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Reliability Sensitivity Analysis with Random and Interval Variables

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SUMMARY

In reliability analysis and reliability-based design, sensitivity analysis identifies the relationship between the change in reliability and the change in the characteristics of uncertain variables. Sensitivity analysis is also used to identify the most significant uncertain variables that have the highest contributions to reliability. Most of the current sensitivity analysis methods are applicable for only random variables. In many engineering applications, however, some of uncertain variables are intervals. In this work, a sensitivity analysis method is proposed for the mixture of random and interval variables. Six sensitivity indices are defined for the sensitivity of the average reliability and reliability bounds with respect to the averages and widths of intervals, as well as with respect to the distribution parameters of random variables. The equations of these sensitivity indices are derived based on the First Order Reliability Method (FORM). The proposed reliability sensitivity analysis is a byproduct of FORM without any extra function calls after reliability is found. Once FORM is performed, the sensitivity information is obtained automatically. Two examples are used for demonstration.

1. INTRODUCTION

In reliability analysis [1~3] and reliability-based design [4~7], sensitivity analysis provides information about the relationship between reliability and the distribution parameters of a random variable. Sensitivity analysis can therefore identify the most significant uncertain variables that have the highest contribution to reliability. When only random variables are involved, sensitivity analysis is usually performed for the probabilistic characteristics of a limit-state function, such as its moment, probability density function, and reliability. Such sensitivity analysis is usually named *probabilistic sensitivity analysis* (PSA). Various PSA approaches have been reported in a wide range of literature, including differential analysis [8, 9], variance-based methods [10], and sampling-based methods [10]. These types of probabilistic sensitivity analysis are briefly reviewed below.

(1) Differential analysis (probability sensitivity coefficient)

The probability-based sensitivity measure is defined as the rate of change in a probability (P) (reliability or the probability of failure) due to the change in a distribution parameter (q_i) of a random input, namely $\partial P / \partial q_i$. $\partial P / \partial q_i$ can be calculated by the finite difference method given by [2]:

$$S_{q_i} = \frac{P(q_i + \Delta q_i) - P(q_i)}{\Delta q_i} \tag{1}$$

where q_i is a distribution parameter, such as the mean or the variance of a random variable; Δq_i is a small step size of q_i.

Various probability sensitivity measures have been proposed in literature [11~14]. Wu [11] and Wu and Mohanty [12] propose a normalized *cumulative density function*

(CDF)-based sensitivity coefficient for the probability of failure with respect to the distribution parameters of random variables. The sensitivity is defined by:

$$S_{q_i} = \frac{\partial p_f / p_f}{\partial q_i / q_i} = \int \cdots \int_{\Omega} \frac{q_i}{f_{\mathbf{X}}(\mathbf{X})} \frac{\partial f_{\mathbf{X}}(\mathbf{X})}{\partial q_i} \left[\frac{f_{\mathbf{X}}(\mathbf{X})}{p_f} \right] d\mathbf{x} = E\left[\frac{q_i \partial f_{\mathbf{X}}(\mathbf{X})}{f_{\mathbf{X}}(\mathbf{X}) \partial q_i} \right]_{\Omega} \tag{2}$$

where $f_{\mathbf{X}}$ is the joint probability density function of all random variables, p_f is the probability of failure, **X** is a vector of random variables, and Ω denotes the failure region. The calculation of this sensitivity measure involves evaluating a multidimensional integral. A sampling method is usually used to estimate this integral, which makes this method computationally expensive. Mavris et al. [13] extend Wu's method to evaluate the sensitivity of any probabilistic characteristics, such as the variance and mean of a limit-state function.

Another sensitivity measure related to reliability is the Most Probable Point (MPP) based sensitivity coefficients [14], defined as the gradient of a limit-state function at the MPP in the standard normal space, normalized by the reliability index. Let G be a response calculated by a limit-state function $G = g(\mathbf{X})$, where **X** is the vector of random variables. After **X** is transformed into standard normal random variables **U**, the MPP, $\mathbf{u}^* = \left(u_1^*, u_2^*, \cdots, u_{nx}^*\right)$, the shortest distance point from the limit state $g(\mathbf{U}) = c$, where c is a limit state, to the origin O is identified. (The equation for the MPP search will be given in Eq. (4).) The sensitivity of reliability with respect to the ith random variables is then calculated by

$$S_i = \frac{\left(u_i^*\right)^2}{\beta^2} \tag{3}$$

where β is the magnitude of $\mathbf{u}^*$ or the reliability index. For the MPP-based reliability analysis, the probability sensitivity coefficient does not require any additional computational efforts after the MPP is found. The sensitivity coefficient S_i is just a byproduct of reliability analysis.

(2) Variance-based methods

Variance-based sensitivity analysis methods rely on the decomposition of the variance of a response into items contributed by various sources of input variations [***Comment 1-17***]. These sources can be classified into two types: main effects and total effects. A main effect refers to the effect of only one random variable, while a total effect is used to include both the individual effect of a random variable and the interaction of the random variable with other random variables. Although the methods provide a global sensitivity measure, their major limitation is that a variance is assumed to be sufficient to describe the uncertainties encountered. This type of methods may lose accuracy when the variance is not a good measure of the distribution dispersion, such as in the case where a response distribution has high skewness and kurtosis [15].

(3) Sampling methods

Sampling approaches, such as Monte Carlo sampling for sensitivity analysis, usually involve three steps: (1) generating samples for uncertain input variables; (2) numerically evaluating a limit-state function and then obtaining samples of response variables; (3) statistically analyzing responses and quantifying their uncertainties, and then exploring the effects of the uncertainty of input variables on responses. Sampling methods are easy to use but computationally expensive when reliability is high. Because the probability of failure is low in this case, a large number of samples are required to capture a failure event.

The current PSA methods handle only random variables that are assumed to follow certain probability distributions. However, in many engineering applications, the information or knowledge might be too insufficient to build probability distributions. As discussed in [16, 17], uncertainty is sometimes represented by intervals due to the lack of knowledge. One example is that the true contact resistance in the vehicle crashworthiness design is hard to know; an interval is then used based on the engineers' best judgment [18]. Another example is in a new design. It is difficult to determine the precise distribution of design variables, such as dimensions. Engineers often define their design variables in the form of nominal values plus and minus certain tolerances, like 10±0.01mm. More examples of intervals can be found in [4, 16]. Sometimes even though a variable is random and follows a non-uniform distribution, only one interval estimate is available due to limited information or sparse samples. In this case, assigning an assumed distribution to the variable may lead to erroneous results [19]. When intervals are involved, the current PSA methods are no longer applicable.

Several methods of dealing with only interval variables have been reported for reliability analysis and reliability-based design [17, 20~34]. A few sensitivity analysis methods [35~38] for epistemic uncertainty (uncertainty due to the lack of knowledge) are potentially capable of dealing with interval variables. These methods use intervals to represent epistemic uncertainty. For example, a sensitivity analysis approach on the basis of belief and plausibility measures is proposed by Bae, et al [35, 36]. The results of this approach can help guide the data collection to improve the accuracy of reliability analysis and distinguish the dominant contributors of uncertainty. A sampling-based sensitivity analysis method is developed by Helton, et al [37]. It consists of three steps: an initial analysis to explore the model behavior, a stepwise analysis to indicate the effects of uncertain variables on belief and plausibility functions, and a summary analysis to show a series of variance-based sensitivity analysis results. Considering the complexity of the mixture of aleatory and epistemic uncertainties, Guo and Du [38] propose an approach to conduct sensitivity analysis with this mixture. In their method, the most important epistemic variables are captured under the framework of the unified uncertainty analysis.

All of the above methods are capable of identifying the most significant interval variables, but they have some limitations. For example, it is difficult to use them to obtain information about how individual intervals impact reliability, especially how reliability bounds will change after narrowing interval bounds. In this work, we propose a sensitivity analysis method to handle the situation where both interval variables and random variables are involved. The intervals are treated as is without any distribution assumptions. With this method, we attempt to answer the following questions:

1) How will the width of the reliability bounds change if the width of an interval is reduced or if the average of the interval is changed?
2) How will the average reliability change if the width of an interval is reduced or if the average of the interval is changed?
3) How will the width of the reliability bounds change if a distribution parameter

of a random variable is changed?

4) How will the average reliability change if a distribution parameter of a random variable is changed?

The answers to the above questions will provide useful information about improving reliability and reducing the impact of intervals and random variables on reliability. Hence, six sensitivity indices are proposed for answering these questions. Equations for the sensitivity indices are then derived and corresponding computational procedures are developed. The calculation of sensitivity indices requires searching the minimum and maximum reliability, or the probabilities of failure, over the intervals. To alleviate the computational burden, we use an efficient FORM-based unified reliability analysis framework [39].

This paper is organized as follows: Sec. 2 provides a brief review of the unified reliability analysis. In Sec. 3, the six sensitivity indices are defined, and the equations for calculating these sensitivity indices are derived. In Sec. 4, two engineering examples are used to illustrate the proposed method. Conclusions and future work are summarized in Sec. 5.

2. UNIFIED RELIABILITY ANALYSIS

Reliability analysis is one of the main steps of reliability sensitivity analysis. The proposed sensitivity analysis is based on the First Order Reliability Method (FORM) [40, 41] which is applicable for random variables, and the unified reliability analysis (URA) [39], which is applicable for the mixture of random and interval variables. Both methods are briefly reviewed in this section.

2.1. Reliability analysis

In the reliability analysis where only random variables **X** are involved, reliability is defined by

$$R = \Pr\{G = g(\mathbf{X}) \geq c\} = 1 - \Pr\{G = g(\mathbf{X}) < c\} = 1 - p_f \quad (4)$$

where $\Pr\{\cdot\}$ denotes a probability, G is a response, c is a specific limit state, $\mathbf{X} = \left(X_1, X_2, \cdots, X_i, \cdots, X_{nx}\right)$ is a vector of random variables, g is a performance function, also called a limit-state function [42], and p_f is the probability of failure.

If the joint *probability density function* (PDF) of **X** is $f_{\mathbf{X}}$, the probability of failure p_f is calculated by

$$p_f = \Pr\{G = g(\mathbf{X}) < c\} = \int_{g(\mathbf{X})<c} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \quad (5)$$

The limit-state function $g(\mathbf{X})$ is usually a nonlinear function of **X**; therefore, the integration boundary is nonlinear. Since the number of random variables is usually high, multidimensional integration is involved. There is rarely a closed-form solution to Eq. (5). The First Order Reliability Method (FORM) is widely used to easily evaluate the integral in Eq. (5).

FORM involves three steps to approximate the probability integral: 1) transforming original random variables **X** into standard normal random variables **U**, 2) searching for the Most Probable Point (MPP), and 3) calculating p_f.

Step 1: Transformation, which is given by

$$u_i = \Phi^{-1}\left\{F_{X_i}(x_i)\right\} \quad (6)$$

where F_{X_i} is the CDF of X_i, and Φ^{-1} is the inverse CDF of a standard normal distribution.

Step 2: MPP search, where the MPP $\mathbf{u}^*$ is identified by

$$\begin{aligned} &\min_{\mathbf{U}} \|\mathbf{U}\| \\ &\text{s.t.} \quad g(\mathbf{U}) = c \end{aligned} \tag{7}$$

in which $\|\cdot\|$ stands for the magnitude of a vector. Geometrically, the MPP is the shortest distance point from the limit state $g(\mathbf{U}) = c$ to the origin in **U**-space. The minimum distance $\beta = \|\mathbf{u}^*\|$ is called the *reliability index.*

Step 3: Estimation of p_f, which is given by

$$p_f = \Phi(-\beta) \tag{8}$$

where Φ is the CDF of a standard normal distribution.

The most computationally intensive work of FORM is the MPP search. The following recursive algorithm [43] is used for the MPP search,

$$\begin{cases} \beta^{(k)} = \beta^{(k-1)} + \dfrac{g(\mathbf{u}^{(k-1)})}{\left\|\nabla g(\mathbf{u}^{(k-1)})\right\|} \\ \mathbf{u}^{(k)} = -\beta^{(k)} \dfrac{\nabla g(\mathbf{u}^{(k-1)})}{\left\|\nabla g(\mathbf{u}^{(k-1)})\right\|} \end{cases} \tag{9}$$

where $\nabla g(\mathbf{u}^{(k)})$ is the gradient of g at $\mathbf{u}^{(k)}$, $\left\|\nabla g(\mathbf{u}^{(k)})\right\|$ is its magnitude, and k is the iteration counter.

2.2. Unified reliability analysis (URA)

When both random and interval variables are present, random variables **X** are characterized by probability distributions while interval variables **Y** reside on $[\mathbf{y}^l, \mathbf{y}^u]$. The unified uncertainty analysis framework and computational method proposed in [39] is applicable to handle this situation. As shown in [39], the cumulative distribution function (CDF) of the response $G = g(\mathbf{X}, \mathbf{Y})$ has its upper and lower bounds, and so does reliability $\Pr\{G \geq c\}$. The unified reliability analysis (URA) [39] is used to find the reliability bounds.

The URA framework is illustrated in Figure 1. The inputs to the framework are random variables **X** defined by a joint PDF and interval variables **Y**. The outputs are CDF bounds and reliability bounds.

Figure 1. The unified reliability analysis framework

The set of intervals **Y** is denoted by $\Delta_{\mathbf{Y}}$, and the event of failure is defined by $g(\mathbf{X,Y}) < c$. According to [39], the upper and lower bounds of the probability of failure, p_f^U and p_f^L, are calculated by

$$p_f^L = \Pr\left\{G_{\max}(\mathbf{X}, \mathbf{Y}) < c \middle| \mathbf{Y} \in \Delta_{\mathbf{Y}}\right\} \tag{10}$$

and

$$p_f^U = \Pr\left\{G_{\min}(\mathbf{X}, \mathbf{Y}) < c \middle| \mathbf{Y} \in \Delta_{\mathbf{Y}}\right\} \tag{11}$$

respectively. $G_{\min}$ and $G_{\max}$ are the global minimum and maximum values, respectively, of G over $\Delta_{\mathbf{Y}}$.

The evaluation of the upper and lower bounds of the probability of failure is essentially the evaluation of the minimum and maximum CDF of the limit-state function. Therefore, traditional probabilistic analysis methods can be used for the unified reliability analysis (URA). The First Order Reliability Method (FORM) is employed for the URA.

Figure 2 depicts the numerical procedure of the URA method. The procedure involves two types of analysis. The first one is *probabilistic analysis* (PA), which is responsible for the MPP search and the calculation of the probability of failure. The second one is *interval analysis* (IA), which is responsible for the search of the maximum and minimum values of G. The direct combination of PA and IA will involve a double loop procedure, where PA is an outer loop and IA is an inner loop. For example, to find the lower bound of p_f, at every iteration of the MPP search in the outer loop, interval analysis inner loops will be called to find the maximum G in terms of $\mathbf{Y}$. This method is inefficient due to the double-loop procedure. The efficient computational method is then developed in [44]. The method involves an efficient sequential single-loop procedure, where PA is decoupled from IA. The flowchart of this efficient procedure is shown in Figure 2 for the p_f^L calculation. The solution is the MPP where G is maximized. The MPP for p_f^L is then named $\mathbf{u}^{*,L}$ in this paper. And the MPP for the maximum probability of failure p_f^U is called $\mathbf{u}^{*,U}$.

Figure 2. Flowchart of sequential single-loop procedure for p_f^L calculation

The probability $\Pr\left\{G_{\max}(\mathbf{X},\mathbf{Y})<c \middle| \mathbf{Y}\in\Delta_{\mathbf{Y}}\right\}$ in Eq. (10) is then computed by

$$\Pr\left\{G_{\max}(\mathbf{X},\mathbf{Y})<c \middle| \mathbf{Y}\in\Delta_{\mathbf{Y}}\right\}=\Phi(-\beta)=\Phi(-\left\|\mathbf{u}^*\right\|). \quad (12)$$

For the p_f^U calculation, the model of the MPP search is the same as in Figure 2 except that IA becomes a minimization problem.

3. RELIABILITY SENSITIVITY ANALYSIS

When only random variables are involved, reliability sensitivity analysis is used to find the rate of change in the probability of failure (or reliability) due to the changes in distribution parameters (usually means and standard deviations). When both random variables and interval variables are involved, reliability analysis will generate two bounds of reliability or of the probability of failure p_f. The gap between the maximum probability of failure p_f^U and the minimum probability of failure p_f^L represents the effect of interval variables on the probability of failure. In addition to the traditional sensitivity analysis in terms of random variables, sensitivity analysis in terms of interval variables is also needed. In this work, six types of sensitivity are proposed with respect to both random variables and interval variables. The proposed sensitivity indexes are summarized in Table I .

Insert Table I here

3.1. Type I sensitivity $\partial\delta_p/\partial\delta_i$

$\partial\delta_p/\partial\delta_i$ is the sensitivity of the width of the p_f bounds, δ_p, with respect to the interval width of the *i*th interval variable Y_i, δ_i. δ_p is defined by

$$\delta_p = p_f^U - p_f^L \tag{13}$$

The width of Y_i is calculated by

$$\delta_i = y_i^U - y_i^L \tag{14}$$

where y_i^L and y_i^U are the lower and upper bounds of y_i, respectively.

To obtain a unique sensitivity index, we define the change of δ_i, $\Delta(\delta_i)$ in such a way that Y_i expands in both directions equally; namely, y_i^L is decreased by $\frac{\Delta(\delta_i)}{2}$ and y_i^U is increased by $\frac{\Delta(\delta_i)}{2}$. There are infinite ways that Y_i can change by $\Delta(\delta_i)$, for example, $[y_i^L, y_i^U]$ can change to $\left[y_i^L - \frac{3\Delta(\delta_i)}{4}, y_i^U + \frac{\Delta(\delta_i)}{4}\right]$ or $\left[y_i^L - \frac{\Delta(\delta_i)}{4}, y_i^U + \frac{3\Delta(\delta_i)}{4}\right]$. Our definition makes the change unique.

This type of sensitivity can identify interval variables that have the largest impact on the width of the p_f bounds. If the gap of the p_f bounds is too wide, decisions will be difficult to make. To narrow the width of p_f bounds efficiently, more information about the important interval variables should be collected, and then their widths can be reduced. Sensitivity analysis will provide a useful guidance to the collection of more information.

To derive the equations for $\partial\delta_p / \partial\delta_i$, we consider all the situations where the maximum or minimum p_f occurs on the lower bound, upper bound, or at an interior point of Y_i. Next we demonstrate how to derive $\partial\delta_p / \partial\delta_i$ when the maximum p_f occurs on the upper bound of Y_i and the minimum p_f occurs on the lower bound of Y_i. The derivations of other cases are given in Appendix B, and the common equations used in derivations are given in Appendix A.

The problem can be stated as:

Given: $G = g(\mathbf{X},\ \mathbf{Y}), \mathbf{Y}_{\sim i} = (Y_1, Y_2, \cdots, Y_{i-1}, Y_{i+1}, \cdots Y_{ny})$, $\bar{y}_i = \dfrac{y_i^L + y_i^U}{2}$, p_f^L occurs at y_i^L, and p_f^U occurs at y_i^U.

Find: $\partial \delta_p / \partial \delta_i$.

$$
\begin{aligned}
\frac{\partial \delta_p}{\partial \delta_i} &= \frac{\partial\left(p_f^U - p_f^L\right)}{\partial \delta_i} = \frac{\partial\left[p_f^U\left(\bar{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) - p_f^L\left(\bar{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial \delta_i} \\
&= \frac{\partial\left[p_f^U\left(\bar{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial\left(\bar{y}_i + \frac{1}{2}\delta_i\right)} \frac{\partial\left(\bar{y}_i + \frac{1}{2}\delta_i\right)}{\partial \delta_i} - \frac{\partial\left[p_f^L\left(\bar{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial\left(\bar{y}_i - \frac{1}{2}\delta_i\right)} \frac{\partial\left(\bar{y}_i - \frac{1}{2}\delta_i\right)}{\partial \delta_i} \\
&= \left(\frac{1}{2}\right) \frac{\partial\left[p_f^U\left(\bar{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial\left(\bar{y}_i + \frac{1}{2}\delta_i\right)} - \left(-\frac{1}{2}\right) \frac{\partial\left[p_f^L\left(\bar{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial\left(\bar{y}_i - \frac{1}{2}\delta_i\right)} \\
&= \frac{1}{2}\left(\frac{\partial p_f^U}{\partial y_i^U} + \frac{\partial p_f^L}{\partial y_i^L}\right)
\end{aligned}
\tag{15}
$$

$\dfrac{\partial p_f^U}{\partial y_i^U}$ and $\dfrac{\partial p_f^L}{\partial y_i^L}$ then need to be calculated. In this case, the MPP's of p_f^L or p_f^U are on one bound of Y_i. Let *h* be the bound and p_f be p_f^U or p_f^L. Then,

$$
\frac{\partial p_f}{\partial h} = \frac{\partial[\Phi(-\beta)]}{\partial h} = -\phi(-\beta)\frac{\partial \beta}{\partial h} \tag{16}
$$

where $\phi(\cdot)$ is the PDF of a standard normal distribution. Next, we will show how to calculate $\dfrac{\partial \beta}{\partial h}$.

Let the MPP be $\mathbf{u}^* = \left(u_1^*, u_2^*, \cdots, u_{nx}^*\right)$ and the corresponding intervals **Y** be **y**. In the U-space after $\mathbf{X}$ are transformed into **U**, the limit-state function becomes $g(\mathbf{U}, \mathbf{Y})$, and at the MPP the limit-state function is $g(\mathbf{u}^*, \mathbf{y})$, where $\mathbf{y}$ is the vector of **Y** at the MPP. Let $\nabla g(\mathbf{u}^*)$ be the gradient of $g(\mathbf{U}, \mathbf{Y})$ in terms of **U** at the MPP; namely,

$\nabla g(\mathbf{u}^*) = \left(\frac{\partial g}{\partial U_1}\bigg|_{\mathbf{u}^*,\mathbf{y}}, \frac{\partial g}{\partial U_2}\bigg|_{\mathbf{u}^*,\mathbf{y}}, \cdots, \frac{\partial g}{\partial U_n}\bigg|_{\mathbf{u}^*,\mathbf{y}}, \right)$. For brevity, without losing generality, we will drop **Y** or **y** in the limit-state function expression in the following derivations. At the MPP, the following equation holds [40, 41],

$$u_i^* = -\beta \frac{\nabla g(\mathbf{u}^*)}{\left\| \nabla g(\mathbf{u}^*) \right\|} \tag{17}$$

$\frac{\nabla g(\mathbf{u}^*)}{\left\| \nabla g(\mathbf{u}^*) \right\|}$ is the unit vector of the gradient, and the gradient is calculated at the MPP, therefore a constant. Then,

$$\frac{\partial u_i}{\partial h} = -\frac{\partial \beta}{\partial h} \frac{\frac{\partial g}{\partial U_i}\Big|_{u_i^*}}{\left\| \nabla g(\mathbf{u}^*) \right\|} \tag{18}$$

Recall that y_i is on one bound h of the interval variable Y_i at the MPP, where $G = g(\mathbf{u}^*, h)$ reaches the limit state and hereby becomes a constant. Then

$$\frac{\partial G}{\partial h} = \sum_{i=1}^{n_x} \frac{\partial g}{\partial U_i} \frac{\partial U_i}{\partial h} + \frac{\partial g}{\partial h} = 0 \tag{19}$$

Therefore, Eq. (19) becomes

$$\sum_{i=1}^{n_x} \frac{\partial g}{\partial U_i} \frac{\partial U_i}{\partial h} + \frac{\partial g}{\partial h} = \sum_{i=1}^{n_x} \frac{\partial g}{\partial U_i} \left(-\frac{\partial \beta}{\partial h} \frac{\frac{\partial g}{\partial U_i}\Big|_{u_i^*}}{\left\| \nabla g(\mathbf{u}^*) \right\|} \right) + \frac{\partial g}{\partial h} = -\frac{\partial \beta}{\partial h} \frac{\sum_{i=1}^{n_x} \left(\frac{\partial g}{\partial U_i}\Big|_{u_i^*} \right)^2}{\left\| \nabla g(\mathbf{u}^*) \right\|} + \frac{\partial g}{\partial h}$$
$$= -\frac{\partial \beta}{\partial h} \left\| \nabla g(\mathbf{u}^*) \right\| + \frac{\partial g}{\partial h} = 0 \tag{20}$$

We then obtain

$$\frac{\partial \beta}{\partial h} = \frac{\frac{\partial g}{\partial h}}{\left\| \nabla g(\mathbf{u}^*) \right\|} \tag{21}$$

Substituting $\partial \beta / \partial h$ in Eq. (16) with Eq. (21) yields

$$\frac{\partial p_f}{\partial h} = -\phi(-\beta)\frac{\partial \beta}{\partial h} = -\phi(-\beta)\frac{\frac{\partial g}{\partial h}}{\left\|\nabla g(\mathbf{u}^*)\right\|} = \frac{-\phi(-\beta)}{\left\|\nabla g(\mathbf{u}^*)\right\|}\frac{\partial g}{\partial h} \tag{22}$$

Using the results from Eqs.(22) and (15), we get the equation of Type I sensitivity when $p_f^{\max}$ occurs on the upper bound of Y_i and $p_f^{\min}$ occurs on the lower bound of Y_i as follows:

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{1}{2}\left(\frac{\partial p_f^U}{\partial y_i^U} + \frac{\partial p_f^L}{\partial y_i^L}\right) = -\frac{1}{2}\left(\frac{\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^U} + \frac{\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right) \tag{23}$$

where β^U is the reliability index at the maximum p_f, β^L is the reliability index at the minimum p_f, $\mathbf{u}^{*,U}$ is the MPP for the maximum p_f, and $\mathbf{u}^{*,L}$ is the MPP for the minimum p_f. The equations of Type I sensitivity for other situations are given in Appendix B.

3.2. Type II sensitivity $\partial \overline{p}_f / \partial \delta_i$

$\partial \overline{p}_f / \partial \delta_i$ is the sensitivity of the average p_f, $\overline{p}_f$, with respect to δ_i. $\overline{p}_f$ is defined by

$$\overline{p}_f = \frac{p_f^U + p_f^L}{2} \tag{24}$$

The relationship among p_f^U, p_f^L, δ_p and $\overline{p}_f$ is illustrated in Figure 3. This type of sensitivity quantifies the rate of change of the mean value of p_f due to the change of the interval width of Y_i. The equations of this type of sensitivity are given in Appendix C.

Figure 3. p_f^U, p_f^L, δ_i, and $\overline{p}_f$

3.3. Type III sensitivity $\partial\delta_p / \partial\overline{y}_i$

$\partial\delta_p / \partial\overline{y}_i$ is the sensitivity of the width of the probability of failure δ_p with respect to the average of the *i*th interval variable, $\overline{y}_i$. $\overline{y}_i$ is defined by

$$\overline{y}_i = \frac{y_i^U + y_i^L}{2} \tag{25}$$

The relationship among y_i^L, y_i^U, δ_i and $\overline{y}_i$ is illustrated in Figure 4. This type of sensitivity is useful when we can control the averages of the interval variables during reliability based-optimization. We can efficiently decrease the reliability gap by shifting averages of interval variables to which the probability of failure is highly sensitive. The equations of this type sensitivity are given in Appendix D.

Figure 4. y_i^L, y_i^U, δ_i, and $\overline{y}_i$

3.4. Type IV sensitivity $\partial\overline{p}_f / \partial\overline{y}_i$

$\partial\overline{p}_f / \partial\overline{y}_i$ is the sensitivity of the average probability of failure $\overline{p}_f$ with respect to $\overline{y}_i$. It tells us how much the average probability of failure will change given the change in the midpoint of an interval variable. The equations of this type of sensitivity are given in Appendix E.

3.5. Type V sensitivity $\partial\delta_p / \partial q_i$

$\partial\delta_p / \partial q_i$ is the sensitivity of the width of the probability of failure δ_p with respect to a distribution parameter, q_i, of random variable X_i. For example, for a normal distribution, q_i would be the mean μ_i or standard deviation σ_i while for uniform distribution, q_i could be one of the interval bounds. As shown previously, the p_f gap δ_p is mainly caused by interval variables [38]. On the other hand, the value of p_f primarily depends on random variables. The equations of this type of sensitivity are given in Appendix F.

3.6. Type VI sensitivity $\partial \overline{p}_f / \partial q_i$

$\partial \overline{p}_f / \partial q_i$ is the sensitivity of the average probability of failure $\overline{p}_f$ with respect to a distribution parameter, q_i, of random variable X_i. The equations of this type of sensitivity are given in Appendix G. The equations of Type V and VI sensitivities for a normal distribution are also given in Appendices F and G, respectively.

3.7. Equations of all the sensitivity indices

The equations for all the above sensitivity indices are summarized in Tables II, III and IV [***Comment 1-17***].

Insert Table II here

Insert Table III here

Insert Table IV here

In the above table, *w* is given in Equation (A11) in Appendix A.

The procedure to calculate the sensitivity indices is illustrated in Figure 5. First, unified reliability analysis is conducted to obtain MPPs and interval variables at p_f^U and p_f^L. Then depending on the location of the interval variables, either interior or on a bound, at the MPP, the corresponding equations from Table II, III, and IV are used to calculate the sensitivity indices.

Figure 5. The procedure to calculate sensitivity indices

4. NUMERICAL EXAMPLES

Two examples are used to demonstrate our proposed sensitivity measures with random and interval variables. The first example deals with normally distributed variables while the second example handles random variables with non-normal distributions.

4.1. Example 1- Adhesive Bonding Example

A double-lap joint design of a rubber-modified epoxy based adhesive [45] is illustrated in Figure 6. The design consists of aluminum outer adherends and an inner steel adherend. The assembly is cured at 250 °F and is stress-free at temperature T_1. The completed bond is subjected to an axial load *P* at a service temperature T_2. The coefficients of thermal expansion for the outer and inner adherend α_o and α_i are 6×10^{-6} and 13×10^{-6} $\text{in/(in}\cdot{}^{\text{o}}\text{F)}$, respectively. The modulus E_o and the thickness t_o, of the outer adherend, and the modulus E_i and the thickness t_i, of the inner adherend [***Comment 1-17***], are random variables. The shear modulus *G*, width *b*, length *L*, of the adhesive, and the lap-shear strength of adhesive S_a are also random variables. Their distributions are given in Table V.

Figure 6. A double-lap joint design of adhesive

Insert Table V here

Because it is difficult to spread the adhesive uniformly over the surface, the thickness of the adhesive is estimated to be in an interval shown in Table VI. The temperature change, $\Delta T = T_2 - T_1$, is difficult to fit into some probability distribution since the temperature field is unknown. An interval is therefore assigned for ΔT as listed in Table VI.

Insert Table VI here

The limit-state function is the safety margin for strength requirement of the joint, which is defined by the difference between the lap-shear strength of adhesive and the maximum shear stress $\tau_{\max}$. The equation is obtained at x = 0.5 where the maximum shear stress occurs. The function is given by

$$G = g\left(\mathbf{X}, \mathbf{Y}\right) = S_a - \tau_{\max}$$

where $\tau_{\max} = \tau(0.5)$, and

$$\tau(x) = \frac{P\omega}{4b\sinh(\omega L/2)}\cosh(\omega x) + \left[\frac{P\omega}{4b\cosh(\omega L/2)}\left(\frac{2E_o t_o - E_i t_i}{2E_o t_o + E_i t_i}\right) + \frac{(\alpha_i - \alpha_o)\Delta T \omega}{(1/E_o t_o + 2/E_i t_i)\cosh(\omega L/2)}\right]\sinh(\omega x)$$

and $\omega = \sqrt{\frac{G}{h}\left(\frac{1}{E_o t_o} + \frac{2}{E_i t_i}\right)}$.

The failure event is defined by $F = \left\{\mathbf{X}, \mathbf{Y} \middle| g\left(\mathbf{X}, \mathbf{Y}\right) < 0\right\}$.

The analysis results are listed in Tables VII, VIII, IX and X. To verify the proposed method, additional reliability analyses are also conducted. The results are shown as “Numerical verification” in Table VIII (for interval variables) and Table X (for random variables). Each parameter (the average or width of an interval variable, or a distribution parameter of a random variable), with respect to which a sensitivity index would be calculated, is increased by a small step size. An additional reliability analysis for that parameter is then performed. The rate of change in the reliability analysis results with respect to the parameter was computed. The rate should be very close to the sensitivity index calculated from the proposed method. Both Tables VIII and X show good consistency and verify the accuracy of the proposed method.

The sign of a sensitivity index gives a possible direction for improvement. For example, in Table VIII $\partial\delta_p / \partial\delta_1$ and $\partial\delta_p / \partial\delta_2$ are both positive while $\partial\delta_p / \partial\overline{y}_1$ and $\partial\delta_p / \partial\overline{y}_2$ are both negative. Therefore, if we wish to reduce the bounds of p_f, we could narrow the intervals of thickness of adhesive (δ_1) and the temperature change (δ_2) or increase their averages of them ($\overline{y}_1$ and $\overline{y}_2$). A similar conclusion can be drawn for $\partial\delta_p / \partial\overline{y}_i$ and $\partial\overline{p}_f / \partial\overline{y}_i$.

To better interpret the sensitivity analysis results, the percentage change in Table IX is also included. $\Delta_{y_i}^{+1\%}$ indicates the change in δ_p or $\overline{p}_f$ corresponding to the 1% increase in δ_i or $\overline{y}_i$, respectively. For instance, if δ_1 increased by 1%, or δ_1 increased by $\left(y_1^U - y_1^L\right) \times 1\% = \left(0.0205 - 0.0195\right) \times 1\% = 1.0 \times 10^{-5}$ inch, the width of the probability of failure bounds δ_p would increase by $\left(5.009\times10^{-2}\right)\times\left(1.0\times10^{-5}\right) = 5.009\times10^{-7}$, where the multiplier is the change in δ_1 while the multiplicand is the Type I sensitivity index. Similarly, the average probability of failure $\overline{p}_f$ would change by $\left(-2.494\times10^{-2}\right)\times\left(1.0\times10^{-5}\right) = -2.494\times10^{-7}$. Since the sign is negative, $\overline{p}_f$ would actually decrease. This example indicates how the change in input uncertainty impacts reliability or the probability of failure. A sensitivity index also tells us the relative importance of uncertain variables. For example, Y_1 has higher $\Delta^{+1\%}$ of Type I ~ IV sensitivity indices than those of Y_2; Y_1 is therefore more significant than Y_2 in terms of its impact on δ_p and $\overline{p}_f$.

Insert Table VII here

Insert Table VIII here

Insert Table IX here

Table X shows sensitivities in terms of the mean and standard deviation of random variables. The positive signs of $\partial\delta_p/\partial q$ and $\partial\overline{p}_f/\partial q$ imply that the distribution parameters need to be lowered to reduce δ_p and $\overline{p}_f$. And the negative ones suggest that distribution parameters need to be increased to reduce δ_p and $\overline{p}_f$. From this table, it can be concluded that X_7 has the highest impact on δ_p and $\overline{p}_f$ because it has the highest sensitivity index values. Given the positive signs of Type V and VI sensitivity indices of X_7, reducing the mean and variance of X_7 would be more efficient than adjusting other random variables in order to lower δ_p and $\overline{p}_f$.

Insert Table X here

4.2. Example 2- Cantilever Tube

In Example 1, all random variables are normally distributed. In this example, some random variables follow non-normal distributions. The cantilever tube shown in Figure 7 is subject to external forces F_1, F_2, and P, and torsion T [44]. The limit-state function is defined as the difference between the yield strength S and the maximum stress $\sigma_{\max}$, namely,

$$G = g(\mathbf{X}, \mathbf{Y}) = S - \sigma_{\max}$$

where $\sigma_{\max}$ is the maximum von Mises stress on the top surface of the tube at the origin and is given by

$$\sigma_{\max} = \sqrt{\sigma_x^2 + 3\tau_{zx}^2}\ .$$

Figure 7. Cantilever tube

The normal stress σ_x is calculated by

$$\sigma_x = \frac{P + F_1 \sin\theta_1 + F_2 \sin\theta_2}{A} + \frac{Mc}{I}$$

where the first term is the normal stress due to the axial forces, and the second term is the normal stress due to the bending moment *M*, which is given by

$$M = F_1 L_1 \cos\theta_1 + F_2 L_2 \cos\theta_2$$

and

$$A = \frac{\pi}{4}\left[d^2 - \left(d - 2t \right)^2 \right]$$

$$c = d/2 ,$$

$$I = \frac{\pi}{64}\left[d^4 - \left(d - 2t \right)^4 \right]$$

The torsional stress τ_{zx} at the same point is calculated by

$$\tau_{zx} = \frac{Td}{2J}$$

where $J = 2I$.

The random and interval variables are given in Tables XI and XII, respectively.

Insert Table XI here

Insert Table XII here

The results of reliability analysis and sensitivity are listed in Table XIII, XIV, and XV. It is noted that sensitivity indices of $\partial\delta_p / \partial\delta_i$, and $\partial\delta_p / \partial\overline{y}_i$ are all positive while sensitivity indices of $\partial\overline{p}_f / \partial\delta_i$ and $\partial\overline{p}_f / \partial\overline{y}_i$ are all negative. In this case, the direction of the change in δ_p will be opposite to the direction of change in $\overline{p}_f$ whenever we adjust δ_i and $\overline{y}_i$. For instance, decreasing δ_1 will result in a lower δ_p and a higher $\overline{p}_f$.

Insert Table XIII here

Insert Table XIV here

Insert Table XV here

In this example, uniform distributions and a Gumbel distribution are involved. In Table XVI, the sensitivities in terms of the parameters of these two distributions are also calculated. It is indicated that Type V and VI sensitivities of uniformly distributed variables, X_3 and X_4, are all positive. Hence, if we raise or lower the bounds of X_3 and X_4, the change of δ_p and $\overline{p}_f$ will follow the same direction.

Insert Table XVI here

5. CONCLUSIONS

When information or knowledge is not adequate to build probability distributions, interval variables may be used. In this case, probabilistic sensitivity analysis approaches are no longer applicable. An effective sensitivity analysis method is proposed to handle the mixture of random variables and interval variables.

With the presence of both random and interval variables, reliability and the probability of failure resides between their lower and upper bounds. In this work, based on the unified uncertainty analysis framework [39], we have explored various sensitivity indices with respect to both random and interval variables. Four new types of sensitivity for interval variables include the sensitivities of the width and average of the probability of failure bounds with respect to the interval width and with respect to the mean of each interval variable. Two new types of sensitivity for random variables include the sensitivities of the width and average of the probability of failure with respect to the distribution parameters of each random variable. Equations for the six sensitivity indices are derived. Through the unified reliability analysis and the First Order Reliability Method (FORM), the sensitivity indices are calculated after reliability analysis is completed without calling the limit-state function again. The sensitivity indices are therefore a byproduct of reliability analysis.

The advantages of the proposed methods are as follows: (1) The method is easy to use because it employs the First Order Reliability Method (FORM), which is widely used in industry. (2) Sensitivity information is just a byproduct of reliability analysis. (3) Both random and interval variables can be handled by reliability analysis at the same time. And (4) the computation is efficient without a double-loop procedure or Monte Carlo simulation involved.

The method has some limitations. Since it is based on only the First Order Reliability Method (FORM), the method cannot be directly extended to the Second Order Reliability Method (SORM). The method assumes the global optimal solution if optimization is used for interval analysis. The method may not provide an accurate solution if a global optima is not reached. It is well known that FORM may not be accurate when multiple MPPs exist. The proposed method exhibits the same behavior for the multiple MPPs situation.

Future work would be the further improvement of efficiency and the inclusion of more sensitivity indices. For higher efficiency, the efficient interval arithmetic could be used for interval analysis. Other sensitivity methods, such as those suggested in [46], could also be incorporated.

APPENDIX A: COMMON EQUATIONS

1. Derivative of p_f with respect to one bound of an interval variable Y_i

$\frac{\partial p_f}{\partial h}$ is given in Eq. (22) and is rewritten below.

$$\frac{\partial p_f}{\partial h} = \frac{-\phi(-\beta\ )}{\left\|\nabla g(\mathbf{u}^*)\right\|}\frac{\partial g}{\partial h} \tag{A1}$$

where p_f could be p_f^L or p_f^U, and β could be β^L and β^U.

2. Derivative of p_f with respect to the width of an interval, δ_i

If p_f occurs at y_i^U,

$$\frac{\partial p_f}{\partial \delta_i} = \frac{\partial p_f\left(y_i^U, \mathbf{Y}_{\sim i}\right)}{\partial \delta_i} = \frac{\partial p_f\left(\bar{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial \delta_i} = \frac{\partial p_f\left(\bar{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial(\bar{y}_i + \frac{1}{2}\delta_i)}\frac{\partial(\bar{y}_i + \frac{1}{2}\delta_i)}{\partial \delta_i} \tag{A2}$$

Eq. (A2) can then be simplified to

$$\frac{\partial p_f}{\partial \delta_i} = \frac{1}{2}\frac{\partial p_f\left(\bar{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial(\bar{y}_i + \frac{1}{2}\delta_i)} = \frac{1}{2}\frac{\partial p_f}{\partial y_i^U} \tag{A3}$$

Similarly, if p_f occurs at y_i^L, the equation becomes

$$\frac{\partial p_f}{\partial \delta_i} = -\frac{1}{2}\frac{\partial p_f\left(\bar{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial(\bar{y}_i - \frac{1}{2}\delta_i)} = -\frac{1}{2}\frac{\partial p_f}{\partial y_i^L} \tag{A4}$$

If p_f occurs at an interior point $\hat{y}_i$, which is not a function of δ_i, it can then be shown that

$$\frac{\partial p_f}{\partial \delta_i} = \frac{\partial p_f\left(\hat{y}_i, \mathbf{Y}_{\sim i}\right)}{\partial \delta_i} = \frac{\partial p_f\left(\hat{y}_i, \mathbf{Y}_{\sim i}\right)}{\partial \hat{y}_i}\frac{\partial \hat{y}_i}{\partial \delta_i} = \frac{\partial p_f\left(\hat{y}_i, \mathbf{Y}_{\sim i}\right)}{\partial \hat{y}_i} \cdot 0 = 0 \qquad \text{(A5)}$$

3. Derivative of p_f with respect to the average of an interval, $\overline{y}_i$

If p_f occurs at y_i^U, one can obtain

$$\frac{\partial p_f}{\partial \overline{y}_i} = \frac{\partial p_f\left(y_i^U, \mathbf{Y}_{\sim i}\right)}{\partial \overline{y}_i} = \frac{\partial p_f\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial \overline{y}_i} = \frac{\partial p_f\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial\left(\overline{y}_i + \frac{1}{2}\delta_i\right)}\frac{\partial\left(\overline{y}_i + \frac{1}{2}\delta_i\right)}{\partial \overline{y}_i} \qquad \text{(A6)}$$

and therefore

$$\frac{\partial p_f}{\partial \overline{y}_i} = \frac{\partial p_f\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial\left(\overline{y}_i + \frac{1}{2}\delta_i\right)} = \frac{\partial p_f}{\partial y_i^U} \qquad \text{(A7)}$$

Similarly, if p_f occurs at y_i^L,

$$\frac{\partial p_f}{\partial \overline{y}_i} = \frac{\partial p_f\left(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial\left(\overline{y}_i - \frac{1}{2}\delta_i\right)} = \frac{\partial p_f}{\partial y_i^L} \qquad \text{(A8)}$$

If p_f occurs at an interior point $\hat{y}_i$,

$$\frac{\partial p_f}{\partial \overline{y}_i} = \frac{\partial p_f\left(\hat{y}_i, \mathbf{Y}_{\sim i}\right)}{\partial \overline{y}_i} = \frac{\partial p_f\left(\hat{y}_i, \mathbf{Y}_{\sim i}\right)}{\partial \hat{y}_i}\frac{\partial \hat{y}_i}{\partial \overline{y}_i} = \frac{\partial p_f\left(\hat{y}_i, \mathbf{Y}_{\sim i}\right)}{\partial \hat{y}_i} \cdot 0 = 0 \qquad \text{(A9)}$$

4. Derivative of p_f bound with respect to a distribution parameter q_i

$$\frac{\partial p_f}{\partial q_i} = \frac{\partial \Phi(-\beta\ )}{\partial q_i} = -\phi(-\beta\ )\frac{\partial \beta}{\partial u_i^*}\frac{\partial u_i^*}{\partial q_i} \qquad \text{(A10)}$$

If the CDF of X_i is $F_{X_i}\left(x_i\right)$, then

$$u_i^* = \Phi^{-1}\left[F_{X_i}\left(x_i^*\right)\right] = w(q_1, q_2, \cdots, q_i, \cdots, q_n) \qquad \text{(A11)}$$

where n is the number of distribution parameters.

Then from $\beta = \left\|\mathbf{u}_i^*\right\|$, one obtains

$$\frac{\partial p_f}{\partial q_i} = -\phi(-\beta\ )\frac{u_i^*}{\beta}\frac{\partial w}{\partial q_i} \tag{A12}$$

APPENDIX B: EQUATIONS FOR TYPE I SENSITIVITY $\partial\delta_p / \partial\delta_i$

Case 1: p_f^L occurs at y_i^L and p_f^U occurs at y_i^U (see Section 3).

Case 2: p_f^L occurs at y_i^U and p_f^U occurs at y_i^L.

$$\frac{\partial\delta_p}{\partial\delta_i} = \frac{\partial\left[p_f^U\left(\overline{y}_i - \frac{1}{2}\delta_i, y_{\sim i}\right) - p_f^L\left(\overline{y}_i + \frac{1}{2}\delta_i, y_{\sim i})\right)\right]}{\partial\delta_i} \tag{B1}$$

Using Eqs. (A3) and (A4) gives

$$\frac{\partial\delta_p}{\partial\delta_i} = -\frac{1}{2}\left(\frac{\partial p_f^U}{\partial y_i^L} + \frac{\partial p_f^L}{\partial y_i^U}\right) \tag{B2}$$

Then from Eq. (A1),

$$\frac{\partial\delta_p}{\partial\delta_i} = -\frac{1}{2}\left(\frac{\partial p_f^U}{\partial y_i^L} + \frac{\partial p_f^L}{\partial y_i^U}\right) = -\frac{1}{2}\left[\frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^L} + \frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^U}\right] \tag{B3}$$

Case 3: p_f^L occurs at an interior point $\hat{y}_i$ and p_f^U occurs at y_i^U.

$$\frac{\partial\delta_p}{\partial\delta_i} = \frac{\partial\left[p_f^U\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) - p_f^L\left(\hat{y}_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial\delta_i} \tag{B4}$$

Using Eqs. (A3) and (A5), one obtains

$$\frac{\partial\delta_p}{\partial\delta_i} = \frac{1}{2}\frac{\partial p_f^U}{\partial y_i^U}. \tag{B5}$$

Applying the results from Eq. (A1) yields

$$\frac{\partial\delta_p}{\partial\delta_i} = \frac{1}{2}\frac{\partial p_f^U}{\partial y_i^U} = \frac{1}{2}\left[\frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^U}\right] \tag{B6}$$

Case 4: p_f^L occurs at an interior point $\hat{y}_i$ and p_f^U occurs at y_i^L.

$$\frac{\partial\delta_p}{\partial\delta_i} = \frac{\partial\left[p_f^U\left(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) - p_f^L\left(\hat{y}_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial\delta_i} \tag{B7}$$

Using Eqs. (A4) and (A5) yields

$$\frac{\partial \delta_p}{\partial \delta_i} = -\frac{1}{2}\frac{\partial p_f^U}{\partial y_i^L} \tag{B8}$$

Applying Eq. (A1) yields

$$\frac{\partial \delta_p}{\partial \delta_i} = -\frac{1}{2}\left[\frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right] \tag{B9}$$

Case 5: p_f^L occurs at y_i^U and p_f^U occurs at an interior point $\hat{y}_i$.

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{\partial\left[p_f^U(\hat{y}_i, \mathbf{Y}_{\sim i}) - p_f^L\left(\bar{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial \delta_i} \tag{B10}$$

Using Eqs. (A3) and (A5), one obtains

$$\frac{\partial \delta_p}{\partial \delta_i} = -\frac{1}{2}\frac{\partial p_f^L}{\partial y_i^U} \tag{B11}$$

Using Eq. (A1) yields

$$\frac{\partial \delta_p}{\partial \delta_i} = -\frac{1}{2}\left[\frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^U}\right] \tag{B12}$$

Case 6: p_f^L occurs at y_i^L and p_f^U occurs at an interior point $\hat{y}_i$.

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{\partial\left[p_f^U(\hat{y}_i, \mathbf{Y}_{\sim i}) - p_f^L\left(\bar{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial \delta_i} \tag{B13}$$

Using Eqs. (A4) and (A5) gives

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{1}{2}\frac{\partial p_f^L}{\partial y_i^L} = \frac{1}{2}\left[\frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right] \tag{B14}$$

Using Eq. (A1) yields

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{1}{2}\left[\frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right] \tag{B15}$$

Case 7: p_f^L and p_f^U occur at two interior points $\hat{y}_{i1}$ and $\hat{y}_{i2}$, respectively.

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{\partial\left[p_f^U(\hat{y}_{i1}, \mathbf{Y}_{\sim i}) - p_f^L\left(\hat{y}_{i2}, \mathbf{Y}_{\sim i}\right)\right]}{\partial \delta_i} \tag{B16}$$

Using Eq. (A5) yields

$$\frac{\partial \delta_p}{\partial \delta_i} = 0\,. \tag{B17}$$

APPENDIX C: EQUATIONS FOR TYPE II SENSITIVITY $\partial\overline{p}_f / \partial\delta_i$

Case 1: p_f^L occurs at y_i^L and p_f^U occurs at y_i^U.

$$\frac{\partial\overline{p}_f}{\partial\delta_i} = \frac{\partial\left\{\frac{1}{2}\left[p_f^U\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) + p_f^L\left(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]\right\}}{\partial\delta_i} \tag{C1}$$

Using Eqs. (A3) and (A4) yields

$$\frac{\partial\overline{p}_f}{\partial\delta_i} = \frac{1}{4}\left(\frac{\partial p_f^U}{\partial y_i^U} - \frac{\partial p_f^L}{\partial y_i^L}\right) \tag{C2}$$

From Eq. (A1)

$$\frac{\partial\overline{p}_f}{\partial\delta_i} = \frac{1}{4}\left(\frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^U} - \frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^L}\right) \tag{C3}$$

Case 2: p_f^L occurs at y_i^U and p_f^U occurs at y_i^L.

$$\frac{\partial\overline{p}_f}{\partial\delta_i} = \frac{\partial\left\{\frac{1}{2}\left[p_f^U\left(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) + p_f^L\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]\right\}}{\partial\delta_i} \tag{C4}$$

Using Eqs. (A3) and (A4) yields

$$\frac{\partial\overline{p}_f}{\partial\delta_i} = \frac{1}{4}\left(-\frac{\partial p_f^U}{\partial y_i^L} + \frac{\partial p_f^L}{\partial y_i^U}\right) \tag{C5}$$

From Eq. (A1)

$$\frac{\partial\overline{p}_f}{\partial\delta_i} = \frac{1}{4}\left[-\frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^L} + \frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^U}\right] \tag{C6}$$

Case 3: p_f^L occurs at an interior point $\hat{y}_i$ and p_f^U occurs at y_i^U.

$$\frac{\partial\overline{p}_f}{\partial\delta_i} = \frac{\partial\left\{\frac{1}{2}\left[p_f^U\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) + p_f^L(\hat{y}_i, \mathbf{Y}_{\sim i})\right]\right\}}{\partial\delta_i} \tag{C7}$$

Using Eqs. (A3) and (A5) yields

$$\frac{\partial\overline{p}_f}{\partial\delta_i} = \frac{1}{4}\frac{\partial p_f^U}{\partial y_i^U} \tag{C8}$$

From Eq. (A1)

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{1}{4}\left[\frac{-\phi(-\beta^U)}{\left\| \nabla g(\mathbf{u}^{*,U}) \right\|} \frac{\partial g}{\partial Y_i}\Big|_{y_i^U} \right] \tag{C9}$$

Case 4: p_f^L occurs at an interior point $\hat{y}_i$ and p_f^U occurs at y_i^L.

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{\partial \left\{ \frac{1}{2}\left[p_f^U\left(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) + p_f^L\left(\hat{y}_i, \mathbf{Y}_{\sim i} \right) \right] \right\}}{\partial \delta_i} \tag{C10}$$

Using Eqs.(A4) and (A5) yields

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = -\frac{1}{4}\frac{\partial p_f^U}{\partial y_i^L} \tag{C11}$$

Applying Eq. (A1), one obtains

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = -\frac{1}{4}\left[\frac{-\phi(-\beta^U)}{\left\| \nabla g(\mathbf{u}^{*,U}) \right\|} \frac{\partial g}{\partial Y_i}\Big|_{y_i^L} \right] \tag{C12}$$

Case 5: p_f^L occurs at y_i^U and p_f^U occurs at an interior point $\hat{y}_i$

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{\partial \left\{ \frac{1}{2}\left[p_f^U(\hat{y}_i, \mathbf{Y}_{\sim i}) + p_f^L(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}) \right] \right\}}{\partial(\delta_i)} \tag{C13}$$

Using Eqs. (A3) and (A5) gives

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{1}{4}\frac{\partial p_f^L}{\partial y_i^U} \tag{C14}$$

Applying Eq. (A1) yields

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{1}{4}\left[\frac{-\phi(-\beta^L)}{\left\| \nabla g(\mathbf{u}^{*,L}) \right\|} \frac{\partial g}{\partial Y_i}\Big|_{y_i^U} \right] \tag{C15}$$

Case 6: p_f^L occurs at y_i^L and p_f^U occurs at an interior point $\hat{y}_i$.

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{\partial \left\{ \frac{1}{2}\left[p_f^U(\hat{y}_i, \mathbf{Y}_{\sim i}) + p_f^L(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}) \right] \right\}}{\partial \delta_i} \tag{C16}$$

Using Eqs. (A4) and (A5) yields

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = -\frac{1}{4}\frac{\partial p_f^L}{\partial y_i^L} \tag{C17}$$

From Eq. (A1)

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = -\frac{1}{4}\left[\frac{-\phi(-\beta^L)}{\left\| \nabla g(\mathbf{u}^{*,L}) \right\|} \frac{\partial g}{\partial Y_i}\Big|_{y_i^L} \right] \tag{C18}$$

Case 7: p_f^L and p_f^U occur at two interior points $\hat{y}_{i1}$ and $\hat{y}_{i2}$, respectively.

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{\partial \left\{ \frac{1}{2}\left[p_f^U(\hat{y}_{i1}, \mathbf{Y}_{\sim i}) + p_f^L\left(\hat{y}_{i2}, \mathbf{Y}_{\sim i}\right)\right]\right\}}{\partial \delta_i} \tag{C19}$$

Using Eq. (A5) yields

$$\frac{\partial \delta_p}{\partial \delta_i} = 0 \tag{C20}$$

APPENDIX D: EQUATIONS FOR TYPE III SENSITIVITY $\partial \delta_p / \partial \overline{y}_i$

Case 1: p_f^L occurs at y_i^L and p_f^U occurs at y_i^U.

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial\left(p_f^U - p_f^L\right)}{\partial \overline{y}_i} = \frac{\partial\left[p_f^U\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) - p_f^L\left(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial \overline{y}_i} \tag{D1}$$

Using Eqs. (A7) and (A8) yields

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial p_f^U}{\partial \overline{y}_i} - \frac{\partial p_f^L}{\partial \overline{y}_i} = \frac{\partial p_f^U}{\partial y_i^U} - \frac{\partial p_f^L}{\partial y_i^L} \tag{D2}$$

From Eq. (A1)

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{-\phi(-\beta^U)}{\left\| \nabla g(\mathbf{u}^{*,U}) \right\|} \frac{\partial g}{\partial Y_i}\Big|_{y_i^U} - \frac{-\phi(-\beta^L)}{\left\| \nabla g(\mathbf{u}^{*,L}) \right\|} \frac{\partial g}{\partial Y_i}\Big|_{y_i^L} \tag{D3}$$

Case 2: p_f^L occurs at y_i^U and p_f^U occurs at y_i^L.

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial\left(p_f^U - p_f^L\right)}{\partial \overline{y}_i} = \frac{\partial\left[p_f^U\left(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) - p_f^L\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial \overline{y}_i} \tag{D4}$$

Using Eqs. (A7) and (A8) gives

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial p_f^U}{\partial \overline{y}_i} - \frac{\partial p_f^L}{\partial \overline{y}_i} = \frac{\partial p_f^U}{\partial y_i^L} - \frac{\partial p_f^L}{\partial y_i^U} \tag{D5}$$

Applying the results of Eq. (A1) yields

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|} \frac{\partial g}{\partial Y_i}\Big|_{y_i^L} - \frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|} \frac{\partial g}{\partial Y_i}\Big|_{y_i^U} \tag{D6}$$

Case 3: p_f^L occurs at an interior point $\widehat{y}_i$ and p_f^U occurs at y_i^U.

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial (p_f^U - p_f^L)}{\partial \overline{y}_i} = \frac{\partial \left[p_f^U \left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) - p_f^L \left(\widehat{y}_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \overline{y}_i} \tag{D7}$$

Using Eqs. (A7) and (A9), one obtains

$$\frac{\partial \delta_p}{\partial \overline{y}_i} == \frac{\partial p_f^U}{\partial \overline{y}_i} = \frac{\partial p_f^U}{\partial y_i^U} \tag{D8}$$

By Eq. (A1)

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|} \frac{\partial g}{\partial Y_i}\Big|_{y_i^U} \tag{D9}$$

Case 4: p_f^L occurs at an interior point $\widehat{y}_i$ and p_f^U occurs at y_i^L.

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial \left(p_f^U - p_f^L \right)}{\partial \overline{y}_i} = \frac{\partial \left[p_f^U \left(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) - p_f^L \left(\widehat{y}_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \overline{y}_i} \tag{D10}$$

Using Eqs. (A8) and (A9) gives

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial p_f^U}{\partial \overline{y}_i} = \frac{\partial p_f^U}{\partial y_i^L} \tag{D11}$$

By Eq. (A1)

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|} \frac{\partial g}{\partial Y_i}\Big|_{y_i^L} \tag{D12}$$

Case 5: p_f^L occurs at y_i^U and p_f^U occurs at an interior point $\widehat{y}_i$.

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial \left(p_f^U - p_f^L \right)}{\partial \overline{y}_i} = \frac{\partial \left[p_f^U \left(\widehat{y}_i, \mathbf{Y}_{\sim i} \right) - p_f^L \left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \overline{y}_i} \tag{D13}$$

Using Eqs. (A7) and (A9) gives

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = -\frac{\partial p_f^L}{\partial \overline{y}_i} = -\frac{\partial p_f^L}{\partial y_i^U} \tag{D14}$$

Using Eq. (A1) yields

$$\frac{\partial \delta_p}{\partial \overline{y}_i} == -\frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^U} \tag{D15}$$

Case 6: p_f^L occurs at y_i^L and p_f^U occurs at an interior point $\hat{y}_i$.

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial\left(p_f^U - p_f^L\right)}{\partial \overline{y}_i} = \frac{\partial\left[p_f^U\left(\hat{y}_i, \mathbf{Y}_{\sim i}\right) - p_f^L\left(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial \overline{y}_i} \tag{D16}$$

Using Eqs. (A8) and (A9) gives

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = -\frac{\partial p_f^L}{\partial \overline{y}_i} = -\frac{\partial p_f^L}{\partial y_i^L} \tag{D17}$$

Using Eq. (A1) yields

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = -\frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^L} \tag{D18}$$

Case 7: p_f^L and p_f^U occur at two interior points $\hat{y}_{i1}$ and $\hat{y}_{i2}$, respectively.

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial\left[p_f^U(\hat{y}_{i1}, \mathbf{Y}_{\sim i}) - p_f^L\left(\hat{y}_{i2}, \mathbf{Y}_{\sim i}\right)\right]}{\partial \overline{y}_i} \tag{D19}$$

Using Eq. (A9) yields

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = 0 \tag{D20}$$

APPENDIX E: EQUATIONS FOR TYPE IV SENSITIVITY $\partial \overline{p}_f / \partial \overline{y}_i$

Case 1: p_f^L occurs at y_i^L and p_f^U occurs at y_i^U.

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{\partial\left(\dfrac{p_f^U + p_f^L}{2}\right)}{\partial \overline{y}_i} = \frac{\partial\left\{\dfrac{1}{2}\left[p_f^U\left(\overline{y}_i + \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) + p_f^L\left(\overline{y}_i - \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]\right\}}{\partial \overline{y}_i} \tag{E1}$$

Using Eqs. (A7) and (A9) gives

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\left(\frac{\partial p_f^U}{\partial \overline{y}_i} + \frac{\partial p_f^L}{\partial \overline{y}_i}\right) = \frac{1}{2}\left(\frac{\partial p_f^U}{\partial y_i^U} + \frac{\partial p_f^L}{\partial y_i^L}\right) \tag{E2}$$

Using Eq. (A1) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\left[\frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^U} + \frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right]. \tag{E3}$$

Case 2: p_f^L occurs at y_i^U and p_f^U occurs at y_i^L.

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{\partial\left(\dfrac{p_f^U + p_f^L}{2}\right)}{\partial \overline{y}_i} = \frac{\partial\left\{\dfrac{1}{2}\left[p_f^U\left(\overline{y}_i - \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) + p_f^L\left(\overline{y}_i + \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]\right\}}{\partial \overline{y}_i} \tag{E4}$$

Using Eqs. (A7) and (A8) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\left(\frac{\partial p_f^U}{\partial \overline{y}_i} + \frac{\partial p_f^L}{\partial \overline{y}_i}\right) = \frac{1}{2}\left(\frac{\partial p_f^U}{\partial y_i^L} + \frac{\partial p_f^L}{\partial y_i^U}\right) \tag{E5}$$

Using Eq. (A1) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\left[\frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^L} + \frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^U}\right] \tag{E6}$$

Case 3: p_f^L occurs at an interior point $\hat{y}_i$ and p_f^U occurs at y_i^U.

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{\partial\left(\dfrac{p_f^U + p_f^L}{2}\right)}{\partial \overline{y}_i} = \frac{\partial\left\{\dfrac{1}{2}\left[p_f^U\left(\overline{y}_i + \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) + p_f^L\left(\hat{y}_i, \mathbf{Y}_{\sim i}\right)\right]\right\}}{\partial \overline{y}_i} \tag{E7}$$

Using Eqs. (A7) and (A9) gives

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\frac{\partial p_f^U}{\partial \overline{y}_i} = \frac{1}{2}\frac{\partial p_f^U}{\partial y_i^U} \tag{E8}$$

Using Eq. (A1) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2} \frac{-\phi(-\beta^U)}{\left\| \nabla g(\mathbf{u}^{*,U}) \right\|} \frac{\partial g}{\partial Y_i}\Big|_{y_i^U} \tag{E9}$$

Case 4: p_f^L occurs at an interior point $\widehat{y}_i$ and p_f^U occurs at y_i^L.

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{\partial \left(\dfrac{p_f^U + p_f^L}{2} \right)}{\partial \overline{y}_i} = \frac{\partial \left\{ \dfrac{1}{2} \left[p_f^U \left(\overline{y}_i - \dfrac{1}{2} \delta_i, \mathbf{Y}_{\sim i} \right) + p_f^L \left(\widehat{y}_i, \mathbf{Y}_{\sim i} \right) \right] \right\}}{\partial \overline{y}_i} \tag{E10}$$

Using Eqs. (A8) and (A9) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2} \frac{\partial p_f^U}{\partial \overline{y}_i} = \frac{1}{2} \frac{\partial p_f^U}{\partial y_i^L} \tag{E11}$$

Using Eq. (A1) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2} \frac{-\phi(-\beta^U)}{\left\| \nabla g(\mathbf{u}^{*,U}) \right\|} \frac{\partial g}{\partial Y_i}\Big|_{y_i^L} \tag{E12}$$

Case 5: p_f^L occurs at y_i^U and p_f^U occurs at an interior point $\widehat{y}_i$.

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{\partial \left(\dfrac{p_f^U + p_f^L}{2} \right)}{\partial \overline{y}_i} = \frac{\partial \left\{ \dfrac{1}{2} \left[p_f^U \left(\widehat{y}_i, \mathbf{Y}_{\sim i} \right) + p_f^L \left(\overline{y}_i + \dfrac{1}{2} \delta_i, \mathbf{Y}_{\sim i} \right) \right] \right\}}{\partial \overline{y}_i} \tag{E13}$$

Using Eqs. (A7) and (A9) gives

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2} \frac{\partial p_f^L}{\partial \overline{y}_i} = \frac{1}{2} \frac{\partial p_f^L}{\partial y_i^U} \tag{E14}$$

Using Eq. (A1) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2} \frac{-\phi(-\beta^L)}{\left\| \nabla g(\mathbf{u}^{*,L}) \right\|} \frac{\partial g}{\partial Y_i}\Big|_{y_i^U} \tag{E15}$$

Case 6: p_f^L occurs at y_i^L and p_f^U occurs at an interior point $\widehat{y}_i$.

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{\partial \left(\dfrac{p_f^U + p_f^L}{2} \right)}{\partial \overline{y}_i} = \frac{\partial \left\{ \dfrac{1}{2} \left[p_f^U \left(\widehat{y}_i, \mathbf{Y}_{\sim i} \right) + p_f^L \left(\overline{y}_i - \dfrac{1}{2} \delta_i, \mathbf{Y}_{\sim i} \right) \right] \right\}}{\partial \overline{y}_i} \tag{E16}$$

Using Eq. (A8) and (A9), one obtains

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\frac{\partial p_f^L}{\partial \overline{y}_i} = \frac{1}{2}\frac{\partial p_f^L}{\partial y_i^L} \tag{E17}$$

Applying Eq. (A1) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^L} \tag{E18}$$

Case 7: p_f^L and p_f^U occur at two interior points $\hat{y}_{i1}$ and $\hat{y}_{i2}$, respectively.

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{\partial\left\{\frac{1}{2}\left[p_f^U(\hat{y}_{i1}, \mathbf{Y}_{\sim i}) + p_f^L\left(\hat{y}_{i2}, \mathbf{Y}_{\sim i}\right)\right]\right\}}{\partial \overline{y}_i} \tag{E19}$$

Using Eq. (A9) gives

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = 0 \tag{E20}$$

APPENDIX F: EQUATIONS FOR TYPE V SENSITIVITY $\partial\delta_p / \partial q_i$

$$\frac{\partial \delta_p}{\partial q_i} = \frac{\partial(p_f^U - p_f^L)}{\partial q_i} = \frac{\partial p_f^U}{\partial q_i} - \frac{\partial p_f^L}{\partial q_i} \tag{F1}$$

Using Eq. (A12) gives

$$\frac{\partial \delta_p}{\partial q_i} = -\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{\partial w}{\partial q_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{\partial w}{\partial q_i} \tag{F2}$$

where $u_i^{*,U}$ is the MPP at p_f^U and $u_i^{*,L}$ is the MPP at p_f^L.

Specifically, for a normal distributed random variable $X_i \sim N(\mu_i, \sigma_i)$,

$$w(\mu_i, \sigma_i) = \Phi^{-1}\left[F_{X_i}\left(x_i^*\right)\right] = \Phi^{-1}\left[\Phi\left(\frac{x_i^* - \mu_i}{\sigma_i}\right)\right] = \frac{x_i^* - \mu_i}{\sigma_i}, \tag{F3}$$

so it can be obtained that

$$\frac{\partial w}{\partial \mu_i} = -\frac{1}{\sigma_i}, \frac{\partial w}{\partial \sigma_i} = -\frac{x_i^* - \mu_i}{\sigma_i^2} = -\frac{u_i^*}{\sigma_i}. \tag{F4}$$

Therefore, from Eq. (F2), we can obtain the following sensitivities.

1) $q_i = \mu_i$

$$\frac{\partial \delta_p}{\partial \mu_i} = -\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{\partial w}{\partial \mu_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{\partial w}{\partial \mu_i} = \phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U \sigma_i} - \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L \sigma_i} \quad \text{(F5)}$$

2) $q_i = \sigma_i$

$$\begin{aligned} \frac{\partial \delta_p}{\partial \sigma_i} &= -\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{\partial w}{\partial \sigma_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{\partial w}{\partial \sigma_i} \\ &= \phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{u_i^{*,U}}{\sigma_i} - \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{u_i^{*,L}}{\sigma_i} = \phi(-\beta^L)\frac{(u_i^{*,U})^2}{\beta^L \sigma_i} - \phi(-\beta^L)\frac{(u_i^{*,L})^2}{\beta^L \sigma_i} \end{aligned} \quad \text{(F6)}$$

APPENDIX G: EQUATIONS FOR TYPE VI SENSITIVITY $\partial \overline{p}_f / \partial q_i$

$$\frac{\partial(\overline{p}_f)}{\partial q_i} = \frac{\partial\left(\frac{p_f^U + p_f^L}{2}\right)}{\partial q_i} = \frac{1}{2}\left(\frac{\partial p_f^U}{\partial q_i} + \frac{\partial p_f^L}{\partial q_i}\right) \quad \text{(G1)}$$

Using Eq. (A12), it can be easily shown that

$$\frac{\partial \overline{p}_f}{\partial q_i} = -\frac{1}{2}\left[\phi(-\beta^U)\frac{u_i^{*,L}}{\beta^U}\frac{\partial w}{\partial q_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{\partial w}{\partial q_i}\right] \quad \text{(G2)}$$

Applying the results from Eq. (F4) for a normal distributed random variable $X_i \sim N(\mu_i, \sigma_i)$, the following sensitivities are obtained.

1) $q_i = \mu_i$

$$\begin{aligned} \frac{\partial \overline{p}_f}{\partial \mu_i} &= -\frac{1}{2}\left[\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{\partial w}{\partial \mu_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{\partial w}{\partial \mu_i}\right] \\ &= \frac{1}{2}\left[\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U \sigma_i} + \phi(-\beta^U)\frac{u_i^{*,L}}{\beta^U \sigma_i}\right] \end{aligned} \quad \text{(G3)}$$

2) $q_i = \sigma_i$

$$\begin{aligned} \frac{\partial \overline{p}_f}{\partial \sigma_i} &= -\frac{1}{2}\left[\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{\partial w}{\partial \sigma_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{\partial w}{\partial \sigma_i}\right] \\ &= \frac{1}{2}\left[\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{u_i^{*,U}}{\sigma_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{u_i^{*,L}}{\sigma_i}\right] = \frac{1}{2}\left[\phi(-\beta^U)\frac{(u_i^{*,U})^2}{\beta^U \sigma_i} + \phi(-\beta^L)\frac{(u_i^{*,L})^2}{\beta^L \sigma_i}\right] \end{aligned} \quad \text{(G4)}$$

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Table I. Six sensitivity indices

Sensitivity type	Description	Input
Type I $\partial\delta_p / \partial\delta_i$	Sensitivity of the width of the p_f bounds, δ_p, with respect to the width of interval variable Y_i, δ_i	Interval
Type II $\partial\overline{p}_f / \partial\delta_i$	Sensitivity of the average p_f, $\overline{p}_f$, with respect to the width of interval variable Y_i, δ_i	Interval
Type III $\partial\delta_p / \partial\overline{y}_i$	Sensitivity of the width of the p_f bounds, δ_p, with respect to the average of interval variable Y_i, $\overline{y}_i$	Interval
Type IV $\partial\overline{p}_f / \partial\overline{y}_i$	Sensitivity of the average p_f, $\overline{p}_f$, with respect to the average of interval variable Y_i, $\overline{y}_i$	Interval
Type V $\partial\delta_p / \partial q_i$	Sensitivity of the width of the p_f bounds, δ_p, with respect to a distribution parameter, q_i, of random variable X_i	Random
Type VI $\partial\overline{p}_f / \partial q_i$	Sensitivity of the average p_f, $\overline{p}_f$, with respect to a distribution parameter, q_i, of random variable X_i	Random

Table II. Type I and II sensitivities for intervals

Case		Type I $\partial \delta_p / \partial \delta_i$ (Appendix B)	Type II $\partial \overline{p}_f / \partial \delta_i$ (Appendix C)
1	p_f^U occurs at y_i^U; p_f^L occurs at y_i^L	$-\frac{1}{2}\left[\frac{\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^U} + \frac{\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^L}\right]$	$-\frac{1}{4}\left[\frac{\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^U} - \frac{\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^L}\right]$
2	p_f^U occurs at y_i^L; p_f^L occurs at y_i^U	$\frac{1}{2}\left[\frac{\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^L} + \frac{\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^U}\right]$	$\frac{1}{4}\left[\frac{\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^L} - \frac{\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^U}\right]$
3	p_f^U occurs at y_i^U; p_f^L occurs at an interior point	$-\frac{1}{2}\left[\frac{\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^U}\right]$	$-\frac{1}{4}\left[\frac{\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^U}\right]$
4	p_f^U occurs at y_i^L; p_f^L occurs at an interior point	$-\frac{1}{2}\left[\frac{\phi(\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^L}\right]$	$\frac{1}{4}\left[\frac{\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^L}\right]$
5	p_f^L occurs at y_i^U; p_f^U occurs at an interior point	$\frac{1}{2}\left[\frac{\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^U}\right]$	$-\frac{1}{4}\left[\frac{\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^U}\right]$
6	p_f^L occurs at y_i^L; p_f^U occurs at an interior point	$-\frac{1}{2}\left[\frac{\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^L}\right]$	$\frac{1}{4}\left[\frac{\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\Big\|_{y_i^L}\right]$
7	p_f^U and p_f^L both occurs at interior points	0	0

Table III. Type III and IV sensitivities for intervals

Case		Type III $\partial\delta_p / \partial\overline{y}_i$ (Appendix D)	Type IV $\partial\overline{p}_f / \partial\overline{y}_i$ (Appendix E)
1	p_f^U occurs at y_i^U; p_f^L occurs at y_i^L	$\frac{-\phi(-\beta^U)}{\left\lVert\nabla g(\mathbf{u}^{*,U})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^U} + \frac{\phi(-\beta^L)}{\left\lVert\nabla g(\mathbf{u}^{*,L})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^L}$	$-\frac{1}{2}\left[\frac{\phi(-\beta^U)}{\left\lVert\nabla g(\mathbf{u}^{*,U})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^U} + \frac{\phi(-\beta^L)}{\left\lVert\nabla g(\mathbf{u}^{*,L})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^L}\right]$
2	p_f^U occurs at y_i^L; p_f^L occurs at y_i^U	$\frac{-\phi(-\beta^U)}{\left\lVert\nabla g(\mathbf{u}^{*,U})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^L} + \frac{\phi(-\beta^L)}{\left\lVert\nabla g(\mathbf{u}^{*,L})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^U}$	$-\frac{1}{2}\left[\frac{\phi(-\beta^U)}{\left\lVert\nabla g(\mathbf{u}^{*,U})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^L} + \frac{\phi(-\beta^L)}{\left\lVert\nabla g(\mathbf{u}^{*,L})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^U}\right]$
3	p_f^U occurs at y_i^U; p_f^L occurs at an interior point	$\frac{-\phi(-\beta^U)}{\left\lVert\nabla g(\mathbf{u}^{*,U})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^U}$	$-\frac{1}{2}\left[\frac{\phi(-\beta^U)}{\left\lVert\nabla g(\mathbf{u}^{*,U})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^U}\right]$
4	p_f^U occurs at y_i^L; p_f^L occurs at an interior point	$\frac{-\phi(-\beta^U)}{\left\lVert\nabla g(\mathbf{u}^{*,U})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^L}$	$-\frac{1}{2}\left[\frac{\phi(-\beta^U)}{\left\lVert\nabla g(\mathbf{u}^{*,U})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^L}\right]$
5	p_f^L occurs at y_i^U; p_f^U occurs at an interior point	$\frac{\phi(-\beta^L)}{\left\lVert\nabla g(\mathbf{u}^{*,L})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^U}$	$-\frac{1}{2}\left[\frac{\phi(-\beta^L)}{\left\lVert\nabla g(\mathbf{u}^{*,L})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^U}\right]$
6	p_f^L occurs at y_i^L; p_f^U occurs at an interior point	$\frac{\phi(-\beta^L)}{\left\lVert\nabla g(\mathbf{u}^{*,L})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^L}$	$-\frac{1}{2}\left[\frac{\phi(-\beta^L)}{\left\lVert\nabla g(\mathbf{u}^{*,L})\right\rVert}\frac{\partial g}{\partial Y_i}\Big\rvert_{y_i^L}\right]$
7	p_f^U and p_f^L both occurs at interior points	0	0

Table IV. Type V and VI sensitivities for random variables

Case	Type V $\partial\delta_p / \partial q$ (Appendix F)	Type VI $\partial\overline{p}_f / \partial q$ (Appendix G)
General	$-\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{\partial w}{\partial q_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{\partial w}{\partial q_i}$	$-\frac{1}{2}\left[\phi(-\beta^U)\frac{u_i^{*,L}}{\beta^U}\frac{\partial w}{\partial q_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{\partial w}{\partial q_i}\right]$

Table V. Random Variables X

Variable	Mean	Standard deviation	Distribution
X_1 (E_o)	10×10^{6}psi	0.1×10^{6}psi	Normal
X_2 (E_i)	30×10^{6}psi	0.3×10^{6}psi	Normal
X_3 (t_o)	0.15 in	0.0015 in	Normal
X_4 (t_i)	0.10 in	0.001 in	Normal
X_5 (G)	0.2×10^{6}psi	0.002×10^{6}psi	Normal
X_6 (b)	1 in	0.01 in	Normal
X_7 (L)	1.1 in	0.011 in	Normal
X_8 (P)	2000 psi	20 psi	Normal
X_9 (S_a)	4100 psi	41 psi	Normal

Table VI. Interval variables

Variable	Lower bound	Upper bound
Y_1 (h)	0.0195 in	0.0205 in
Y_2 (ΔT)	-131.0 °F	-129.0 °F

Table VII. Bounds of the probability of failure

Probability of Failure	p_f^L	p_f^U	$\overline{p}_f$	δ_p
p_f	7.797×10^{-5}	1.067×10^{-2}	5.374×10^{-3}	1.059×10^{-2}

Table VIII. Sensitivity with respect to interval variables

Type of sensitivity	Proposed method		Numerical verification	
	Y_1	Y_2	Y_1	Y_2
Type I $\partial\delta_p / \partial\delta_i$	5.009×10^{-2}	9.309×10^{-6}	5.071×10^{-2}	9.379×10^{-6}
Type II $\partial\overline{p}_f / \partial\delta_i$	-2.494×10^{-2}	-4.655×10^{-6}	-2.525×10^{-2}	-4.669×10^{-6}
Type III $\partial\delta_p / \partial\overline{y}_i$	-9.978×10^{-2}	-1.862×10^{-5}	-1.001×10^{-1}	-1.868×10^{-5}
Type IV $\partial\overline{p}_f / \partial\overline{y}_i$	5.009×10^{-2}	9.309×10^{-6}	5.071×10^{-2}	9.379×10^{-6}

Table IX. The change of δ_p and $\overline{p}_f$ with 1% increases in δ_i and $\overline{y}_i$

Type of sensitivity	$\Delta_1^{+1\%}$	$\Delta_2^{+1\%}$
Type I $\partial\delta_p / \partial\delta_i$	5.009×10^{-7}	1.862×10^{-7}
Type II $\partial\overline{p}_f / \partial\delta_i$	-2.494×10^{-7}	-9.310×10^{-8}
Type III $\partial\delta_p / \partial\overline{y}_i$	-1.996×10^{-3}	2.241×10^{-5}
Type IV $\partial\overline{p}_f / \partial\overline{y}_i$	1.002×10^{-3}	-1.210×10^{-5}

Table X. Sensitivity with respect random Variables

	Proposed method		Numerical validation	
	Type V $\partial\delta_p/\partial q$	Type VI $\partial\overline{p}_f/\partial q$	Type V $\partial\delta_p/\partial q$	Type VI $\partial\overline{p}_f/\partial q$
$X_1(\mu_1)$	1.091×10^{-10}	5.475×10^{-11}	1.097×10^{-10}	5.509×10^{-11}
$X_1(\sigma_1)$	4.566×10^{-11}	2.295×10^{-11}	4.631×10^{-11}	2.328×10^{-11}
$X_2(\mu_2)$	-2.097×10^{-11}	-1.054×10^{-11}	-2.083×10^{-11}	-1.046×10^{-11}
$X_2(\sigma_2)$	5.066×10^{-12}	2.550×10^{-12}	4.997×10^{-12}	2.515×10^{-12}
$X_3(\mu_3)$	7.270×10^{-3}	3.650×10^{-3}	7.315×10^{-3}	3.673×10^{-3}
$X_3(\sigma_3)$	3.044×10^{-3}	1.530×10^{-3}	3.087×10^{-3}	1.552×10^{-3}
$X_4(\mu_4)$	-6.292×10^{-3}	-3.161×10^{-3}	-6.250×10^{-3}	-3.139×10^{-3}
$X_4(\sigma_4)$	1.520×10^{-3}	7.650×10^{-4}	1.499×10^{-3}	7.546×10^{-4}
$X_5(\mu_5)$	9.818×10^{-9}	4.931×10^{-9}	9.780×10^{-9}	4.911×10^{-9}
$X_5(\sigma_5)$	7.402×10^{-9}	3.723×10^{-9}	7.330×10^{-9}	3.687×10^{-9}
$X_6(\mu_6)$	-1.913×10^{-3}	-9.608×10^{-4}	-1.912×10^{-3}	-9.601×10^{-4}
$X_6(\sigma_6)$	1.405×10^{-3}	7.068×10^{-4}	1.404×10^{-3}	7.060×10^{-4}
$X_7(\mu_7)$	-8.018×10^{-3}	-4.026×10^{-3}	-8.011×10^{-3}	-4.022×10^{-3}
$X_7(\sigma_7)$	2.715×10^{-2}	1.365×10^{-2}	2.731×10^{-2}	1.373×10^{-2}
$X_8(\mu_8)$	9.421×10^{-7}	4.731×10^{-7}	9.428×10^{-7}	4.735×10^{-7}
$X_8(\sigma_8)$	6.815×10^{-7}	3.428×10^{-7}	6.817×10^{-7}	3.429×10^{-7}
$X_9(\mu_9)$	-1.079×10^{-6}	-5.417×10^{-7}	-1.078×10^{-6}	-5.411×10^{-7}
$X_9(\sigma_9)$	1.832×10^{-6}	9.213×10^{-7}	1.831×10^{-6}	9.210×10^{-7}

Table XI. Random Variables

Variable	Parameter 1	Parameter 2	Distribution
X_1 (t)	5 mm (mean)	0.1 mm (std*)	Normal
X_2 (d)	42 mm (mean)	0.5 mm (std)	Normal
X_3 (L_1)	119.75 mm (lb**)	120.25 mm (ub***)	Uniform
X_4 (L_2)	59.75 mm (lb)	60.25 mm (ub)	Uniform
X_5 (F_1)	3.0 kN (mean)	0.3 kN (std)	Normal
X_6 (F_2)	3.0 kN (mean)	0.3 kN (std)	Normal
X_7 (P)	12.0 kN (mean)	1.2 kN (std)	Gumbel
X_8 (T)	90.0 N·m (mean)	9.0 N·m (std)	Normal
X_9 (S_y)	220.0 MPa (mean)	22.0 MPa (std)	Normal

*: std – standard deviation

**: lb – the lower bound of a uniform distribution

***: ub – the upper bound of a uniform distribution

Table XII. Interval Variables

Variable	Lower bound	Upper bound
Y_1 (θ_1)	0°	10°
Y_2 (θ_2)	5°	15°

Table XIII. Bounds of Probability of Failure

Probability of Failure	p_f^L	p_f^U	$\overline{p}_f$	δ_p
p_f	1.437×10^{-4}	1.631×10^{-4}	1.530×10^{-4}	1.940×10^{-5}

Table XIV. Sensitivity of Interval Variables

Type of sensitivity	Proposed Method		Numerical Validation	
	Y_1	Y_2	Y_1	Y_2
Type I $\partial\delta_p / \partial\delta_i$	1.038×10^{-4}	5.861×10^{-5}	1.034×10^{-4}	5.837×10^{-5}
Type II $\partial\overline{p}_f / \partial\delta_i$	-5.192×10^{-5}	-2.930×10^{-5}	-5.170×10^{-5}	-2.919×10^{-5}
Type III $\partial\delta_p / \partial\overline{y}_i$	2.077×10^{-4}	1.172×10^{-4}	2.068×10^{-4}	1.167×10^{-4}
Type IV $\partial\overline{p}_f / \partial\overline{y}_i$	-1.038×10^{-4}	-5.861×10^{-5}	-1.034×10^{-4}	-5.837×10^{-5}

Table XV. The Change of δ_p and $\overline{p}_f$ with 1% Increases in δ_i and $\overline{y}_i$

Type of sensitivity	$\Delta_{y_1}^{+1\%}$	$\Delta_{y_2}^{+1\%}$
Type I $\partial\delta_p / \partial\delta_i$	1.038×10^{-5}	5.861×10^{-6}
Type II $\partial\overline{p}_f / \partial\delta_i$	-5.192×10^{-6}	-2.930×10^{-6}
Type III $\partial\delta_p / \partial\overline{y}_i$	1.039×10^{-5}	5.860×10^{-6}
Type IV $\partial\overline{p}_f / \partial\overline{y}_i$	-5.190×10^{-6}	-2.931×10^{-6}

Table XVI. Sensitivity of random variables

	Proposed Method		Numerical Validation	
	Type V $\partial\delta_p / \partial q$	Type VI $\partial\overline{p}_f / \partial q$	Type V $\partial\delta_p / \partial q$	Type VI $\partial\overline{p}_f / \partial q$
$X_1(\mu_1)$	-5.822×10^{-2}	-4.886×10^{-1}	-5.820×10^{-2}	-4.886×10^{-1}
$X_1(\sigma_1)$	1.614×10^{-2}	1.457×10^{-1}	1.615×10^{-2}	1.458×10^{-1}
$X_2(\mu_2)$	-2.413×10^{-2}	-1.888×10^{-1}	-2.413×10^{-2}	-1.888×10^{-1}
$X_2(\sigma_2)$	1.393×10^{-2}	1.088×10^{-1}	1.394×10^{-2}	1.089×10^{-1}
$X_3(a_3)$	1.093×10^{-3}	8.412×10^{-3}	1.093×10^{-3}	8.413×10^{-3}
$X_3(b_3)$	1.130×10^{-3}	8.697×10^{-3}	1.137×10^{-3}	8.742×10^{-3}
$X_4(a_4)$	1.123×10^{-3}	7.893×10^{-3}	1.124×10^{-3}	7.894×10^{-3}
$X_4(b_4)$	1.162×10^{-3}	8.143×10^{-3}	1.167×10^{-3}	8.163×10^{-3}
$X_5(\mu_5)$	7.630×10^{-8}	6.197×10^{-7}	7.631×10^{-8}	6.197×10^{-7}
$X_5(\sigma_5)$	8.347×10^{-8}	7.033×10^{-7}	8.355×10^{-8}	7.040×10^{-7}
$X_6(\mu_6)$	3.908×10^{-8}	3.117×10^{-7}	3.908×10^{-8}	3.117×10^{-7}
$X_6(\sigma_6)$	2.192×10^{-8}	1.779×10^{-7}	2.193×10^{-8}	1.780×10^{-7}
$X_7(\mu_7)$	5.002×10^{-9}	4.256×10^{-8}	5.002×10^{-9}	4.256×10^{-8}
$X_7(\sigma_7)$	5.139×10^{-10}	5.670×10^{-9}	5.143×10^{-10}	5.674×10^{-9}
$X_8(\mu_8)$	5.678×10^{-8}	5.050×10^{-7}	5.688×10^{-8}	5.049×10^{-7}
$X_8(\sigma_8)$	1.363×10^{-9}	1.402×10^{-8}	1.363×10^{-9}	1.402×10^{-8}
$X_9(\mu_9)$	-2.887×10^{-12}	-2.457×10^{-11}	-2.886×10^{-12}	-2.457×10^{-11}
$X_9(\sigma_9)$	8.708×10^{-12}	8.108×10^{-11}	8.740×10^{-12}	8.146×10^{-11}