discipline is involved.

Multidisciplinary Design Optimization (MDO) (Balling and Sobieski 1994) has become a systematic approach to optimization of complex, coupled engineering systems. “Multidisciplinary” refers to the different aspects that must be included in designing a system. The design involves multiple interacting disciplines, such as those found in aircraft, spacecraft, automobiles, and industrial manufacturing applications. Numerous successful examples of MDO applications have been reported in many areas (Giesing, et al., 1998; Salas and Townsend, 1998), such as in aerospace engineering (Xiong et al., 2004; Laban, 2004; Maute and Allen, 2004), vehicle design (Kodiyalam, S. et al., 2004), coupled thermal-structural problem (Autio, 2001), and fluid-structure problem (Lund, et al., 2003)

To ensure the high reliability in complex systems design, techniques of reliability analysis and RBD under the MDO framework have been developed recently (Sues, et al. 1995; Sues and Cesare 2000; Koch, et al. 2000; Du and Chen, 2002; Padmanabhan and Batill, 2002a and 2002b; Padmanabhan et al., 2003; Agarwal et al, 2003; Ahn et al., 2004). In the work of Sues, et al. (Sues, et al. 1995), response surface models of system output are created at the system level to replace the computationally expensive simulation models. Using the response surface models, reliability analysis is conducted for MDO under uncertainty. The use of response surface models may be costly if high accuracy is required and a large number of variables are involves. A framework for reliability-based MDO (RBMDO) is proposed by Sues and Cesare (2000). In their work, reliability analysis is decoupled from optimization. Reliabilities are computed initially before the first execution of the optimization loop and then updated after the optimization loop is executed. To alleviate the computational burden, in the optimization loop, approximate forms of reliability constraints are used. To integrate the existing reliability analysis techniques with the MDO framework, a multi-stage, parallel implementation strategy of probabilistic design optimization is utilized by Koch, et al. (2000). Padmanabhan et al. (2003) demonstrate the use of Monte Carlo Simulation in MDO. The Concurrent Subsystem Optimization techniques have been used to search the Most Probable Point (MPP) (Padmanabhan and Batill, 2002a and 2002b). The collaborative reliability analysis has also been proposed, where the MPP and multidisciplinary analysis are conducted concurrently (Du and Chen, 2002). To reduce computational effort of performing RBD for multidisciplinary systems, Agarwal et al (2003) employ a decomposition approach, which uses the method of Simultaneous Analysis and Design (SAND) as an optimization drive within a single-level RBD strategy. Ahn et al. (2004) propose a new strategy named Sequential Approach on Reliability Analysis under Multidisciplinary Analysis Systems. In their approach, reliability analysis and multidisciplinary analysis are decomposed and arranged in a sequential manner, making a recursive loop. The integration of these efficient reliability analysis methods into the MDO framework can potentially improve the overall performance of RBMDO.

Engineering applications of RBMDO have been reported (Xiao et al, 1999; Pettit and Grandhi, 2000; Sues et al, 2001; Hirohata et al, 2004). Xiao et al(1999) apply a framework of RBD for an aircraft wing design with the maximum cruise range, involving coupled aero-structural analysis. Pettit and Grandhi (2000) show that the RBMDO solution provides an optimum design, which is improved over the deterministic design in terms of robustness and reliability. Sues et al (2001) apply the framework of RBMDO for a full-scale wing design. Another example is the research conducted by Hirohata et al. (2004). In their work, a RBMDO is performed for CPU module packaging in order to identify the reliability relationship between packaging solutions and the statistical trade-off mechanism among multi-objectives. It is verified that the applied method can assist in the selection of a suitable packaging solution for the required specifications and reliability.

In all of the existing RBMDO frameworks, most computations are consumed on reliability analysis during the optimization process. The efficiency of reliability analysis dominates the overall efficiency of the entire design process. Since reliability analysis is usually conducted at the system-level multidisciplinary analysis, double loops or triple loops of iterative computations will be involved. As a result, RBMDO becomes much less affordable compared to deterministic MDO. In this work, we propose a sequential optimization and reliability assessment method for MDO to improve the efficiency.

This paper is organized as follows. The background of reliability analysis, reliability-based design, and multidisciplinary design optimization is presented in Section 2. RBMDO is reviewed in section 3. In Section 4, the proposed methodology, the sequential optimization and reliability assessment for multidisciplinary systems design, is discussed in detail. Examples are given in Section 5, followed by the conclusions in Section 6.

2 Review of Reliability-Based Design

In the process of RBD, reliability analysis is called repeatedly to evaluate the reliability of each of the probabilistic constraints at every design point generated by the optimizer. Reliability analysis is therefore a critical component of RBD. Next, a brief review of the reliability analysis is provided first followed by the review of RBD.

2.1 Reliability Analysis

Reliability is calculated by the following multidimensional integral

$$R = \Pr\{G(\mathbf{x}) < 0\} = \int_{G(\mathbf{x})<0} f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x}, \tag{1}$$

where $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ is a vector of independent random variables, $f_{\mathbf{x}}(\mathbf{x})$ is the joint probability density function of **x,** and $G(\mathbf{x})$ is a performance function.

The performance function $G(\mathbf{x}) = 0$ divides the random variable space into two regions, namely, the safe region where $G(\mathbf{x}) < 0$ and the failure region where $G(\mathbf{x}) > 0$.

Because it is difficult to obtain an analytical solution to the probability integration in (1), approximation methods, such as the First Order Reliability Method (FORM) (Hasofer and Lind, 1974) and the Second Order Reliability Method (SORM) (Breitung, 1984), are often the method of the choice. The procedure of the two methods is as follows.

At first, the original random variables $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ (in *x*-space) are transformed into a set of random variables $\mathbf{u} = \{u_1, u_2, \cdots, u_n\}$ (in *u*-space) whose elements follow a standard normal distribution. The transformation form **x** to **u** is based on the condition that the cumulative distribution functions (CDF) of the random variables remain the same before and after the transformation. This type of transformation is called Rosenblatt transformation (Rosenblatt, 1952), which is expressed by

$$F_{X_i}(x_i) = \Phi(u_i),\ i = 1, 2, \cdots, n \tag{2}$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution.

The transformed standard normal variable is then given by

$$\mathbf{u}_i = \Phi^{-1}\left[F_{X_i}(\mathbf{x}_i)\right] \tag{3}$$

For example, for a normally distributed random variable $\mathbf{x}_i \sim N(\mu_i, \sigma_i)$, Eq. 3 yields

$$\mathbf{u}_i = \Phi^{-1}\left[F_{X_i}(\mathbf{x}_i)\right] = \Phi^{-1}\left[\Phi\left(\frac{\mathbf{x}_i - \mu_i}{\sigma_i}\right)\right] = \frac{\mathbf{x}_i - \mu_i}{\sigma_i} \tag{4}$$

or

$$\mathbf{x}_i = \mu_i + \sigma_i \mathbf{u}_i \tag{5}$$

Next, the performance function $G(\mathbf{u})$ is approximated with a linear form (in FORM) or a quadratic form (in SORM) at the so-called Most Probable Point (MPP). After the two-step simplification and approximation, the probability integration in (1) can be solved analytically. To reduce the accuracy loss, the expansion point is selected at the MPP, which has the highest contribution to the probability integration. Maximizing the joint probability density function (PDF) of the random variables on the hyper surface of the integration region $G(\mathbf{u})=0$ in *u*-space results in the following optimization model for locating the MPP.

$$\begin{aligned}&\min_{\mathbf{u}}\|\mathbf{u}\|\\&\text{s.t. } G(\mathbf{u})=0\end{aligned}\tag{6}$$

where $\|\mathbf{u}\|$ stands for the magnitude of a vector.

After the solution MPP $\mathbf{u}^*$ is identified, the reliability is simply computed in FORM as

$$R=\Phi(\beta)\tag{7}$$

where $\beta=\|\mathbf{u}^*\|$ is the shortest distance from the surface $G(\mathbf{u})=0$ to the origin in *u*-space and is called *reliability index*. The formulation of the second order reliability method (SORM) can be found in (Breitung, 1984).

Besides using the direct reliability evaluation as shown in (6) and (7), inverse methods have also been proposed to assess the reliability, such as inverse reliability strategy (Li and Foschi, 1998; Tu, et al., 1999; Choi and Youn, 2001; Du et al., 2004), safety-factor strategy (Wu, et al, 2001), and probabilistic sufficiency factor approach (Qu and Haftka, 2004). In this work, we use the inverse reliability strategy to model a RBD problem because the inverse reliability strategy is more efficient than a direct reliability evaluation. In using the inverse reliability strategy, the percentile value of the performance function is calculated, the percentile G^R is the value that corresponds to a given reliability $R=\Phi(\beta)$, namely,

$$\Pr\{G(\mathbf{x})<G^R\}=R\,.\tag{8}$$

If FORM is used, the MPP for the inverse reliability problem is identified by the following model,

$$\begin{aligned} &\max_{\mathbf{u}} G(\mathbf{u}) \\ &\text{s.t. } \|\mathbf{u}\| = \beta \end{aligned} \tag{9}$$

Then G^R is the function value calculated at the MPP $\mathbf{u}^*$ by

$$G^R = G(\mathbf{u}^*) \ . \tag{10}$$

2.2 Reliability-Based Design

Reliability-based design (RBD) ensures the reliability higher than an acceptable level. The following typical RBD model formulates the trade-off between a higher reliability and a lower cost:

$$\begin{cases} \min_{\mathbf{d}} \ v(\mathbf{d}, \mathbf{x}) \\ \text{s.t. } \Pr\{G_i(\mathbf{d}, \mathbf{x}) < 0\} > R_i, \ i = 1, 2, \cdots, n_G \\ \quad g_j(\mathbf{d}, \mathbf{x}) < 0, \ j = 1, 2, \cdots, n_g \end{cases} \tag{11}$$

where v is a cost-type objective function; $\mathbf{d}$ is the vector of deterministic design variables; $\mathbf{x}$ is the vector of random design variables; $G_i(\mathbf{d}, \mathbf{x})$ is a constraint function that is subject to the reliability requirement; R_i is the required reliability for $G_i(\mathbf{d}, \mathbf{x})$; $g_j(\mathbf{d}, \mathbf{x})$ is a deterministic constraint function; n_G is the number of $G_i(\mathbf{d}, \mathbf{x})$; n_g is the number of $g_j(\mathbf{d}, \mathbf{x})$.

The conventional approach for solving a RBD problem is to employ a double-loop strategy. The optimization loop (outer loop) calls the reliability analysis (inner loop) repeatedly as shown in Fig. 1.

Insert Fig.1 here

Fig. 1 A double-loop procedure of RBD

As the double-loop strategy is computationally expensive, various techniques have been developed to improve efficiency. These techniques can be classified into two categories: one is to improve the efficiency of reliability analysis (Du and Chen, 2001; Choi, and Youn 2003), and the other is to formulate the design problem in such a way that the RBD problem can be solved efficiently. The latter includes (1) the use of inverse reliability formulation to reduce the computational demand of reliability analysis (Li and Foschi, 1998; Tu, et al., 1999; Choi and Youn, 2001) and (2) sequential single-loop and single-loop procedures (Chen and Hasselman, 1997; Wu and Wang, 1998; Wu, et al., 2001; Du and Chen, 2004; Du and Sudjianto, 2004; Du, et al., 2004; Qu and Haftka, 2004; Patel, et al. 2005).

3 Reliability-Based Multidisciplinary Design Optimization

In this section, we discuss the general RBMDO problems and models. We use a three-discipline system as an example for discussion. The extension of the discussion to a general multidisciplinary system will be obvious. A system with three coupled disciplines (subsystems) is shown in Fig. 2. The notations in the figure are explained below.

Insert Fig.2 here

Fig. 2 Multidisciplinary systems with random parameters

$\mathbf{d}_s$: shared design variables, which are common design variables of all disciplines

$\mathbf{d}_i$: local design variables of discipline *i*

$\mathbf{x}_s$: shared random parameters, which are common input variables to all disciplines

$\mathbf{x}_i$: local random input variables to discipline *i*

$\mathbf{y}_{ij}$: coupling variables, which are output of discipline *i* and the input of discipline *j*

$\mathbf{z}_i$: output of discipline i

The complete set of output coupling variables from discipline i is expressed by

$$\mathbf{y}_{i\cdot} = (\mathbf{y}_{i\cdot},\ j = 1,2,3,\ j \neq i) = \mathbf{y}_{i\cdot}(\mathbf{d}_s, \mathbf{d}_i, \mathbf{x}_s, \mathbf{x}_i, \mathbf{y}_{\cdot i}), \tag{12}$$

where $\mathbf{y}_{i\cdot}$ represents dependent variables as on the left-hand side of (12) and also the functional relationships between dependent variables and independent variables. We use the same way for other dependent variables in the rest of the paper. $\mathbf{y}_{\cdot i}$ in (12) is the vector of coupling variables, which are the inputs to discipline i and the outputs from other disciplines, i.e.

$$\mathbf{y}_{\cdot i} = \left\{\mathbf{y}_{ji},\ j = 1,2,3,\ j \neq i\right\}. \tag{13}$$

Expanding (12) over all disciplines, we obtain the following simultaneous equations that determine the consistency over the interface among the disciplines.

$$\begin{aligned}
\mathbf{y}_{12} &= \mathrm{y}_{12}(\mathbf{d}_s, \mathbf{d}_1, \mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{\cdot 1}) \\
\mathbf{y}_{13} &= \mathrm{y}_{13}(\mathbf{d}_s, \mathbf{d}_1, \mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{\cdot 1}) \\
\mathbf{y}_{21} &= \mathrm{y}_{21}(\mathbf{d}_s, \mathbf{d}_2, \mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_{\cdot 2}) \\
\mathbf{y}_{23} &= \mathrm{y}_{23}(\mathbf{d}_s, \mathbf{d}_2, \mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_{\cdot 2}) \\
\mathbf{y}_{31} &= \mathrm{y}_{31}(\mathbf{d}_s, \mathbf{d}_3, \mathbf{x}_s, \mathbf{x}_3, \mathbf{y}_{\cdot 3}) \\
\mathbf{y}_{32} &= \mathrm{y}_{32}(\mathbf{d}_s, \mathbf{d}_3, \mathbf{x}_s, \mathbf{x}_3, \mathbf{y}_{\cdot 3})
\end{aligned} \tag{14}$$

The output $\mathbf{z}_i$ consists of three parts as follows:

$$\mathbf{z}_i = (\mathbf{v}_i, \mathbf{g}_i, \mathbf{G}_i) \tag{15}$$

where $\mathbf{v}_i$ are part of the system level objective functions, $\mathbf{g}_i$ are local deterministic constraint functions, and $\mathbf{G}_i$ are local constraint functions subject to reliability requirements. They can be expressed in subsystems as

$$\begin{aligned}
&\mathbf{v}_i = \mathbf{v}_i(\mathbf{d}_s, \mathbf{d}_i, \mathbf{x}_s, \mathbf{x}_i, \mathbf{y}_{\cdot i}) \\
&\mathbf{g}_i = \mathbf{g}_i(\mathbf{d}_s, \mathbf{d}_i, \mathbf{x}_s, \mathbf{x}_i, \mathbf{y}_{\cdot i}) \\
&\mathbf{G}_i = \mathbf{G}_i(\mathbf{d}_s, \mathbf{d}_i, \mathbf{x}_s, \mathbf{x}_i, \mathbf{y}_{\cdot i})
\end{aligned} \tag{16}$$

We can solve (16) and obtain the outputs $(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3)$ given a set of inputs $(\mathbf{d}_s, \mathbf{x}_s, \mathbf{d}_1, \mathbf{x}_1, \mathbf{d}_2, \mathbf{x}_2, \mathbf{d}_3, \mathbf{x}_3)$. This is the task of multidisciplinary analysis (MDA). As shown in Fig.2, the major difficulty in dealing with uncertainty is that *a reliability analysis in one discipline will need to consider uncertainties propagated from other disciplines* due to the data flow among disciplines (Batill, et al., 2000).

The general RBD model under the multidisciplinary environment is given by

$$\begin{aligned}
&\min_{(\mathbf{d}_s, \mathbf{d})} \ v(\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3) \\
&\text{s. t.} \quad \Pr\{\mathbf{G}_i(\mathbf{d}_s, \mathbf{d}_i, \mathbf{x}_s, \mathbf{x}_i, \mathbf{y}_{\cdot i}) \le 0\} > \mathbf{R}_i, i = 1,2,3 \\
&\qquad \mathbf{g}_j(\mathbf{d}_s, \mathbf{d}_i, \mathbf{x}_s, \mathbf{x}_i, \mathbf{y}_{\cdot i}) \le 0, \ j = 1,2,3
\end{aligned} \tag{17}$$

In the above optimization model, $\mathbf{d} = (\mathbf{d_1}, \mathbf{d_2}, \mathbf{d_3})$ is the vector of local design variables; the objective function v is the function of the nominal values of the individual disciplinary outputs $(\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3)$. In this paper, $\bar{\bullet}$ denotes a nominal value of a response variable and a mean value of a basic input random variable. $\bar{\mathbf{v}}_i$ is given by

$$\bar{\mathrm{v}}_i = \mathbf{v}_i(\mathbf{d}_s, \mathbf{d}_i, \bar{\mathbf{x}}_s, \bar{\mathbf{x}}_i, \bar{\mathbf{y}}_{\cdot i}). \tag{18}$$

The nominal values of coupling variables $\bar{\mathbf{y}}_{\cdot i}$ are determined by

$$\bar{\mathbf{y}}_{\cdot i} = (\bar{\mathbf{y}}_{ji}) = \mathbf{y}_{ji}(\mathbf{d}_s, \mathbf{d}_i, \bar{\mathbf{x}}_s, \bar{\mathbf{x}}_i, \bar{\mathbf{y}}_{\bullet j}), i = 1,2,3; \ j = 1,2,3; i \ne j \tag{19}$$

where $\bar{\mathbf{y}}_{\bullet j}$ is the vector of coupling variables passed into the jth discipline from all the other disciplines.

As demonstrated in Fig. 3, solving a RBMDO problem given in the above optimization model involves a triple loop procedure, where the outer loop is the overall optimization, the middle loop is reliability analysis, and the inner loop is the multidisciplinary analysis (MDA). Under this framework, the number of reliability analyses is equal to the number of function evaluations consumed by the overall optimization. Reliability analysis also needs to call MDA repeatedly. The total number of disciplinary analyses is therefore very high. In this work, we introduce the Sequential Optimization and Reliability Assessment (SORA) to solve a general RBMDO problem efficiently.

Insert Fig.3 here

Fig. 3 Reliability-based MDO

4 Sequential Optimization and Reliability Assessment under Multidisciplinary Design Environment

4.1 The Strategy

The focus of this work is to model and solve RBMDO problems efficiently. We propose to achieve high efficiency in the following two aspects.

(1) *Model a RBMDO problem that the problem can be solved efficiently*.

The sequential single-loop methods (Chen and Hasselman, 1997; Wu and Wang, 1998; Wu, et al., 2001; Du and Chen, 2004; Qu and Haftka, 2004; Patel, et al. 2005) have been successfully applied to single-disciplinary RBD. We use the same strategy for RBD under a multidisciplinary environment. The central idea is to employ a series of cycles of MDO and reliability analysis. In each cycle MDO and reliability analysis are decoupled from each other; reliability analysis is only conducted after the MDO. The idea is outlined in Fig. 4. It is seen that through this procedure, the number of reliability analyses is equal to the number of cycles. The design is likely to converge in a few cycles, and therefore the computational efficiency will be much higher than a procedure where reliability analysis is applied directly with MDO.

Insert Fig.4 here

Fig. 4 The Procedure of SORA

(2) *Perform reliability analysis only up to the necessary level*

The inverse reliability strategy (percentile performance) is used because evaluating a percentile performance is more efficient than evaluating an actual reliability (Tu, et al., 1999; Du and Chen, 2002a). Since the system failure modes are correlated due to the sharing factors (materials, dimensions, and loads), if the reliability constraints of some failure modes are satisfied, other reliability constraints may never be active. However, those never-active reliability constraints may unfortunately dominate the computational effort in the RBD process (Murotsu, et al., 1994). To this end, we seek a procedure which *evaluates a percentile performance only up to a needed level*. The optimization model with percentile performance formulation is discussed in the next section.

4.2 The Procedure

As mentioned previously (Fig. 4), the overall optimization is conducted with sequential cycles of optimization and reliability analysis where the deterministic MDO is followed by the reliability analysis. In each cycle, four steps are involved and are shown as follows.

Step 1: Solve the deterministic MDO. In the first cycle, the optimization is the conventional MDO without any uncertainty. From the second cycle, based on the information obtained from the previous cycle, the deterministic MDO is formulated in such a way that the optimality and reliability requirement will be gradually achieved.

Step 2: Perform reliability analysis. At the optimal point obtained in Step 1, the MPPs of all the reliability constraints are identified, and the percentile performance values corresponding to required reliabilities are calculated.

Step 3: Check the convergence. If reliability requirements are satisfied and the system objective function becomes stable, the entire optimization process stops; otherwise, proceed to Step 4.

Step 4: Formulate a new deterministic MDO model for the next cycle based on the MPP information from Step 3.

The percentile formulation of the MDO problem in Cycle k is given by

$$\begin{aligned} &\min_{(\mathbf{d}_s,\mathbf{d})} \ v(\bar{\mathbf{v}}_1,\bar{\mathbf{v}}_2,\bar{\mathbf{v}}_3) \\ &\text{s. t. } \mathbf{G}_i(\mathbf{d}_s,\mathbf{d}_i,\mathbf{x}_s^{*,(i),k-1},\mathbf{x}_i^{*,(i),k-1},\mathbf{y}_{\cdot i}^{*,(i),k-1}) \le 0, i=1,2,3 \\ &\qquad \mathbf{g}_i(\mathbf{d}_s,\mathbf{d}_i,\bar{\mathbf{x}}_s,\bar{\mathbf{x}}_i,\bar{\mathbf{y}}_{\cdot i}) \le 0, i=1,2,3 \end{aligned} \tag{20}$$

where $\mathbf{x}_s^{*,(i),k-1}$, $\mathbf{x}_i^{*,(i),k-1}$ and $\mathbf{y}_{\cdot i}^{*,(i),k-1}$ are the components of the MPP of $\mathbf{x}_s$, $\mathbf{x}_i$ and $\mathbf{y}_{\cdot i}$ obtained from reliability analysis in the ith discipline in Cycle k-1, and $\mathbf{y}_{\cdot i}^{*,(i),k-1}$ is the vector of coupling variables at the MPP.

Next, we will discuss how to perform the deterministic MDO and reliability analysis under the MDO environment.

4.3 Deterministic MDO

The task of the deterministic MDO is to solve the optimization model specified in (20). The SORA strategy discussed in the last subsection is applicable to any MDO schemes. Herein we employ two methods to perform the deterministic MDO: Individual Disciplinary Feasible approach (IDF) and Multidisciplinary Feasible approach (MDF) (Cramer, et al, 1994; Allison, et al, 2005). The former accommodates the coupling variables as part of design variables and includes system consistency as part of system constraints. The latter excludes the coupling variables from the design variables, and the coupling variables are solved out by the system level analyses.

4.3.1 Individual Disciplinary Feasible Approach (*IDF*)

In IDF, the deterministic MDO model in *Cycle k* is given by

$$\begin{aligned}
&\min_{(\mathbf{d}_s,\mathbf{d},\bar{\mathbf{y}},\mathbf{y}^*)} \ v(\bar{\mathbf{v}}_1,\bar{\mathbf{v}}_2,\bar{\mathbf{v}}_3) \\
&\text{s. t.}\ \ \mathbf{G}_i(\mathbf{d}_s,\mathbf{d}_i,\mathbf{x}_s^{*,(i),k-1},\mathbf{x}_i^{*,(i),k-1},\mathbf{y}_{\cdot i}^{*,(i),k-1}) \le 0, i=1,2,3 \\
&\qquad \mathbf{g}_i(\mathbf{d}_s,\mathbf{d}_i,\bar{\mathbf{x}}_s,\bar{\mathbf{x}}_i,\bar{\mathbf{y}}_{\cdot i}) \le 0, i=1,2,3 \\
&\qquad \mathbf{h}_{\mathbf{aux}1}(\mathbf{d},\mathbf{x},\bar{\mathbf{y}}) = \bar{\mathbf{y}}_{ij} - \mathbf{y}_{ij}(\mathbf{d}_s,\mathbf{d}_i,\bar{\mathbf{x}}_s,\bar{\mathbf{x}}_i,\bar{\mathbf{y}}_{\cdot i}) = 0, i,j=1,2,3, i\ne j \\
&\qquad \mathbf{h}_{\mathbf{aux}2}(\mathbf{d},\mathbf{x},\mathbf{y}^*) = \mathbf{y}_{jm}^{*,k-1} - \mathbf{y}_{jm}(\mathbf{d}_s,\mathbf{d}_j,\mathbf{x}_s^{*,(i),k-1},\mathbf{x}_j^{*,(i),k-1},\mathbf{y}_{\cdot j}^{*,k-1}) = 0, i,j,m=1,2,3, j\ne m
\end{aligned} \tag{21}$$

where $\mathbf{x}_s^{*,(i),k-1}$, $\mathbf{x}_i^{*,(i),k-1}$ and $\mathbf{y}_{\cdot i}^{*,(i),k-1}$ are the components of the MPP (see the explanation for Eq. 20).

In the above model, the design variables also include the nominal values of all the coupling variables, which are given by

$$\bar{\mathbf{y}} = \left\{ \bar{\mathbf{y}}_{ij},\ i=1,\ 2,\ 3, j=1,\ 2,\ 3,\ i\ne j \right\} \tag{22}$$

and all the coupling variables $\mathbf{y}^{*,k-1}$ at the MPPs of the reliability constraints, which are given by

$$\mathbf{y}_{jm}^{*,k-1} = (\mathbf{y}_{\cdot j}^{*,(1),k-1}, \mathbf{y}_{\cdot j}^{*,(2),k-1}, \mathbf{y}_{\cdot j}^{*,(3),k-1}) = 0,\ j, m = 1,2,3,\ j \neq m \tag{23}$$

where $\mathbf{y}_{jm}^{*,(i),k-1}$ is the vector of the coupling variables outputted from discipline j inputted to discipline m, corresponding to each of the reliability constraints in discipline i (i =1, 2, 3) at the MPPs of Cycle k-1.

$\mathbf{h}_{\mathbf{aux}1}(\mathbf{d},\mathbf{x},\bar{\mathbf{y}})$ are the system consistency constraint for coupling variables at the mean values of random variables, and $\mathbf{h}_{\mathbf{aux}2}(\mathbf{d}, \mathbf{x},\mathbf{y}^*)$ are the system consistency constraints for coupling variables at the MPPs.

In IDF approach, individual discipline feasibility is maintained at each optimization iteration. The coupling variables are treated as extra design variables, and interdisciplinary equilibrium is treated as constraints. The system consistency is then maintained at the convergence of optimization (Cramer, et al, 1994; Allison, et al, 2005).

4.3.2 Multidisciplinary Feasible Approach (*MDF*)

If many coupling variables and reliability constraint functions are involved, the number of design variables in IDF will become too large to handle and may cause severe difficulties in efficiency and convergence. In this case, MDF becomes an alternative method. MDF requires a double-loop procedure and only takes the original design variables without any additional design variables in its outer loop. The coupling variable are identified in its inner loop where the system consistency equations are solved out. The outer loop deterministic MDO model in Cycle k is given by

$$
\begin{aligned}
&\min_{(\mathbf{d}_s,\mathbf{d})}\; v(\overline{\mathbf{v}}_1,\overline{\mathbf{v}}_2,\overline{\mathbf{v}}_3)\\
&\text{s. t. } \mathbf{G}_i(\mathbf{d}_s,\mathbf{d}_i,\mathbf{x}_s^{*,(i),k-1},\mathbf{x}_i^{*,(i),k-1},\mathbf{y}_{\cdot i}^{*,k-1})\le 0, i=1,2,3\\
&\qquad \mathbf{g}_i(\mathbf{x}_s^d,\overline{\mathbf{x}}_s^r,\mathbf{x}_i^d,\overline{\mathbf{x}}_i^r,\overline{\mathbf{y}}_{\cdot i})\le 0, i=1,2,3
\end{aligned}
\tag{24}
$$

Since the coupling variables are not included in the outer loop optimization model, they have to be obtained from the inner loop system consistency equations. The inner loop multidisciplinary analysis (MDA) for solving coupling variables at mean values of random variables and the MPPs is given by the system of simultaneous equations

$$
\begin{aligned}
&\overline{\mathbf{y}}_{jm}=\mathbf{y}_{jm}(\mathbf{x}_s^d,\overline{\mathbf{x}}_s^r,\mathbf{x}_i^d,\overline{\mathbf{x}}_i^{r,},\overline{\mathbf{y}}_{\cdot i})\\
&\mathbf{y}_{jm}^{*,k-1}=\mathbf{y}_{jm}(\mathbf{x}_s^d,\mathbf{x}_s^{r,*,(i),k-1},\mathbf{x}_i^d,\mathbf{x}_i^{r,*,(i),k-1},\mathbf{y}_{\cdot j}^{*,k-1}),\; i,j,m=1,2,3,\; j\ne m
\end{aligned}
\tag{25}
$$

Depending on the problems, different iterative methods (Heinkenschloss, et al, 1998; Arian, 1997), such as Gauss-Seidel-type methods (Ortega and Rheinboldt, 1970), Jacobi method (Acton, F.S., 1990), Newton type methods (Dennis and Schnabel, 1996; Kelly, 1995), Broyden’s method (Dennis and Schnabel, 1996; Kelly, 1995) and Powell’s dogleg method (Powell, 1970), can be employed to solve this system of equations. In the examples of this paper, we use a variant of Powell’s dogleg method that is provided by Matlab.

4.4 Reliability Analysis under MDO Environment

Once the optimal solution is obtained from the deterministic MDO, the reliability analysis is conducted. Reliability analysis evaluates the percentile values of reliability constraint functions and checks whether the reliability requirement is met. It also provides the reliability information for building the deterministic MDO model for the next cycle if the reliability requirement is not satisfied.

Reliability analysis by FORM is essentially an MDO problem. Therefore, similar to the deterministic MDO, the same methods, IDF and MDF, can be used for reliability analysis.

4.4.1 Individual Disciplinary Feasible Approach

In IDF, the system consistency is also included in the optimization model for the MPP search. The design variables in the MPP search contain the shared random variables and disciplinary random variables, and all the random coupling variables as well. Since the system consistency is treated as constraints, there is no need to perform MDA as an inner loop. The optimization model for reliability constraint G_i is given by

$$\begin{aligned}
&\max_{(\mathbf{u}_s^{(i)},\mathbf{u}_i^{(i)},\mathbf{y}^{(i)})} \; G_i(\mathbf{d},\mathbf{u}_s^{(i)},\mathbf{u}^{(i)},\mathbf{y}_{\cdot i}^{(i)}) \\
&\quad \text{s. t.} \; \left\|(\mathbf{u}_s^{(i)},\mathbf{u}^{(i)})\right\| = \beta \\
&\qquad \mathbf{h}_{\mathbf{aux}}(\mathbf{d},\mathbf{u},\mathbf{y}) = \mathbf{y}_{jm}^{*,(i)} - \mathbf{y}_{jm}^{*,(i)}(\mathbf{d}_s,\mathbf{d}_j,\mathbf{u}_s^{(i)},\mathbf{u}^{(i)},\mathbf{y}_{\cdot j}^{(i)}) = 0, i, j, m = 1,2,3, j \neq m
\end{aligned} \tag{26}$$

where $\mathbf{u}^{(i)} = (\mathbf{u}_1^{(i)},\mathbf{u}_2^{(i)},\mathbf{u}_3^{(i)})$, and β is the reliability index, which is associated with the required reliability and can be calculated by (7).

The solution is the MPP in *u*-space $(\mathbf{u}_s^{*,(i)}, \mathbf{u}^{*,(i)})$ and the coupling variables $\mathbf{y}_{jm}^{*}$ at the MPP. After the MPP in *x*-space $(\mathbf{x}_s^{*,(i)}, \mathbf{x}^{*,(i)})$ is obtained from the transformation from $(\mathbf{u}_s^{*,(i)}, \mathbf{u}^{*,(i)})$, the percentile value of the constraint function is calculated at $(\mathbf{x}_s^{*,(i)}, \mathbf{x}^{*,(i)})$. $(\mathbf{x}_s^{*,(i)}, \mathbf{x}^{*,(i)})$ will also be used for formulating the deterministic MDO model for the next cycle, as shown in (21).

4.4.2 Multidisciplinary Feasible Approach

The design variables in MDF are the shared random variables and the disciplinary random variables in *u*-space. The optimization model of the outer loop is given by

$$\begin{aligned}
&\max_{(\mathbf{u}_s^{(i)},\mathbf{u}^{(i)})} \; G_i(\mathbf{d},\mathbf{u}_s^{(i)},\mathbf{u}^{(i)},\mathbf{y}_{\cdot i}^{(i)}) \\
&\quad \text{s. t.} \left\|(\mathbf{u}_s^{(i)},\mathbf{u}^{(i)})\right\| = \beta
\end{aligned}$$

The inner loop solves the coupling variables and is given by the following system of simultaneous equations

$$\mathbf{y}_{jm}^{*,(i)} = \mathbf{y}_{jm}^{*,(i)}(\mathbf{d}_s, \mathbf{d}_j, \mathbf{u}_s^{(i)}, \mathbf{u}^{(i)}, \mathbf{y}_{\cdot j}^{(i)}), i, j, m = 1,2,3, j \neq m$$

Similarly to the deterministic MDO, for the purpose of demonstration, the inner loop problem is solved by a variant of Powell's dogleg method.

5 Demonstrative Examples

In this section, two examples are used to demonstrate the proposed methods.

5.1 Example 1 – A Mathematical Problem

A conventional RBD problem is given by

$$\begin{aligned}
\min_{(d_s, d_1, d_2)} \quad & v(\mathbf{d}, \overline{\mathbf{x}}) = \left(d_s + \overline{x}_s\right)^2 + d_1^2 + d_2^2 \\
\text{s. t.} \quad & \Pr\left\{G_1(\mathbf{d}, \mathbf{x}) = x_1 - d_s - x_s - d_1 - d_2 \le 0\right\} \ge R_1 \\
& \Pr\left\{G_2(\mathbf{d}, \mathbf{x}) = d_s + x_s - 2d_1 + d_2 - x_2 \le 0\right\} \ge R_2 \\
& 0 \le d_s, d_1, d_2 \le 5
\end{aligned}$$

where $x_s \sim N(0, 0.3)$, $x_1 \sim N(5, 0.5)$, $x_2 \sim N(1, 0.1)$, $R_1 = R_2 = \Phi(\beta) = 0.9987$, $\beta = 3$. $N(\mu, \sigma)$ stands for a normal distribution with a mean of μ and standard deviation of σ .

The solution to this RBD problem is $\mathbf{d} = \left(d_s, d_1, d_2\right) = (2.2498, 2.2498, 2.2497)$ and $v = 15.1843$. SORA and double-loop method (Du and Chen, 2004) are used to solve the problem.

For demonstration, the problem is artificially decomposed into two subsystems (disciplines) and then formulated as a RBMDO problem as shown in Fig. 5.

Insert Fig.5 here

Fig. 5 Example 1

Method 1 - IDF

First, we use IDF to solve the problem. In each cycle, for the deterministic MDO, the MPP $\left(u_s^{*,(1)}, u_1^{*,(1)}, u_2^{*,(1)}\right)$ for G_1 and the MPP $\left(u_s^{*,(2)}, u_1^{*,(2)}, u_2^{*,(2)}\right)$ for G_2 are obtained from reliability analysis in the previous cycle. The deterministic MDO is formulated as

$$\begin{aligned}
&\min_{\mathbf{DV}} \; v = \left(\overline{v}_1 + \overline{v}_2\right) = (d_s + \overline{x}_s)^2 + d_1^2 + d_2^2 \\
&\text{s. t.} \;\; G_1 = x_1^{*,(1)} - \left(d_s + x_s^{*,(1)} + 2d_1 + 2y_{21}^{*,(1)}\right) \le 0 \\
&\qquad y_{12}^{*,(1)} = d_s + x_s^{*,(1)} + d_1 + y_{21}^{*,(1)} \\
&\qquad y_{21}^{*,(1)} = d_s + x_s^{*,(1)} + d_2 - y_{12}^{*,(1)} \\
&\qquad G_2 = 5d_s + 5x_1^{*,(2)} + 3d_2 - 4y_{12}^{*,(2)} - x_2^{*,(2)} \le 0 \\
&\qquad y_{12}^{*,(2)} = d_s + x_s^{*,(2)} + d_1 + y_{21}^{*,(2)} \\
&\qquad y_{21}^{*,(2)} = d_s + x_s^{*,(2)} + d_2 - y_{12}^{*,(2)}
\end{aligned}$$

The design variables **DV** = $\left(d_s, d_1, d_2;\, \overline{y}_{12}, \overline{y}_{21};\, y_{12}^{*,(1)}, y_{21}^{*,(1)};\, y_{12}^{*,(2)}, y_{21}^{*,(2)}\right)$.

The optimal point $\left(d_s, d_1, d_2\right)$ from the above MDO model is then used in reliability analysis. The model for the MPP search for G_1 is given by

$$\begin{aligned}
&\max_{\mathbf{DV}} \; G_1 = x_1^{(1)} - \left(d_s + x_s^{(1)} + 2d_1 + 2y_{21}^{(1)}\right) \\
&\text{s. t.} \;\; \sqrt{\left(u_s^{(1)}\right)^2 + \left(u_1^{(1)}\right)^2 + \left(u_2^{(1)}\right)^2} = \beta \\
&\qquad y_{12}^{(1)} = d_s + \left(\mu_s + u_s^{(1)}\sigma_s\right) + d_1 + y_{21}^{(1)} \\
&\qquad y_{21}^{(1)} = d_s + \left(\mu_s + u_s^{(1)}\sigma_s\right) + d_2 - y_{12}^{(1)}
\end{aligned}$$

The design variables **DV** = $\left(u_s^{(1)}, u_1^{(1)}, u_2^{(1)};\, y_{12}^{(1)}, y_{21}^{(1)}\right)$.

The solution, MPP $(u_s^{*,(1)}, u_1^{*,(1)}, u_2^{*,(1)})$, is then transformed into x-space, namely,

$$\begin{aligned}
x_s^{*,(1)} &= \mu_s + u_s^{*,(1)}\sigma_s \\
x_1^{*,(1)} &= \mu_1 + u_1^{*,(1)}\sigma_1
\end{aligned}$$

$$x_2^{*,(1)} = \mu_2 + u_2^{*,(1)}\sigma_2$$

The model of reliability analysis for G_2 can be derived in the same way, and the solution is the MPP in x-space $(x_s^{*,(2)}, x_1^{*,(2)}, x_2^{*,(2)})$.

The optimal solution from IDF is $\mathbf{d} = \left(d_s, d_1, d_2\right) =$ (2.2498, 2.2498, 2.2498) and $v = 15.1843$, which is identical to the solution of the conventional RBD problem.

Method 2 - MDF

The problem is also solved by MDF. The outer loop deterministic MDO is then formulated as

$$\begin{aligned}
\min_{(d_s, d_1, d_2)} \quad & v = \left(\bar{v}_1 + \bar{v}_2\right) = (d_s + \bar{x}_s)^2 + d_1^2 + d_2^2 \\
\text{s. t.} \quad & G_1 = x_1^{*,(1)} - \left(d_s + x_s^{*,(1)} + 2d_1 + 2y_{21}^{(1)}\right) \le 0 \\
& G_2 = 5d_s + 5x_1^{*,(2)} + 3d_2 - 4y_{12}^{(2)} - x_2^{*,(2)} \le 0
\end{aligned}$$

The following inner loop is used for solving coupling variables.

$$\begin{aligned}
y_{12}^{(1)} &= d_s + x_s^{(1)} + d_1 + y_{21}^{(1)} \\
y_{21}^{(1)} &= d_s + x_s^{(1)} + d_2 - y_{12}^{(1)} \\
y_{12}^{(2)} &= d_s + x_s^{(2)} + d_1 + y_{21}^{(2)} \\
y_{21}^{(2)} &= d_s + x_s^{(2)} + d_2 - y_{12}^{(2)}
\end{aligned}$$

For reliability analysis, the deterministic design variables $\left(d_s, d_1, d_2\right)$ are given by the above deterministic MDO. The outer loop of the MPP search for G_1 is given by

$$\max_{(u_s, u_1, u_2)} G_1 = \left(\mu_1 + u_1^{(1)}\sigma_1\right) - \left[\ d_s + (\mu_1 + u_1^{(1)}\sigma_1) + 2d_1 + 2y_{21}^{(1)}\ \right]$$

$$\text{s. t.} \ \sqrt{\left(u_s^{(1)}\right)^2 + \left(u_1^{(1)}\right)^2 + \left(u_2^{(1)}\right)^2} = \beta$$

The following inner loop is embedded in the outer loop for solving the coupling variables.

$$
\begin{aligned}
y_{12}^{(1)} &= d_s + \left(\mu_s + u_s^{(1)}\sigma_s\right) + d_1 + y_{21}^{(1)} \\
y_{21}^{(1)} &= d_s + \left(\mu_s + u_s^{(1)}\sigma_s\right) + d_2 - y_{12}^{(1)}
\end{aligned}
$$

The reliability analysis model for constraint G_2 can be derived in the same way.

The identical result is obtained from MDF. To verify the accuracy of the proposed methods, the problem is also solved by the conventional RBMDO method. By the conventional RBMDO, we mean that the traditional RBD method is directly applied to the MDO framework. As mentioned in the introduction section, this direct application results in an expensive triple-loop procedure. The procedure is outlined in Fig. 3 and detailed in Fig. 6.

Insert Fig.6 here

Fig. 6 Conventional RBD for MDO

Table 1 Optimal results

Problem	Method	$(d_s,\ d_1,\ d_2)$	v	G_1 at MPP	G_2 at MPP	n_1	n_2
Original	Sequential Single-loops	(2.252,2.252,2.252)	15.2144	0.0	-0.047695	695	
	Double loop	(2.2519,2.252,2.252)	15.2144	0.0	-0.047755	1992	
MDO	Conventional	(2.25,2.2494,2.2499)	15.1843	0.0	-0.050212	186600	186600
	IDF	(2.2497,2.2498,2.2498)	15.1843	0.0	-0.051335	451	635
	MDF	(2.2498,2.2498,2.2498)	15.1843	0.0	-0.051316	1836	1836

The results of the original single-disciplinary and MDO problems from all the methods are summarized in Table 1. The efficiency of each method, however, is quite different. If we use the number of disciplinary analyses to measure the efficiency, for this example, IDF is more efficient than MDF since the former uses fewer disciplinary analyses than the latter. It is also noted that the percentile value of G_1 is zero at the optimal point. This means that the G_1 is active and the reliability of G_1 is exactly the same as the required reliability. The percentile value of G_2 is less than zero, and therefore the reliability of G_2 is greater than the required reliability.

5.2 Example 2 – Compound Cylinder Design

For a further illustration, we present an engineering application example. Though the problem is simple, it sufficiently demonstrates the use and effectiveness of our proposed methods. A compound cylinders design is modeled as a multidisciplinary design optimization problem, where the inner and outer cylinders are considered as subsystems 1 and 2, respectively.

A system of multiple cylinders can resists relatively large pressures more efficiently when it is designed properly, i.e., it requires less material than a single cylinder (Ugural and Fenster, 1975). Fig. 7 shows the sketches of the outer and inner cylinders of a compound cylinders design.

Insert Fig. 7 here

Fig.7 The compound cylinder system

The internal and external radii of the inner cylinder are a and b, and the internal and external radii of the outer cylinder are c and d. The internal pressure is P_0. The objective is to maximize the volume capacity, or the base area, i.e.

$$v = \pi a^2$$

The inner and outer cylinders are designed by two design groups, and the two designs are coupled through the common radii, contact pressure, and radial stresses at the interface between the two cylinders. The corresponding multidisciplinary systems and notations are given in Fig. 8.

Insert Fig.8 here

Fig. 8 System structure of the compound cylinders design

Shared variables: $\mathbf{d}_s = \phi$, an empty set.

Random shared random variables: $\mathbf{x}_s = (E, S, \rho)$, where E is the modulus of elasticity, S is the allowable stress, and ρ is the Poisson's ratio.

Subsystem 1: the inner cylinder

Deterministic disciplinary design variables: $\mathbf{d}_1 = (a, b)$.

Random disciplinary random variables: $\mathbf{x}_1 = (p_0)$.

Input coupling variables: $\mathbf{y}_{21} = (p, c, d)$, where p is the contact stress at the interface.

Output coupling variables: $\mathbf{y}_{12} = (\delta_1, b)$, where δ_1 is the radial deformation of the inner cylinder at radius b, which is given by

$$\delta_1 = \frac{pb}{E}\left(\frac{b^2 + a^2}{b^2 - a^2} - \rho\right).$$

Output: $\mathbf{z}_1 = (v, \sigma_a, \sigma_b)$, where v is the previously defined objective, and σ_a and σ_b are the tangential stresses at the internal radius a and external radius b of the inner cylinder, respectively, which are given by

$$\sigma_a = \frac{-2pb^2}{b^2 - a^2} + \frac{(a^2 + d^2)p_0}{d^2 - a^2}$$

and

$$\sigma_b = \frac{-p(b^2 + a^2)}{b^2 - a^2} + \frac{a^2(b^2 + d^2)p_0}{(d^2 - a^2)b^2},$$

respectively.

The constraints in Subsystem 1 are as follows:

The reliability constraints are given by

$$\Pr\{G_1(1) = \sigma_a - S < 0\} > R_1(1),$$

and

$$\Pr\{G_1(2) = \sigma_b - S < 0\} > R_1(2),$$

where $R_1(1)$ and $R_1(2)$ are required reliabilities.

The deterministic constraints are given by

$$g_1(1) = a - 1.2b \le 0,$$

$$g_1(2) = b - c \le 0,$$

and

$$g_1(3) = c - b \le 0.$$

Subsystem 2: the outer cylinder

Deterministic disciplinary design variables: $\mathbf{d}_2 = (c, d)$

Random disciplinary variables: $\mathbf{x}_1 = (\delta)$, where δ is the total shrinkage allowance of the two cylinders at the interface.

Input coupling variables: $\mathbf{y}_{12} = (\delta_1, b)$.

Output coupling variables: $\mathbf{y}_{21} = (p, c, d)$, where p is given by

$$p = \frac{\delta_2 E}{c} \Bigg/ \left(\frac{c^2 + d^2}{d^2 - c^2} + \rho \right),$$

in which

$$\delta_2 = \frac{pc}{E} \left(\frac{c^2 + d^2}{d^2 - c^2} + \rho \right).$$

The constraints in Subsystem 2 are as follows:

The reliability constraints are given by

$$\Pr\{G_2(1) = \sigma_c - S < 0\} > R_2(1),$$

and

$$\Pr\{G_2(2) = \sigma_d - S < 0\} > R_2(2),$$

where $R_2(1)$ and $R_2(2)$ are required reliabilities, and the tangential stresses at the internal radius c and external radius d are given by

$$\sigma_c = p\left(\frac{c^2 + d^2}{d^2 - c^2} \right) + \frac{a^2(c^2 + d^2)}{(d^2 - a^2)c^2} p_0,$$

and

$$\sigma_d = \frac{2c^2 p}{d^2 - c^2} + \frac{2a^2 p_0}{d^2 - a^2},$$

respectively.

The deterministic constraints are given by

$$g_2(1) = c - 1.2d \le 0,$$

and

$$g_2(2) = c - b = 0.$$

The distributions of the random variables are given in Table 2.

The required reliabilities are $R_1(1) = R_1(2) = R_2(1) = R_2(2) = \Phi(\beta) = 0.9987$, and the reliability index $\beta = 3$.

Table 2 Distributions of random variables

Variables	Mean	Standard deviation	Distribution
E	30×10^6 psi	3×10^6 psi	Normal
S	40×10^6 psi	4×10^6 psi	Normal
ρ	0.3	0.03	Normal
p_0	20,000 psi	2,000 psi	Normal
δ	0.004 in	0.0001 in	Normal

The Optimal results and convergence history from both IDF and MDF methods are given in Tables 3 and 4, respectively. Both methods generate almost the same results. At the optimal point, the percentile value of constraint $G_1(1)$ is very close to zero. This indicates that the reliability of G_1 is equal to the required reliability. The percentile value of G_2 is negative, and therefore the reliability of G_2 is greater than the required reliability. Both methods converge within 3 cycles. IDF is more efficient than MDF because the former uses fewer disciplinary analyses than the latter.

Table 3 Convergence history of IDF

k	$(d_{11}, d_{12}, d_{21}, d_{22})$	v	$G_1(1)$, $G_1(2)$, $G_2(1)$, $G_2(2)$ at the MPP	n_1	n_2
1	(9.10138, 10.9217, 10.9217, 15)	260.234	0.52011, 0.16963, 0.53322, 0.10437		
2	(7.44443, 10.008, 10.008, 15)	174.1055	1.6688e-3, -0.39349, 1.0308e-3, -0.38524	4721	4141
3	(7.43762, 9.99931, 9.99931, 15)	173.7874	5.4286e-6, -0.39482, -0.0031455, -0.38662		

Table 4 Convergence history of MDF

k	$(d_{11}, d_{12}, d_{21}, d_{22})$	v	$G_1(1), G_1(2), G_2(1), G_2(2)$ at the MPP	n_1	n_2
1	(9.4904, 11.6988, 11.6988, 15)	282.9558	0.59532, 0.19009, 0.53151, 0.15849		
2	(7.44675,10.0183, 10.0183, 15)	174.2143	2.7611e-3, -0.39336, 8.285e-4, -0.38255	43209	43209
3	(7.42897,10.3168, 10.3168, 15)	173.3832	1.3714e-8, -0.39782, -0.003573, -0.38526		

To verify the solutions from IDF and MDF, the problem is also solved by the conventional method (see Fig. 6) . The comparison of the results is given in Table 5. It is seen that the two proposed methods procedure almost identical results to the conventional method and are much more efficient.

Table 5 Comparison of Optimal results by Conventional RBD method, IDF and MDF

Method	$(d_{11}, d_{12}, d_{21}, d_{22})$	v	$G_1(1), G_1(2), G_2(1), G_2(2)$ at the MPP	n_1	n_2
Conv’	(7.43394,9.98739,9.98739,15)	173.6154	-0.001394,-0.39521,-6.8101e-7, -0.38568	2973473	2973473
IDF	(7.43762, 9.99931, 9.99931,15)	173.7874	5.4286e-6, -0.39482, -3.1455e-3, -0.38662	4721	4141
MDF	(7.42897,10.3168, 10.3168,15)	173.3832	1.3714e-8, -0.39782, -3.573e-3, -0.38526	43209	43209

6 Conclusions

A new reliability-based MDO method is developed by the strategy of Sequential Optimization and Reliability Assessment. The entire reliability-based MDO is conducted with sequential cycles of deterministic MDO and reliability analysis. The reliability analysis is decoupled from the deterministic MDO. The model of deterministic MDO is formulated based on the reliability analysis results from the previous cycle such that the violated reliability constraints can be improved.

Two MDO methods, IDF and MDF, are employed for both deterministic MDO and reliability analysis. In IDF, in addition to the original design variables in deterministic MDO and reliability analysis, the coupling variables are also considered as part of design variables. The system consistency is also included as additional constraints in the deterministic MDO and reliability analysis. Because everything is taken care of in a single model for deterministic MDO and reliability analysis, the entire reliability based MDO is performed in a serials of single loop procedure.

On the other hand, in MDF, no additional design variables are considered for either deterministic MDO or reliability analysis. The system consistency for solving the coupling variables is formulated separately as the inner loop and is nested within the outer loop of deterministic MDO and reliability analysis.

As for the selection of method, the following facts may be considered.

1) Theoretically, both MDF and IDF produce the same optimal results. Numerically, IDF may be more robust than MDF. For MDF, if the inner loop generates a non-optimal result or local optimal, the outer loop optimization may diverge. Choosing the appropriate starting point for the inner loop optimization is also a difficult task.

2) The efficiency of the two methods depends on the number of coupling variables and the number of reliability constraint functions. Since all the mean values of coupling variables and the coupling variables at the MPP of each reliability constraint are part of the design variables in IDF, the scale of the problem may become much larger if the number of coupling variables and the number of reliability constraint functions are large. In this case, one should choose MDF. The two examples have illustrated that IDF is more efficient when the number of coupling variables and the number of reliability constraint functions are relatively small. Therefore, one should consider using IDF for the above case.

The structure of the proposed method well suits other frameworks of MDO. Therefore, the proposed methods can be integrated with Concurrent Subsystem Optimization (CSSO) or Collaborative Optimization (CO) by just replacing IDF or MDF with CSSO or CO.

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