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Robust Mechanism Synthesis with Random and Interval Variables

Dr. Xiaoping Du*
Associate Professor
Department of Mechanical and Aerospace Engineering
Missouri University of Science and Technology
U.S.A.

Pavan Kumar Venigella
Graduate Research Assistant
Department of Mechanical and Aerospace Engineering
Missouri University of Science and Technology
U.S.A.

Dr. Deshun Liu
Professor
Hunan University of Science and Technology
P.R. China

***Corresponding Author**
Department of Mechanical and Aerospace Engineering
Missouri University of Science and Technology
1870 Miner Circle
Rolla, MO 65409-4494
Phone: 01-573-341-7249
Fax: 01-573-341-4115
E-mail: dux@mst.edu

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ABSTRACT

Robust mechanism synthesis minimizes the impact of uncertainties on the mechanism performance. It has traditionally been performed by either a probabilistic approach or a worst case approach. Both approaches treat uncertainty as either random variables or interval variables. In reality, uncertainty can be a mixture of both. In this paper, methods are developed for robustness assessment and robust mechanism synthesis when random and interval variables are involved. Monte Carlo simulation is used to perform robustness assessment under an optimization framework for mechanism synthesis.

1. INTRODUCTION

Kinematic synthesis is the process of systematic design of a mechanism to achieve a specific task [1]. The task can be motion generation, path generation, or function generation. Kinematic synthesis is classified into two groups, type synthesis and dimensional synthesis. Type synthesis deals with finding the best suitable mechanism (cam mechanisms, linkages, gear trains, etc), number of links, degree of freedom and so on, to achieve the required performance. Dimensional synthesis deals with determining the significant dimensions of the mechanism to achieve a specific task. In this paper, we concentrate on dimensional synthesis of a mechanism.

Mathematical techniques such as algebraic method, matrix method, and complex numbers are used to model mechanism synthesis problems [1]. After building a mathematical model, optimization techniques can be used to obtain an optimal solution. In the traditional optimization method, the difference between the desired performance and the actual performance of a mechanism is minimized [2-5]. The traditional deterministic mechanism synthesis does not consider any uncertainties in the mechanism and its environment. But in reality, uncertainties exist. Examples of uncertainty include the randomness in dimensions and variations in external forces [6-10]. Due to these uncertainties, the actual mechanism performance is subjected to variations around the designed performance.

Robust design, introduced by Taguchi [11, 12], is a powerful design method for achieving high quality and productivity. Robust design tries to achieve a minimum variation in the performance by controlling design variables without eliminating the cause of uncertainty [13-23]. The objective of robust design is to optimize the mean performance and minimize the performance variation due to uncertainties. The former can be achieved by finding the relation between the mean performance and the design variables. Here the challenge lies in the latter where we need to precisely quantify the performance variation due to the uncertainties, and the task is known as robustness assessment [19]. By assessing the robustness accurately and reducing variations in the performance, robust design ensures that a product perform its intended function regardless the uncertainties.

Every mechanism is subjected to uncertainties. Uncertainties can be in the form of dimensional tolerances in the links, clearances in the joints and so on. The output of the mechanism is affected due to the uncertainties. Probabilistic, fuzzy, and interval methods are generally used to model the uncertainties in an engineering system. The probabilistic method describes an uncertain parameter as a random variable following a specific probability distribution [24-28]. If the information is not sufficient to form a probability distribution, interval approach or fuzzy theory can be used. In interval approach, the uncertainty of a parameter is denoted by a simple range [29-32]. In fuzzy theory, the desirability of using different values within the range is described by a membership function to the range [31, 33]. The interval approach can be conveniently used when there is no sufficient information available about the probability distribution of the uncertain variable.

As shown in literature [34-40], in many engineering applications, uncertain variables can be in the form of random variables and interval variables at the same time. In this paper, methods are developed to quantify robustness in such a situation. A robust mechanism synthesis method is proposed considering uncertainties with both random and interval variables. In Section 2, robustness assessment is presented followed by robust mechanism synthesis in Section 3. Two examples are given in Section 4. Section 5 is the closure of the paper.

2. ROBUSTNESS ASSESSMENT

2.1 Parameter uncertainty

Uncertainty is the difference between the model prediction and actuality. Uncertainty could occur in many ways in a system, for example, in the parameters of a mathematical model of a system or in the sequence of possible events in a discrete event system. Uncertainty is generally distinguished as aleatory uncertainty and epistemic uncertainty [35].

Aleatory uncertainty, also termed as objective or stochastic uncertainty, describes the inherent variation associated with the physical system or the environment under consideration. Epistemic uncertainty is described mainly as the lack of knowledge or information in any phase or operation of a design process [10].

Parameter uncertainty can be aleatory (due to inherent variation) or epistemic (due to limited information) in the physical system or environment in assessing the parameter characteristics [37]. If the parameter uncertainty is aleatory in nature, a probabilistic approach can be used to model uncertainty. In the probabilistic approach, uncertainty is treated as random variables following specific probability distributions. If the parameter uncertainty is epistemic in nature, an interval approach can be used to model the uncertainty. In the interval approach, uncertainty may be denoted by simple ranges.

2.2 Robustness Assessment with Only Random Variables

Mathematically, robustness is measured by the variance (or standard deviation) of the performance [2, 17]. Let a random variable Z be a response variable that represents a performance in mechanism synthesis as shown in Fig. 1 and be in the form of

$$Z = g(\boldsymbol{X}) \tag{1}$$

where $\boldsymbol{X} = \left(X_1, X_2, \cdots, X_{n_X}\right)$ is a vector consisting of n_X random variables.

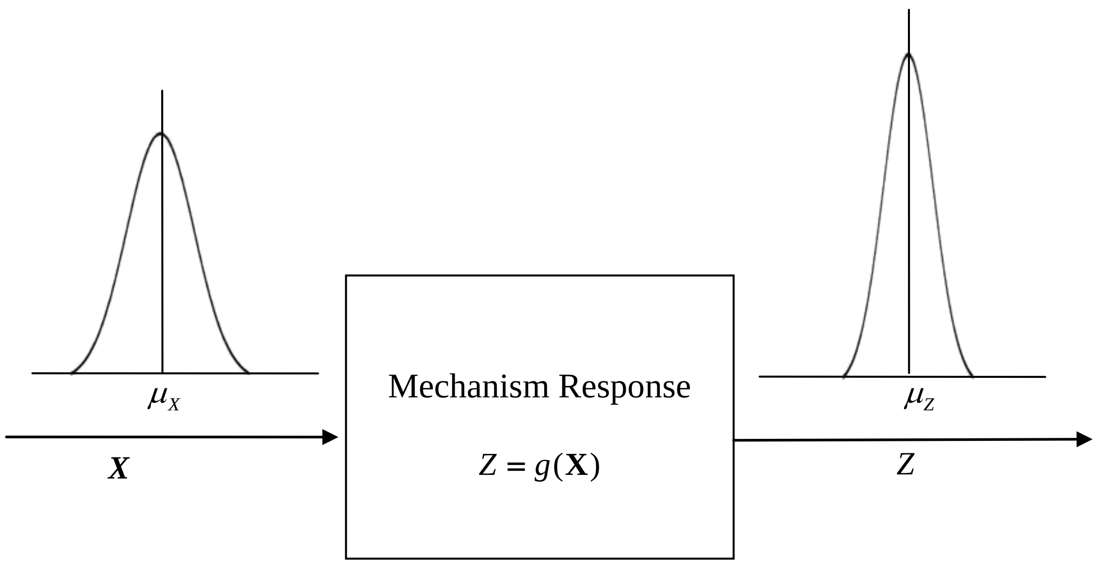

Fig. 1. Robustness assessment with only random variables

In this paper, all the random variables in $\mathbf{X} = \left(X_1, X_2, \cdots, X_{n_X}\right)$ are assumed to be independent. The methods discussed in this paper can be extended to correlated random variables. The elements of $\mathbf{X}$ contain both design variables (e.g. dimensions of a mechanism) that can be controllable by a designer and noise factors that are uncontrollable (e.g. external forces).

Theoretically, the variance σ_Z^2 of Z is calculated by

$$\sigma_Z^2 = E\left[\left(Z-\mu_Z\right)^2\right] = \int_{-\infty}^{\infty}\left[g(\mathbf{x})-\mu_Z\right]^2 f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{2}$$

where E stands for expectation, $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function (PDF) of **X**, and μ_Z is the mean of Z, which is computed by

$$\mu_Z = E[Z] = \int_{-\infty}^{\infty} g(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{3}$$

Due to high dimensionality, analytical solutions to both of the above equations are difficult to obtain. Many approximation methods [17, 19] have been proposed, including Monte Carlo simulation (MCS). MCS is the simplest method and results in accurate estimations. In this paper we use MCS.

The response variable Z can be evaluated from Eq. (1). The estimate of the mean and variance of Z is calculated from the samples of Z obtained from MCS. The equations are:

$$\sigma_Z^2 \cong \frac{1}{N-1}\sum_{i=1}^{N}\left[g(\mathbf{x}_i)-\mu_Z\right]^2 \tag{4}$$

where the mean μ_z is estimated by

$$\mu_Z \cong \frac{1}{N}\sum_{i=1}^{N} g(\mathbf{x}_i) \tag{5}$$

and $\mathbf{x}_i$ $(i = 1, 2, \cdots, N)$ are the samples of random vector **X**, which are drawn from the distributions of **X**. N is the number of samples (simulations). The accuracy of MCS depends on the number of simulations N. A large number of simulations must be performed to achieve an accurate estimate.

The robustness of a system is assessed by the standard deviation σ_Z. For a robust system, a low σ_Z value with μ_Z equal to a desired target value is to be achieved. Consider two designs, Design A and Design B, as shown in Fig. 2. Let the two designs be subjected to similar conditions. Both the designs met the primary requirement of mean value μ_Z, which is equal to the desired target value. From Fig. 2, it is evident that σ_Z of both the designs are different. σ_{Z_A} (standard deviation of Design A) is less than σ_{Z_B} (standard deviation of Design B). This suggests that Design A is more robust than Design B.

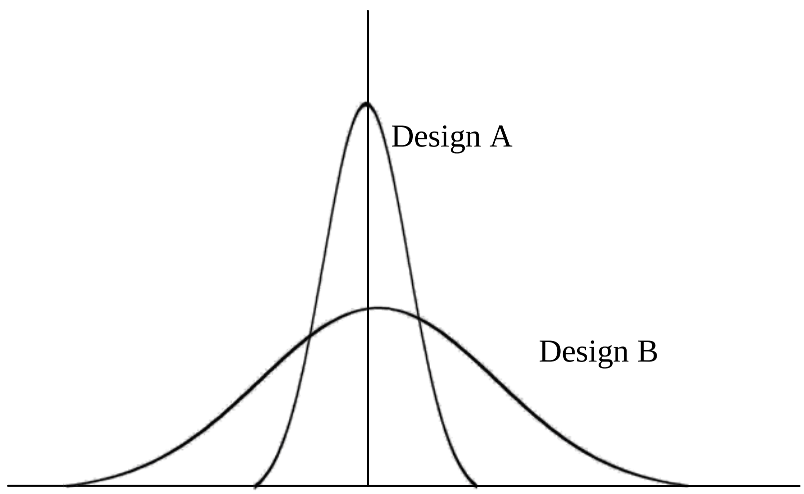

Fig. 2. Robustness comparison between Design A and Design B

2.3 Robustness Assessment with Only Interval Variables

Mathematically, robustness is measured by the width of the interval of the performance [30, 31]. Let Z be a response variable that represents a performance of a mechanism as shown in Fig. 3 and be in the form of

$$Z = g(\mathbf{Y}) \tag{6}$$

where $\mathbf{Y} = (Y_1, Y_2, ..., Y_{n_y})$ is a vector consisting of n_Y interval variables.

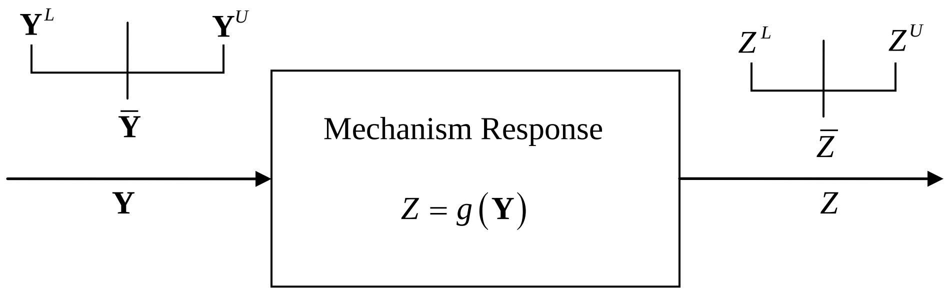

Fig. 3. Robustness assessment with only interval variables

In this paper, all the interval variables in $\mathbf{Y} = (Y_1, Y_2, ..., Y_{n_y})$ are assumed to be independent. The elements of $\mathbf{Y}$ can be design variables and noise factors. $\mathbf{Y}$ resides over its interval $\left[\mathbf{Y}^L, \mathbf{Y}^U\right]$. The midpoint, $\bar{Z}$ and width, δZ, of the interval Z, are calculated by

$$\bar{Z} = \frac{1}{2}\left(Z^U + Z^L\right) \tag{7}$$

and

$$\delta Z = Z^U - Z^L \tag{8}$$

respectively, where Z^U and Z^L represents the upper bound and lower bound of Z, respectively.

The robustness of the system is assessed by δZ. δZ should be as low as possible, while $\bar{Z}$ is equal to the desired target value. Consider two designs as shown in Fig. 4 subjected to similar conditions. The midpoints for both of the designs are equal to $\bar{Z}$, satisfying the mean requirement. Now, comparing the widths of the intervals for the two designs, we see that δZ_B is narrower than δZ_A. Design B is therefore more robust than Design A.

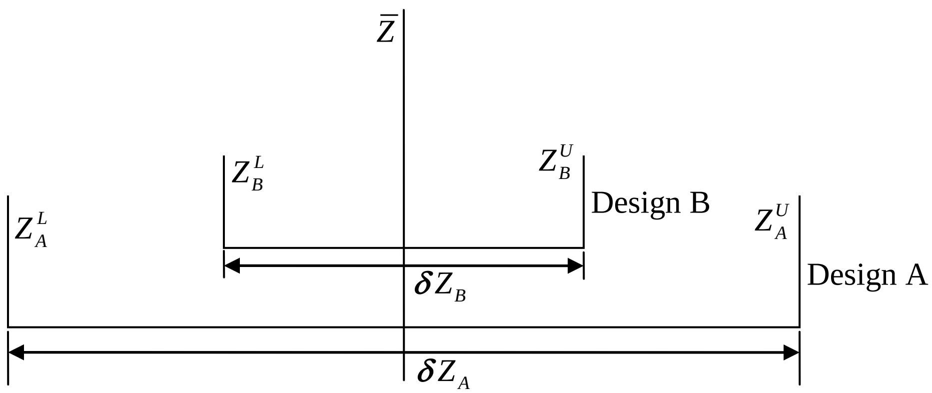

Fig. 4. Robustness assessment between Design A and Design B

2.4 Robustness Assessment with the Mixture of Random and Interval Variables

As shown in the last two subsections, the current robustness assessment methodologies usually treat uncertainties as random variables or as interval variables. However, in many practical engineering applications both random variables and interval variables exist at the same time. The focus of this work is to deal with the presence of both random and interval variables for mechanism synthesis. The details are given below and in Section 3.

When the distributions of the design variables are precisely known, the design variables can be treated as random variables. If the evaluation of the probabilistic characteristics of design variables is prohibitively expensive or may not be precisely known, the design variables can be treated as interval variables.

When both random variables $\mathbf{X} = \left(X_1, X_2, \cdots, X_{n_X}\right)$ and interval variables $\mathbf{Y} = \left(Y_1, Y_2, \cdots, Y_{n_Y}\right)$ exist, the model becomes

$$Z = g(\mathbf{X}, \mathbf{Y}) \tag{9}$$

Fig. 5 explains the existence of both random and interval variables in the design model. Consider a response variable Z, which is dependent on the random variables $\mathbf{X} = \left(X_1, X_2\right)$ and interval variables $\mathbf{Y} = \left(Y_1, Y_2\right)$. Let us exam all of the four combinations of the interval bounds for both the interval variables. In each combination, Z has a probability distribution as shown in Fig. 6. The mean values and standard deviations of Z are also intervals.

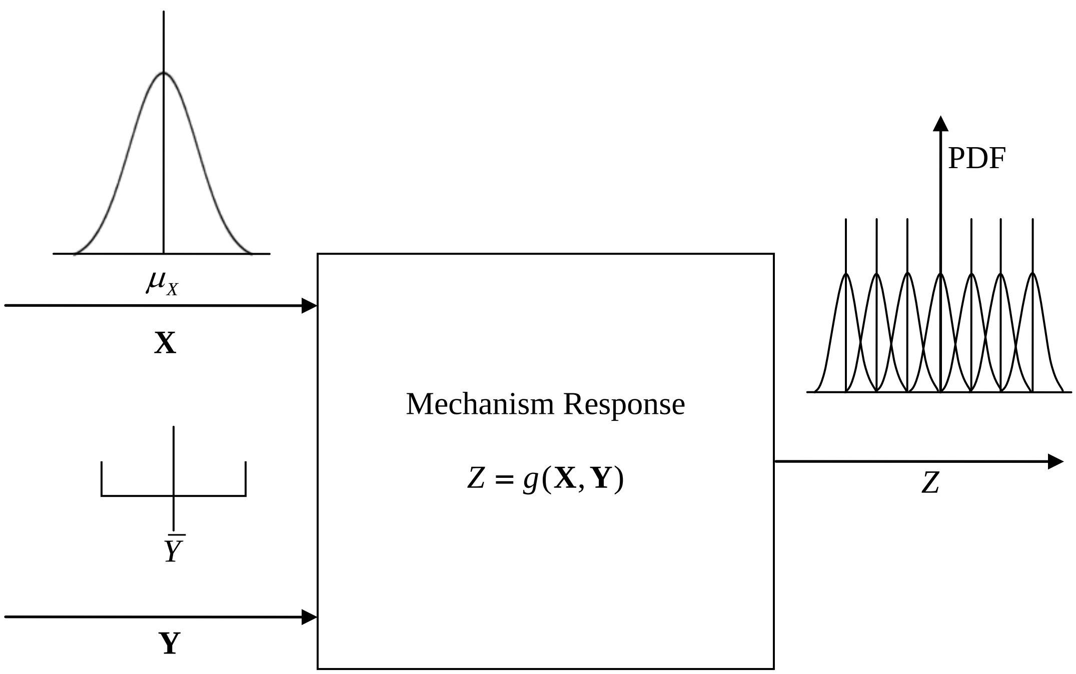

Fig. 5. Mixture of random and interval variables

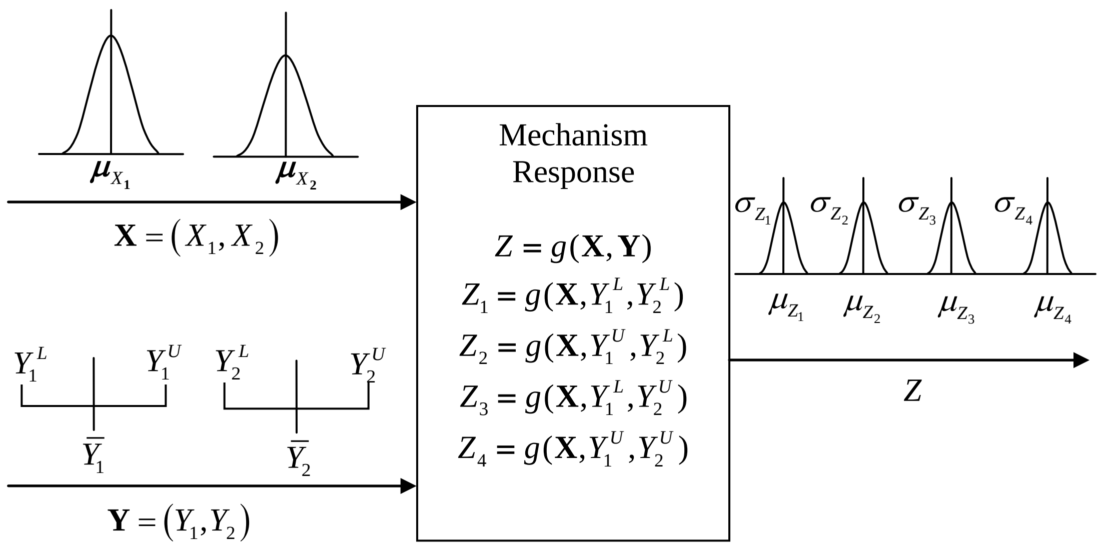

Fig. 6. Example of the mixture of random and interval variables

The probability distributions indicate the uncertainty due to the effect of random variables on Z. The intervals of distributions indicate the effect of interval variables on Z. The average mean value of Z can be calculated by

$$\bar{\mu}_z = \frac{1}{2}\left(\mu_z^{\max} + \mu_z^{\min}\right) \tag{10}$$

where $\mu_z^{\max}$ and $\mu_z^{\min}$ are the maximum and minimum means, respectively. The interval of standard deviations of Z should also be used to assess the robustness. Imagine that if there was no effect of random variables on Z, then Z would be only in the form of intervals due to **Y**. The randomness in Z is therefore due to the random variables **X**. To quantify the effect of randomness on Z, the average of standard deviations of Z is used, which is given by

$$\bar{\sigma}_z = \frac{1}{2}\left(\sigma_z^{\max} + \sigma_z^{\min}\right) \tag{11}$$

where $\sigma_z^{\max}$ and $\sigma_z^{\min}$ are the maximum and the minimum standard deviations, respectively.

Now imagine that if there was no effect on Z from the interval variables **Y**, then Z would be only in the form of a probability distribution. Therefore, the interval of Z is due to the effect of interval variables **Y**. To quantify the effect of interval variables on Z, $\delta\sigma_z$ is used, which is the difference between $\sigma_z^{\max}$ and $\sigma_z^{\min}$ and is computed by

$$\delta\sigma_z = \sigma_z^{\max} - \sigma_z^{\min} \tag{12}$$

In summary, the average standard deviation $\overline{\sigma_z}$ is mainly due to aleatory uncertainty (random variables **X**) while the standard deviation difference $\delta\sigma_Z$ is mainly due to epistemic uncertainty (intervals **Y**). Both a lower value of $\overline{\sigma_z}$ and lower value of $\delta\sigma_z$ are desired for achieving a robust design. To understand this better, consider four designs, which are subjected to similar conditions. Fig. 7 represents the bounds of probability distributions of a mechanism performance. Total uncertainty on the performance can be divided as the uncertainty due to randomness $\left(\bar{\sigma}_Z\right)$ and the uncertainty due to interval variables $\left(\delta\sigma_Z\right)$. The design with less $\bar{\sigma}_Z$ and $\delta\sigma_Z$ is a more robust design. From the distribution curves, comparing $\bar{\sigma}_Z$ for the four designs, we have $\bar{\sigma}_{Z_1} < \bar{\sigma}_{Z_2} < \bar{\sigma}_{Z_3} < \bar{\sigma}_{Z_4}$. The effect of randomness on the performance of Design 1 is the lowest compared with other designs. Design 1 is the most robust design when only randomness is considered. Comparing $\delta\sigma_Z$ for the four designs, we have $\delta\sigma_{Z_1} < \delta\sigma_{Z_3} < \delta\sigma_{Z_2} < \delta\sigma_{Z_4}$. It is seen that Design 1 has the smallest difference between the distribution bounds. Design 1 is therefore the most robust in terms of interval and random variables. It is also seen that Design 4 is the least robust design.

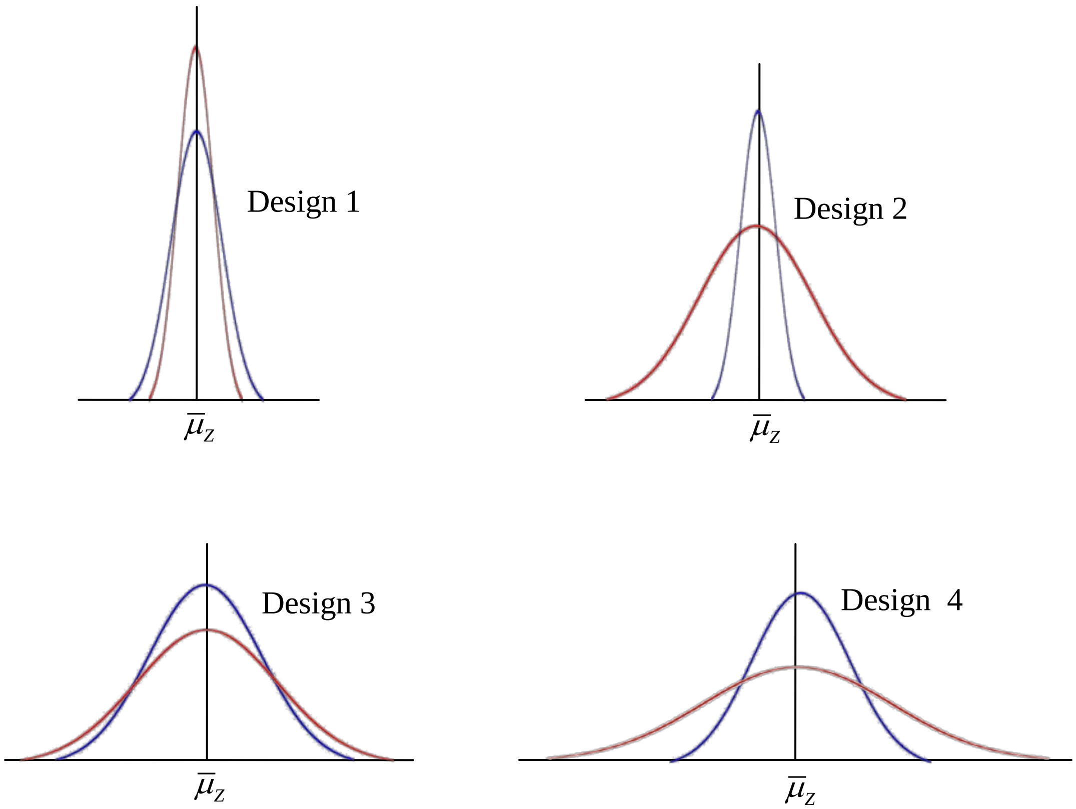

Fig. 7. Robustness assessment with the mixture of random and interval variables

It is however not easy to compare Designs 2 through 3. With the narrower distributions, Design 2 is more robust than Design 3 in terms of randomness. Design 3, however, is more robust than Design 2 in terms of interval variables because the two distribution bounds of Design 3 are closer. In such a case, a decision is left to designers about whether to consider Design 2 or Design 3. A trade-off is usually needed.

As discussed before, the key to mechanism robustness assessment is to calculate $\bar{\sigma}_z$ and $\delta\sigma_z$. A double loop MCS method is proposed to calculate $\bar{\sigma}_z$ and $\delta\sigma_z$. Fig. 8 shows the flowchart of the double loop MCS method. This method consists of an outer loop, which evaluates the effect of interval variables on the variation of the performance. It also includes an inner loop, which evaluates the effect of random variables on the variation of the performance. In the outer loop, all the interval variables are divided into a number of small segments (N_i). The combinations of all intervals are simulated.

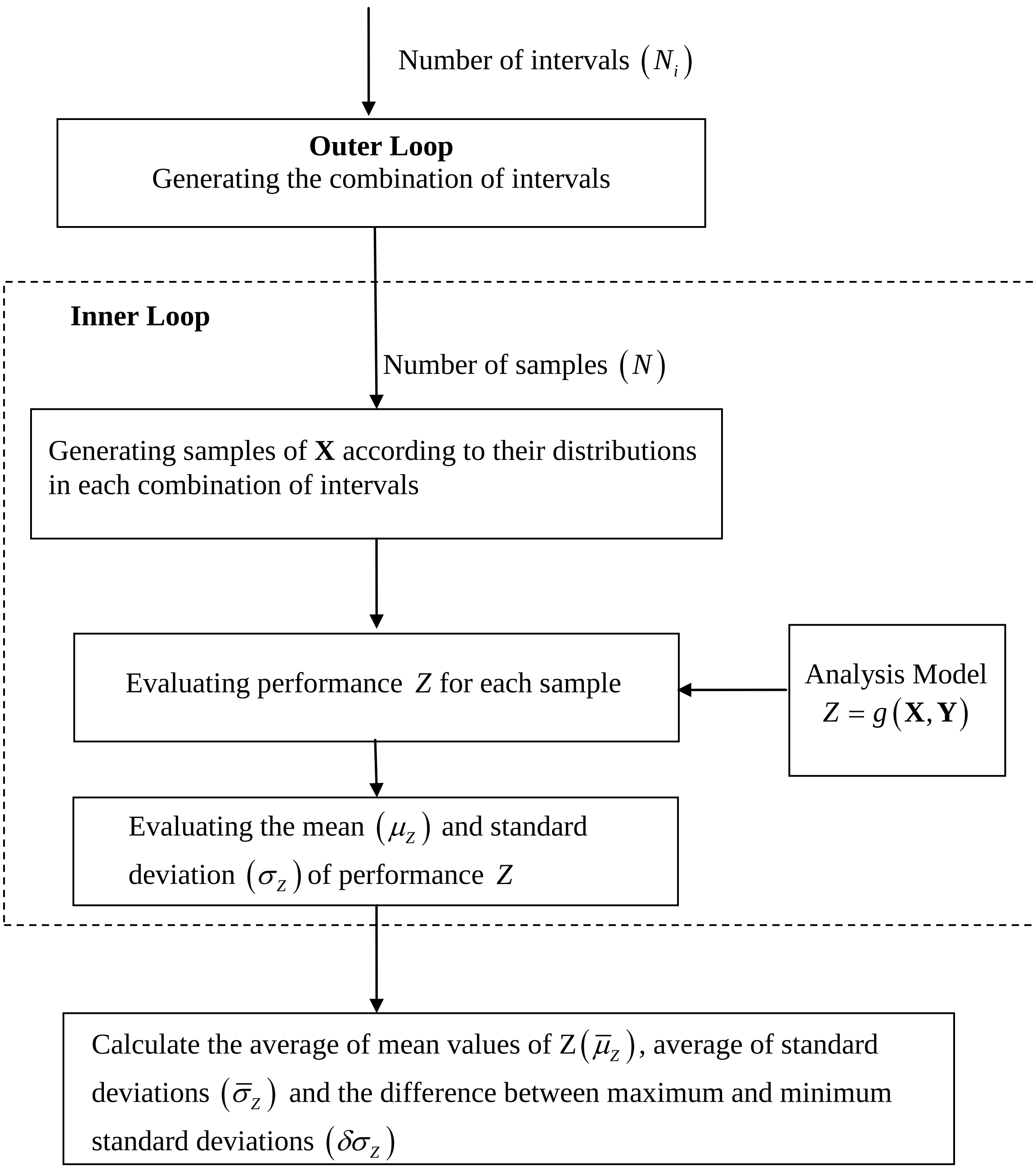

Fig. 8. Double loop Monte Carlo simulation

There are a total of $n_Y \times N_i$ combinations of intervals. For each of the combinations an inner loop is performed. In the inner loop, the samples of random variables **X** are generated according to their distributions. After evaluating Z, for each sample, μ_Z and σ_Z are calculated. After completing the simulations, the output contains $n_Y \times N_i$ samples of μ_Z and σ_Z. If the average of all the means is taken, $\bar{\mu}_Z$ is obtained. $\bar{\mu}_Z$ should be equal to the desired target value. $\sigma_Z^{\max}$ and $\sigma_Z^{\min}$ of σ_Z can also be identified from the obtained σ_Z values. After identifying $\sigma_Z^{\max}$ and $\sigma_Z^{\min}$, $\bar{\sigma}_Z$ and $\delta\sigma_Z$ can be calculated from Eqs. (11) and (12), respectively. From $\bar{\sigma}_Z$ and $\delta\sigma_Z$, the robustness of a system can be assessed. A minimum value of $\bar{\sigma}_Z$ and $\delta\sigma_Z$ is desired for a robust design.

3. MECHANISM SYNTHESIS

In this section, we first review the traditional deterministic mechanism synthesis and mechanism synthesis by probabilistic approach and worst-case approach. We then present the proposed mechanism synthesis when both random variables and intervals exist.

3.1 Deterministic Mechanism Synthesis

Optimization techniques are used for mechanism synthesis. For performing optimization, the objective, the design variables, and the constraints need to be identified. The objective may be the minimization of the difference between the desired performance and the actual performance. The design variables may be the mechanism dimensions, and the constraints may be the existence of crank and a desired transmission angle.

Suppose the objective function $f(\mathbf{d})$ of a mechanism with design variables $\mathbf{d}=(d_1, d_{2,}.....,d_n)$ is to be minimized. Let the mechanism be subjected to design constraints $g_i(\mathbf{d}) \le 0$ $(i=1,2,...,n_g)$ and $h_j(\mathbf{d})=0$ $(j=1,2,...,n_h)$. When uncertainties are not considered, the optimal design model of the synthesis problem is given by

$$\begin{aligned}
&\min_{\mathbf{d}} Z = f(\mathbf{d}) \\
&s.t.\ \ g_i(\mathbf{d}) \le 0, \qquad i=1,\ 2,\ ...,\ n_g \\
&\qquad h_i(\mathbf{d}) = 0, \qquad j=1,\ 2,\ ...,\ n_h \\
&\qquad d_k^L \le d_k \le d_k^U, \quad k=1,\ 2,\ ...,\ n
\end{aligned} \tag{13}$$

where d_k^L and d_k^U are lower and upper bounds of d_k, respectively.

3.2 Robust Mechanism Synthesis with only Random Variables

When design variables are treated as random variables, robustness can be quantified by a standard deviation. We use **X** to represent the random design variables. For a robust mechanism a minimum standard deviation value is to be achieved. The objective of a robust mechanism synthesis would be not only to minimize the difference between the desired performance and actual performance but also to minimize the variation of the performance due to the uncertainties in the design variables. The probabilistic optimization model for the mechanism synthesis can be represented as [14]

$$\begin{aligned}
&\min_{\boldsymbol{\mu}_{\mathbf{x}}} w_1\mu_Z + w_2\sigma_Z \\
&s.t.\ \ \mu_{gi} + k\sigma_{gi} \le 0, \quad i=1,\ 2,\ ...,\ n_g \\
&\qquad h_j(\mathbf{d}) = 0, \qquad j=1,\ 2,\ ...,\ n_h \\
&\qquad \mu_k^L \ \le\ \mu_k \ \le\ \mu_k^U, \quad k=1,\ 2,\ ...,\ n
\end{aligned} \tag{14}$$

where w_1 and w_2 are weighting factors; μ_k^L and μ_k^U are lower and upper bounds of μ_k, respectively; k is a constant, which indicates the probability of constraint satisfaction. The probability is given by $\Phi(k)$, where Φ is the cumulative distribution function of a standard normal variable. For example, if k is 3.0, the probability will be $\Phi(3) = 0.9987$.

Fig. 9 illustrates the difference between the deterministic mechanism synthesis and robust mechanism synthesis. It is seen that the probabilistic mechanism synthesis can achieve a more robust design with the reduced standard deviation.

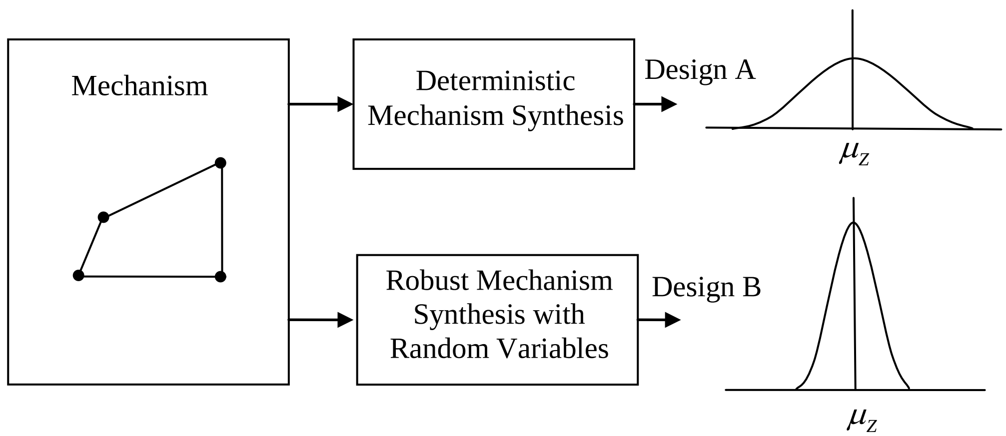

Fig.9. Comparison of deterministic mechanism synthesis and robust mechanism synthesis

3.3 Robust Mechanism Synthesis with only Interval Variables

When uncertainty in design variables is treated as interval variables, an interval approach is used for robust mechanism synthesis. We use **Y** to represent the interval design variables. The robustness is quantified by the width of the interval of the mechanism performance (δZ). The objective of the robust mechanism synthesis would be to minimize the average error between the desired performance and the actual performance of the mechanism and at the same time, to minimize the effect of uncertainty on the mechanism performance. The optimization model for the robust mechanism synthesis with interval variables can be represented as

$$
\begin{aligned}
&\min_{\bar{\mathbf{Y}}} w_1 \bar{Z} + w_2 \delta Z \\
&s.t.\ \ g^{\max} \le 0, && i = 1,\ 2,\ \dots,\ n_g \\
&\qquad h_j(\bar{\mathbf{Y}}) = 0, && j = 1,\ 2,\ \dots,\ n_h \\
&\qquad \bar{Y}_k^L \ \le\ \bar{Y}_k \ \le\ \bar{Y}_k^U, && k = 1,\ 2,\ \dots,\ n
\end{aligned}
\tag{15}
$$

where $\bar{Y}_k^L$ and $\bar{Y}_k^U$ are lower and upper bounds of $\bar{Y}_k$, respectively.

Fig. 10 shows the comparison of the deterministic mechanism synthesis and robust mechanism synthesis with interval variables.

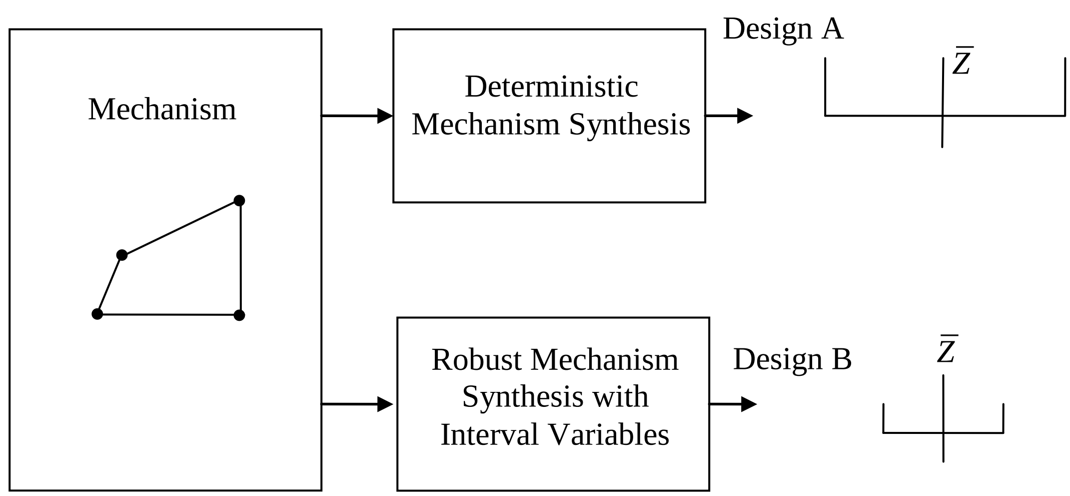

Fig. 10. Comparison of deterministic mechanism synthesis and interval robust mechanism synthesis

3.4 Robust Mechanism Synthesis with Random and Interval Variables

In the real-world engineering systems, uncertainty is in the form of a mixture of random variables and interval variables. In such situations, to quantify robustness, we propose to use a combined method of probabilistic approach and interval approach. The robustness will be quantified by $\bar{\sigma}_Z$ and $\delta\sigma_Z$. Mathematically, the design objective for a robust design is represented as $f(\mathbf{X},\mathbf{Y}) = w_1\bar{\mu}_Z + w_2\bar{\sigma}_Z + w_3\delta\sigma_Z$. The double loop MCS (see section 2.3) is proposed for evaluating $\bar{\sigma}_Z$ and $\delta\sigma_Z$. w_1, w_2, and w_3 are the weighting factors. The flowchart of the design optimization for the robust mechanism synthesis with random and interval variables is shown in Fig. 11.

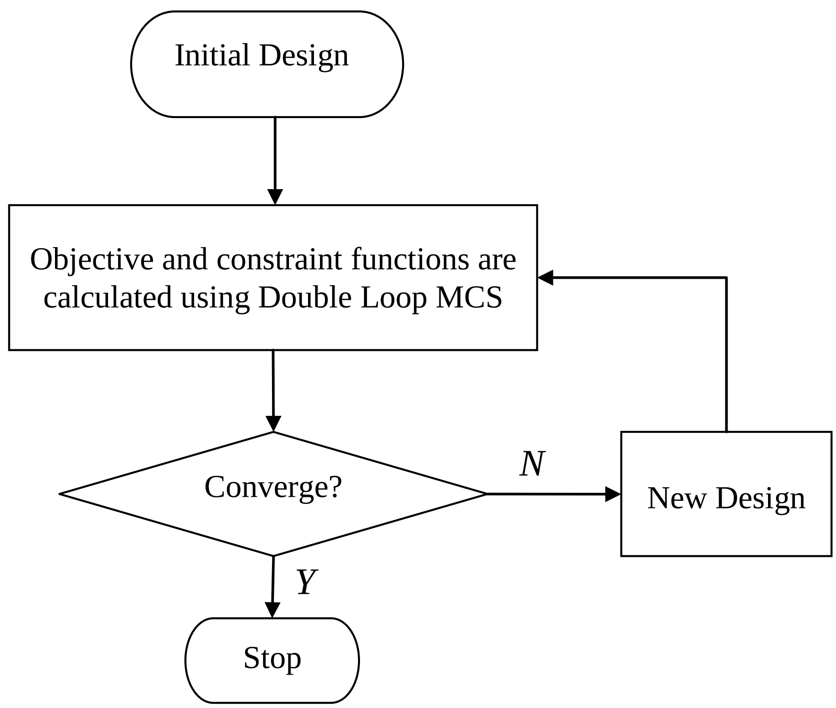

Fig. 11. Optimization for robust mechanism synthesis with random and interval variables

The constraint functions need to be changed to maintain robustness of the design feasibility in the worst case of design variables. A constraint function, therefore, is modified as $\mu_{g_i}^{\max} + k\sigma_{g_i}^{\max} \le 0 \ \left(i = 1, 2, ..., n_i\right)$. $\mu_{g_i}^{\max}$ and $\sigma_{g_i}^{\max}$ are the maximum of the mean value and the maximum of standard deviation of the constraint function $g_i\left(\mathbf{X}, \mathbf{Y}\right)$, respectively. The optimization model for the robust mechanism synthesis with random and interval variables is given by

$$\begin{aligned}
&\min_{\boldsymbol{\mu}_X, \overline{\mathbf{Y}}} w_1\overline{\mu}_Z + w_2\overline{\sigma}_Z + w_3\delta\sigma_Z \\
&s.t. \ \ \mu_{gi}^{\max} + k\sigma_{gi}^{\max} \le 0, \qquad i = 1,\ 2,\ ...,\ n_g \\
&\qquad h_j(\boldsymbol{\mu}_X, \overline{\mathbf{Y}}) = 0, \qquad j = 1,\ 2,\ ...,\ n_h \\
&\qquad \overline{Y}_k^L \le \overline{Y}_k \le \overline{Y}_k^U, \qquad k = 1,\ 2,\ ...,\ n_Y \\
&\qquad \mu_m^L \le \mu_m \le \mu_m^U, \qquad m = 1,\ 2,\ ...,\ n_X
\end{aligned} \tag{16}$$

The flowchart of solving the above optimization model is shown in Fig. 11. There are two major iterative loops. The inner loop is for robustness assessment. The outer loop is the overall design optimization. Double loop MCS is used in the inner loop for robustness assessment. The process is iterated until a design satisfies the design constraints and converges to the optimal objective function.

Fig. 12 shows a comparison between deterministic mechanism synthesis and robust mechanism synthesis. It is evident that the robust mechanism synthesis results in a more robust design compared to the deterministic mechanism synthesis.

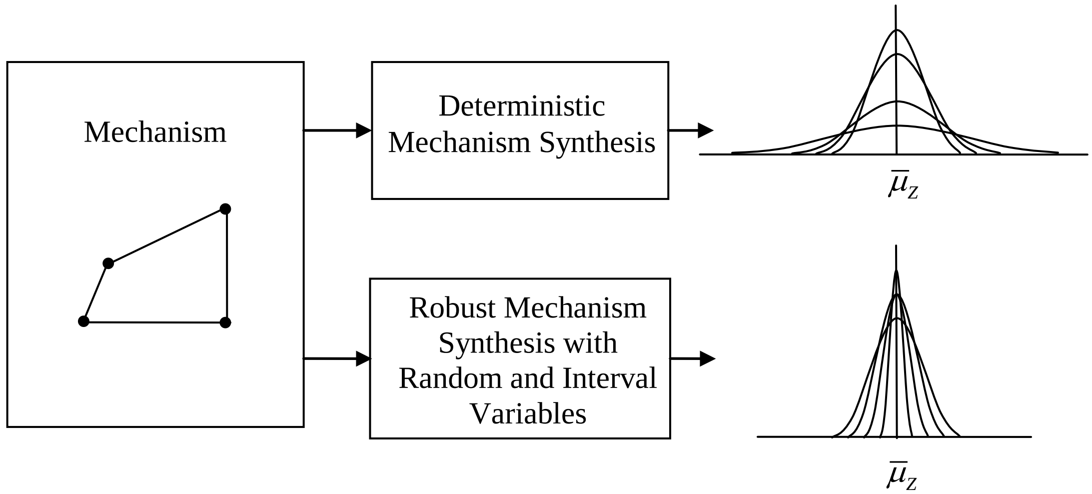

Fig. 12. Comparison of deterministic mechanism synthesis and robust mechanism synthesis with random and interval variables

4. EXAMPLES

The proposed method is validated and demonstrated with two example problems. The first example is a slider crank mechanism design problem, and the second example is a four-bar linkage design problem.

4.1. Example 1 –Slider Crank Mechanism Design

The objective of this example is to design a slider crank mechanism as shown in Fig. 13 such that, for crank angles (θ) of 10˚ and 60˚, the slider distance (s) should be 35.0 mm and 25.0 mm, respectively. The length of the crank (a), the length of the connecting rod (b), and the offset (e) are design variables. Both a and b are random variables, which are given in Table 1. Because different installation positions of the slider are needed, the offset distance e is specified within a tolerance given in Table 2. The distribution of e is not available. Therefore, e is treated as an interval variable. In Table 2, $\bar{e}$ is the average of e.

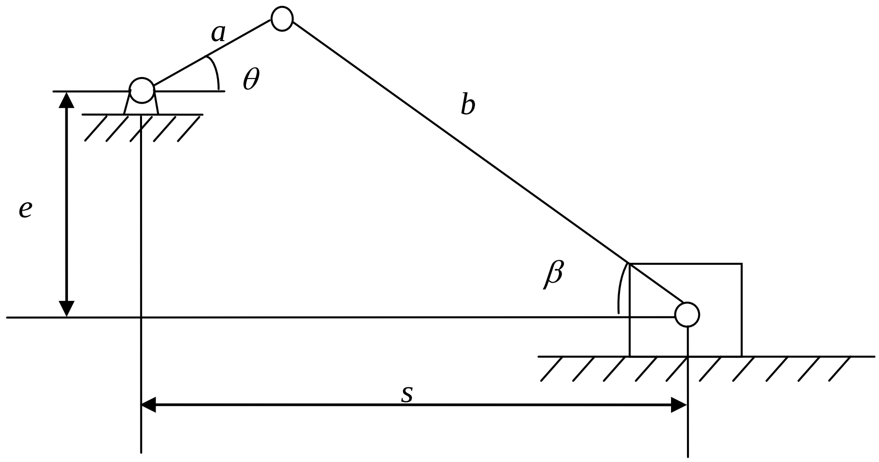

Fig. 13. Slider crank mechanism

Table1. Random Variables

Variable	Mean (μ)	Standard Deviation (σ)	Distribution
$X_1\,(a)$	μ_a	1% of μ_a	Normal
$X_2\,(b)$	μ_b	1% of μ_b	Normal

Table 2. Interval Variable

Variable	Y^L	Y^U
$Y_1\,(e)$	$\overline{e}$ - 5% of $\overline{e}$	$\overline{e}$ + 5% $\overline{e}$

The task is to determine the length of the links a and b, and the offset distance e satisfying the design requirement. First the mechanism synthesis is completed deterministically without considering any uncertainty, and then a robust mechanism synthesis is performed considering the uncertainties in the design variables. Both the designs are then compared.

Deterministic Mechanism Synthesis

When deterministic mechanism synthesis is employed, the nominal values of the design variables are used without considering any uncertainties. The deterministic mechanism synthesis is modeled as

$$\begin{aligned}&\min_{\mathbf{d}} f(\mathbf{d}) = \varepsilon = \sqrt{\left[\varepsilon\left(10^\circ\right)\right]^2 + \left[\varepsilon\left(60^\circ\right)\right]^2} \\ &s.t. \;\; g_1(\mathbf{d}) = e - (b - a) \leq 0 \\ &\quad\;\; g_2(\mathbf{d}) = (e + a) - b\sin 45^\circ \leq 0 \\ &\quad\;\; 0.1 \leq a \leq 20,\; 0.1 \leq b \leq 20,\; 0.1 \leq e \leq 20\end{aligned} \tag{17}$$

where

$$\varepsilon\left(10^\circ\right) = \left[a\cos 10^\circ + \sqrt{b^2 - \left(e + a\sin 10^\circ\right)^2}\right] - 35.0 \tag{18}$$

$$\varepsilon\left(60^\circ\right) = \left[a\cos 60^\circ + \sqrt{b^2 - \left(e + a\sin 60^\circ\right)^2}\right] - 25.0 \tag{19}$$

The task is to find a design having the minimum error at both the positions and satisfying the constraints functions. The design constraints of this mechanism include the existence of the crank (g_1) and the transmission angle greater than 45˚(g_2). The Sequential Quadratic Programming (SQP) is used to perform this optimization. The starting point d =(a ,b ,e)=(4.0, 8.0, 1.0) mm is used. The optimal solution obtained from the deterministic mechanism synthesis is listed in Table 3.

Table 3. Deterministic Optimal Solution

Error (mm)	a (mm)	b (mm)	e (mm)	s at 10˚ (mm)	s at 60˚ (mm)
72.40e^{-12}	11.33	25.31	6.52	35.0	25.0

The design obtained from the deterministic mechanism synthesis results in the slider distances 35.0 mm and 25.0 mm at 10˚ and 60˚, respectively and satisfies the design constraints, such as the existence of crank and the transmission angle (45.16˚) greater than 45.0˚.

Robust Mechanism Synthesis

In the proposed design optimization model, tolerances in the links and the installation error are considered as uncertainty. As described in the preceding section, the average standard deviation $\bar{\sigma}_Z$ and the width of the standard deviation $\delta\sigma_Z$ are minimized to ensure robustness. Herein z denotes the error specified in the objective function, namely, $z_1 = \varepsilon(10^\circ)$ and $z_2 = \varepsilon(60^\circ)$. In this design problem, $\bar{\sigma}_Z$ and $\delta\sigma_Z$ need to be minimized at crank angles of 10˚ and 60˚. The two inequality constraints, the existence of a crank and the transmission of energy are maintained at the worst case of interval variables.

The errors of the actual displacements at 10º and 60º at the means of random variables and the averages of interval variables are also treated as two equality constraints. A double loop MCS is used to evaluate $\bar{\sigma}_Z$ and $\delta\sigma_Z$. In the double loop MCS, 20 intervals (N_i) for the interval variables and 2000 samples (N) for the random variables are taken.

The robust mechanism synthesis is modeled as

$$
\begin{aligned}
&\min_{\boldsymbol{\mu}_x,\mathbf{Y}} \; w_1\overline{\sigma}\left(10^\circ\right)+w_2\delta\sigma\left(10^\circ\right)+w_3\overline{\sigma}\left(60^\circ\right)+w_4\delta\sigma\left(60^\circ\right)\\
&s.t.\; \mu_{g_i}^{\max} \;\; + k\sigma_{g_i}^{\max} \le 0,\; i=1,2\\
&\quad h_1\left(\boldsymbol{\mu}_{\mathbf{x}},\overline{\mathbf{Y}}\right)=\mu_a\cos 10^\circ+\sqrt{\mu_b^2-\left(\overline{e}+\mu_a\sin 10^\circ\right)^2}-35.0\\
&\quad h_2\left(\boldsymbol{\mu}_{\mathbf{x}},\overline{\mathbf{Y}}\right)=\mu_a\cos 60^\circ+\sqrt{\mu_b^2-\left(\overline{e}+\mu_a\sin 60^\circ\right)^2}-25.0\\
&\quad 0.1\le\mu_a\le 20,\; 0.1\le\mu_b\le 20,\; 0.1\le\overline{e}\le 20
\end{aligned}
\tag{20}
$$

Weighting factors are used to formulate the multiple objective functions. $w_1=1/\overline{\sigma}_\varepsilon^*\left(10^\circ\right)$, $w_2=1/\delta\sigma_\varepsilon^*\left(10^\circ\right)$, $w_3=1/\overline{\sigma}_\varepsilon^*\left(60^\circ\right)$, and $w_4=1/\delta\sigma_\varepsilon^*\left(60^\circ\right)$. All w's are calculated at the deterministic optimal solution. For example, in $w_1=1/\overline{\sigma}_\varepsilon^*\left(10^\circ\right)$, $\overline{\sigma}_\varepsilon^*\left(10^\circ\right)$ is the average standard deviation of $\varepsilon\left(10^\circ\right)$ at the deterministic optimal point. w's are used to normalize the multiple objectives.

The robust mechanism synthesis solution is listed in Table 4. It is noted that the nominal displacements are exactly at the required values.

Table 4. Robust Mechanism Synthesis Solution

μ_a (mm)	μ_b (mm)	$\overline{e}$ (mm)	μ_s at 10° (mm)	μ_s at 60° (mm)
13.24	22.21	1.0	35.0	25.0

Next we estimate the robustness of the designs from both deterministic synthesis and robust synthesis. The number of intervals $\left(N_i\right)$ for the interval variable is taken as 20. The Number of samples $\left(N\right)$ is taken as 2000. The solution obtained from the double loop MCS is shown in Table 5. It is seen that the proposed robust mechanism synthesis method results in a more robust design. Both of average standard deviations and the widths of the standard deviations of the slider displacements from robust synthesis are smaller than those from deterministic synthesis.

Table 5 Robustness Assessment of Deterministic Mechanism Synthesis and Robust Mechanism Synthesis

Variables		Deterministic synthesis	Robust synthesis
Deterministic synthesis	Robust synthesis		
a	μ_a	11.33 mm	13.24 mm
b	μ_b	25.31 mm	22.21 mm
e	$\bar{e}$	6.52 mm	1.00mm
$\bar{s}_{10°}$		35.00mm	35.00mm
$\bar{s}_{60°}$		25.00mm	25.00mm
$\bar{\sigma}_{Z_{10°}}$		2.94×10^{-1}mm	2.64×10^{-1} mm
$\delta\sigma_{Z_{10°}}$		2.23×10^{-3} mm	8.18×10^{-5} mm
$\bar{\sigma}_{Z_{60°}}$		3.33×10^{-1} mm	2.70×10^{-1}mm
$\delta\sigma_{Z_{60°}}$		9.67×10^{-3} mm	9.95×10^{-4} mm

The output is also graphically represented. The graphs obtained from both the methods are compared. The family of distribution curves at the crank angle of 10° is shown in Figs. 14 and 15, respectively. The graphics clearly show that the range (band) of the distributions from robust synthesis is much narrower than those from deterministic synthesis.

Similarly, Figs. 16 and 17 show the family of distribution curves at the crank angle of 60˚. It is evident that the design obtained from robust synthesis is more robust compared to the design obtained from the deterministic synthesis.

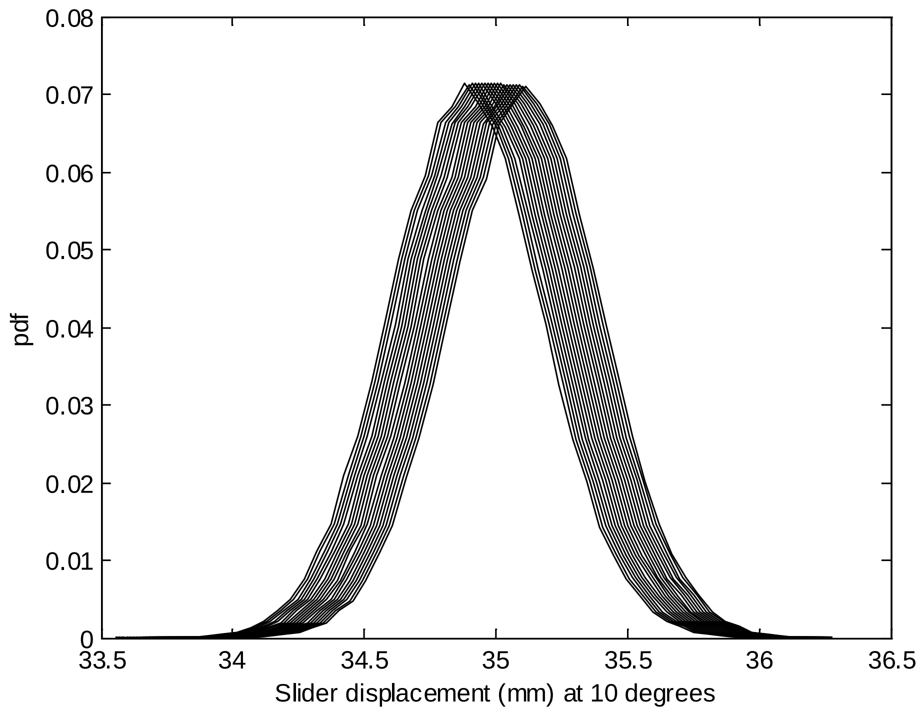

Fig. 14. Distributions at the crank angle of 10˚ from deterministic synthesis

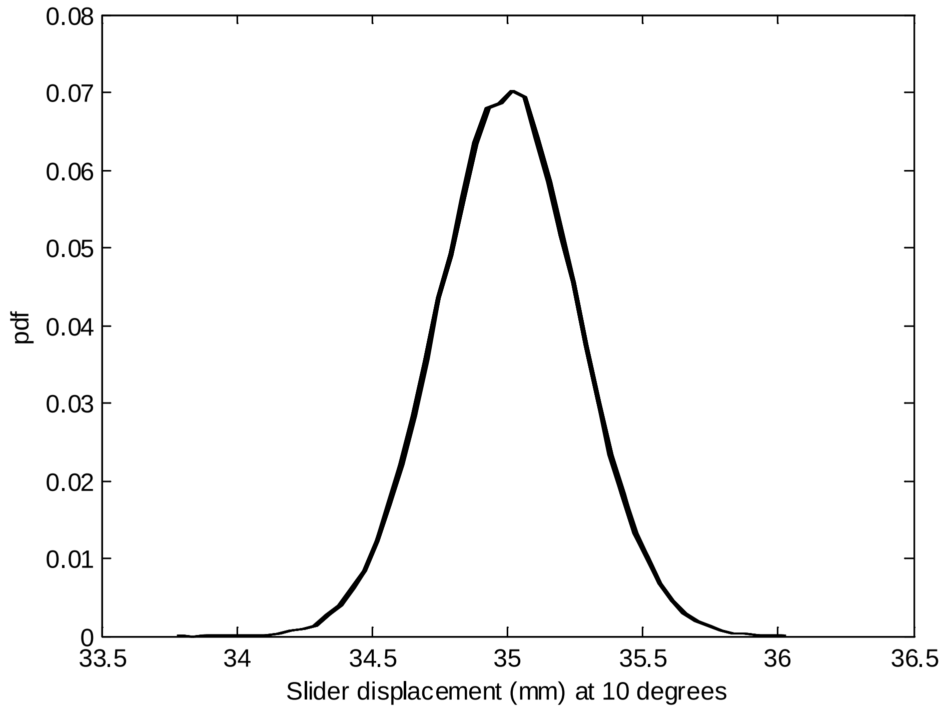

Fig. 15. Distributions at the crank angle of 10˚ from robust mechanism synthesis

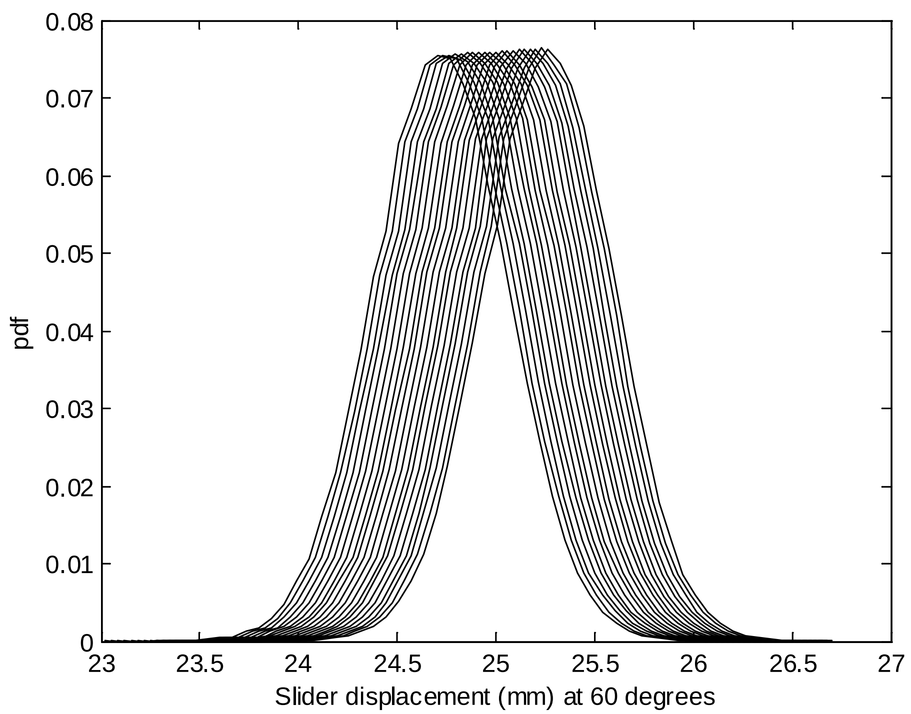

Fig. 16. Distributions at the crank angle of 60˚ from deterministic synthesis

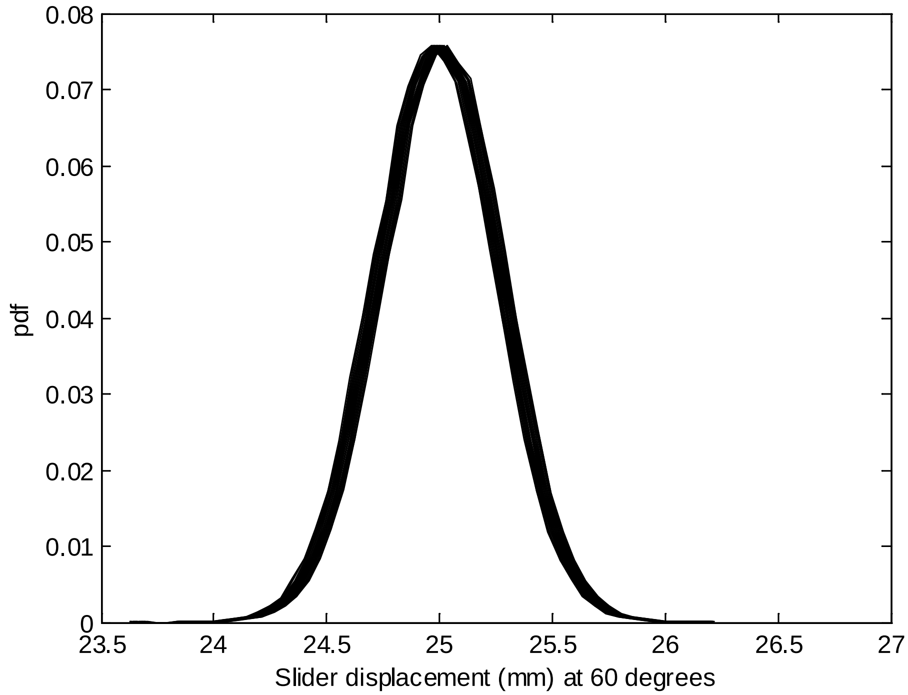

Fig. 17. Distributions at the crank angle of 60˚ from robust synthesis

4.2. Example 2 – A Four-Bar Mechanism Design Problem

A four bar mechanism as shown in Fig. 18 is to be designed such that when the angles (θ_2) of the input link are 10˚ and 60˚, the position of $P\,(x, y)$ should be (55, 108) mm and (45.0, 142.0) mm, respectively.

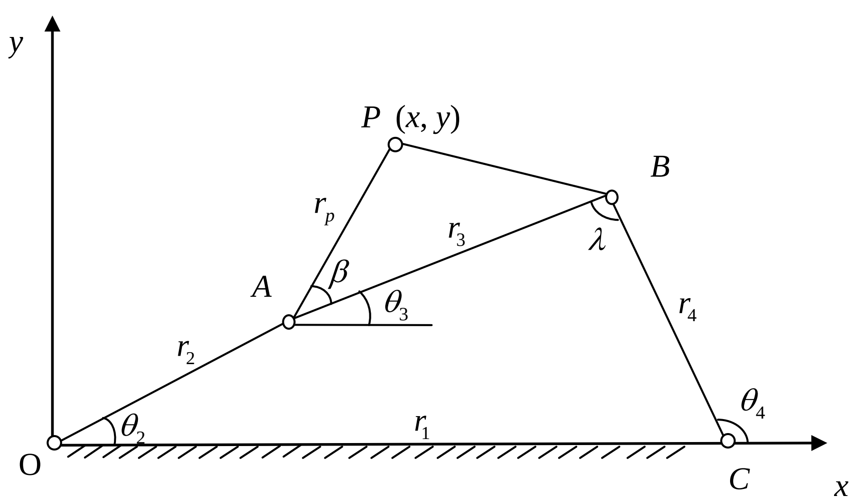

Fig. 18. Four-bar linkage

The lengths of ground link $OC(r_1)$, link $OA(r_2)$, link $AB(r_3)$, link $BC(r_4)$, and link $AP(r_p)$, and the angle (β) are design variables. The links r_2, r_3, r_4 and r_p are random variables, which are given in Table 6. As there is no information available about the type of distribution of the variables r_1 and β, they are considered as interval variables, which are given in Table 7. $\bar{r}$ and $\bar{\beta}$ are the averages of r and β, respectively.

Table 6. Random Variables

Variable	Mean (μ)	Standard Deviation (σ)	Distribution
$X_1\ (r_2)$	μ_{r_2} mm	0.20 mm	Normal
$X_2\ (r_3)$	μ_{r_3} mm	0.20 mm	Normal
$X_3\ (r_4)$	μ_{r_4} mm	0.20 mm	Normal
$X_4\ (r_p)$	μ_{r_p} mm	0.20 mm	Normal

Table 7. Interval Variables

Variable	Y^L	Y^U
$Y_1(r_1)$	$\bar{r}_1 - 0.5$ mm	$\bar{r}_1 + 0.5$ mm
$Y_2(\beta)$	$\bar{\beta} - 1^\circ$	$\bar{\beta} + 1^\circ$

Deterministic Mechanism Synthesis

In deterministic mechanism synthesis, uncertainties in the design variables are not considered. The governing equations for finding the position of $P\,(x, y)$ are given below:

$$P_X = r_2 \cos\theta_2 + r_p \cos(\beta + \theta_3) \tag{21}$$

$$P_Y = r_2 \sin\theta_2 + r_p \sin(\beta + \theta_3) \tag{22}$$

$$r_2 \cos\theta_2 + r_3 \cos\theta_3 - r_4 \cos\theta_4 - r_1 = 0 \tag{23}$$

$$r_2 \sin\theta_2 + r_3 \sin\theta_3 - r_4 \sin\theta_4 = 0 \tag{24}$$

The constraints of this mechanism design include the existence of a crank and the transmission angle constraint, which are given by

$$r_2 + r_1 - (r_2 + r_3) \le 0 \tag{25}$$

$$r_2 + r_3 - (r_4 + r_1) \le 0 \tag{26}$$

$$r_2 + r_4 - (r_1 + r_3) \le 0 \tag{27}$$

$$\lambda_1 = \cos^{-1}\left[\frac{r_3^2 + r_4^2 - (r_1 + r_2)^2}{2r_3 r_4}\right] \ge 40^\circ \tag{28}$$

$$\lambda_2 = \cos^{-1}\left[\frac{r_3^2 + r_4^2 - (r_1 - r_2)^2}{2r_3 r_4}\right] \ge 40^\circ \tag{29}$$

where λ_1 and λ_2 are the possible minimum transmission angles.

The objective is to find the design variables r_1, r_2, r_3, r_4, r_p and β such that, the errors between the actual positions and desired positions of $P(x, y)$ at 10° and 60° of crank angles are minimum. The design should also satisfy the design constraints in Eqs. (25) through (29).

The deterministic mechanism synthesis is then modeled as

$$\min_{\mathbf{d}=(r_1,r_2,r_3,r_4,r_p,\beta)} f(\mathbf{d}) = \varepsilon = \sqrt{[(\varepsilon_x(10^\circ)]^2 + [\varepsilon_y(10^\circ)]^2 + [\varepsilon_x(60^\circ)]^2 + [\varepsilon_y(60^\circ)]^2}$$

$$\begin{aligned}
s.t. \quad & g_1(\mathbf{d}) = r_2 + r_1 - (r_3 + r_4) \le 0 \\
& g_2(\mathbf{d}) = r_2 + r_3 - (r_1 + r_4) \le 0 \\
& g_3(\mathbf{d}) = r_2 + r_4 - (r_1 + r_3) \le 0 \\
& g_4(\mathbf{d}) = \frac{r_3^2 + r_4^2 - (r_1 + r_2)^2}{2r_3 r_4} - \cos 40^\circ \le 0 \\
& g_5(\mathbf{d}) = \frac{r_3^2 + r_4^2 - (r_1 - r_2)^2}{2r_3 r_4} - \cos 40^\circ \le 0 \\
& 20 \le r_1 \le 500, 20 \le r_2 \le 500, 20 \le r_3 \le 500 \\
& 20 \le r_4 \le 500, 20 \le r_p \le 500, 0^\circ \le \beta \le 180^\circ
\end{aligned} \tag{30}$$

where $\varepsilon_x\left(10^\circ\right) = x\left(10^\circ\right) - 55$, $\varepsilon_y\left(10^\circ\right) = y\left(10^\circ\right) - 108$, $\varepsilon_x\left(60^\circ\right) = x\left(60^\circ\right) - 45$, and $\varepsilon_y\left(60^\circ\right) = y\left(60^\circ\right) - 142$. The starting point d= (r_1, r_2, r_3, r_4, r_p, β) = (100, 40, 180, 150, 200, 100).

The optimal solution from the deterministic mechanism synthesis is listed in Table 8. It is seen that the position requirement in the objective function and all the constraints are satisfied at the optimal design point.

Table 8. Deterministic Optimal Solution

Variable	Solution
r_1	258.11 mm
r_2	51.68 mm
r_3	264.17 mm
r_4	154.44 mm
r_p	99.11 mm
β	54.38°
(x, y) at 10°	(55.0, 108.0) mm
(x, y) at 60°	(45.0, 142.0) mm

Robust Mechanism Synthesis

In the proposed robust mechanism synthesis model, the tolerances in the links and the installation error are considered uncertain. Due to the uncertainties, the position $P\left(x, y\right)$, deviates from the desired value. The objective will be not only to maintain the position of $P\left(x, y\right)$ to the desired value but also to minimize the average standard deviation and the difference between the maximum and minimum standard deviations of the performance. The robust mechanism synthesis is modeled as

$$
\begin{aligned}
&\min_{\mathbf{d}} f\left(\mathbf{d}\right) = w_1\bar{\sigma}_x\left(10^\circ\right) + w_2\delta\sigma_x\left(10^\circ\right) + w_3\bar{\sigma}_y\left(10^\circ\right) + w_4\delta\sigma_y\left(10^\circ\right) + w_5\bar{\sigma}_x\left(60^\circ\right) \\
&\qquad\qquad + w_6\delta\sigma_x\left(60^\circ\right) + w_7\bar{\sigma}_y\left(60^\circ\right) + w_8\delta\sigma_y\left(60^\circ\right) \\
&s.t. \quad \mu_{g_i}^{\max} + k\sigma_{g_i}^{\max} \le 0,\ i = 1,2,\ \ldots\ ,5 \\
&\qquad h_1(\boldsymbol{\mu}_{\mathbf{x}},\ \bar{\mathbf{Y}}) = \bar{x}(10^\circ) - 55.0 = 0 \\
&\qquad h_2(\boldsymbol{\mu}_{\mathbf{x}},\ \bar{\mathbf{Y}}) = \bar{y}(10^\circ) - 108.0 = 0 \\
&\qquad h_3(\boldsymbol{\mu}_{\mathbf{x}},\ \bar{\mathbf{Y}}) = \bar{x}(60^\circ) - 45.0 = 0 \\
&\qquad h_4(\boldsymbol{\mu}_{\mathbf{x}},\ \bar{\mathbf{Y}}) = \bar{y}(60^\circ) - 142.0 = 0 \\
&\qquad 20 \le r_1 \le 500,\ 20 \le r_2 \le 500,\ 20 \le r_3 \le 500 \\
&\qquad 20 \le r_4 \le 500,\ 20 \le r_p \le 500,\ 0^\circ \le \beta \le 180^\circ
\end{aligned}
\tag{31}
$$

where $k = 3$, and $\mathbf{d} = (\bar{r}_1,\ \mu_{r2},\ \mu_{r3},\ \mu_{r4},\ \mu_{rp},\ \bar{\beta})$. $\bar{x}$ and $\bar{y}$ are nominal values of the coordinates of point P evaluated at the means of random variables and averages of interval variables.

Weighting factors are used to formulate the multiple objective functions. $w_1 = 1/\bar{\sigma}_x^*\left(10^\circ\right), w_2 = 1/\delta\sigma_x^*\left(10^\circ\right), w_3 = 1/\bar{\sigma}_y^*\left(10^\circ\right), w_4 = 1/\delta\sigma_y^*\left(10^\circ\right), w_5 = 1/\bar{\sigma}_x^*\left(60^\circ\right)$, $w_6 = 1/\delta\sigma_x^*\left(10^\circ\right)$, $w_7 = 1/\bar{\sigma}_y^*\left(60^\circ\right)$, and $w_8 = 1/\delta\sigma_y^*\left(60^\circ\right)$. All w's are calculated at the deterministic optimal solution. For example, in $w_1 = 1/\bar{\sigma}_x^*\left(10^\circ\right)$, $\bar{\sigma}_x^*\left(10^\circ\right)$ is the average standard deviation of $x(10^\circ)$ at the deterministic point. w's are used to normalize the multiple objectives.

The constraint is modified to maintain the robustness of the design feasibility at the worst case of the design variables. In the double loop MCS, 2 intervals are taken for each of the interval variables and 100 samples are taken for the random variables. The optimal solution obtained from the robust mechanism synthesis is listed in Table 9. It is seen that the nominal values of the positions are exactly the desired ones.

Table 9. Optimal Solution Obtained from Robust Mechanism Synthesis

Variable	Solution
$\bar{r}_1$	438.70 mm
μ_{r_2}	51.68 mm
μ_{r_3}	297.06 mm
μ_{r_4}	197.08 mm
μ_{r_P}	99.11 mm
$\bar{\beta}$	59.10°
$(\bar{\mu}_x, \bar{\mu}_y)$ at 10°	(55.0, 108.0) mm
$(\bar{\mu}_x, \bar{\mu}_y)$ at 60°	(45.0,142.0) mm

Robustness Assessment

The double loop MCS is used for assessing the robustness of the designs obtained from the deterministic synthesis and robust synthesis. The comparison of the robustness of the two designs is listed in Table 10.

Table 10. Comparison of Designs Obtained from Deterministic Mechanism Synthesis and Robust Mechanism Synthesis

Variable		Deterministic synthesis	Robust synthesis
Deterministic synthesis	Robust synthesis		
r_1	$\overline{r}_1$	258.11 mm	438.70 mm
r_2	μ_{r_2}	51.68 mm	51.68 mm
r_3	μ_{r_3}	264.17 mm	297.06 mm
r_4	μ_{r_4}	154.44 mm	197.08 mm
r_p	μ_{r_p}	99.11 mm	99.11 mm
β	$\overline{\beta}$	54.38°	59.10°
$\left(\overline{\mu}_x, \overline{\mu}_y\right)$ at 10°		(55.00, 108.0) mm	(55.00, 108.0) mm
$\left(\overline{\mu}_x, \overline{\mu}_y\right)$ at 60°		(45.00, 142.0) mm	(45.00, 142.0) mm
$\left(\overline{\sigma}_x, \overline{\sigma}_y\right)$ at 10°		(2.19×10^{-1}, 2.04×10^{-1}) mm	(1.57×10^{-1}, 2.05×10^{-1}) mm
$\left(\delta\sigma_x, \delta\sigma_y\right)$ at 10°		(5.93×10^{-3}, 7.18×10^{-4}) mm	(1.92×10^{-3}, 5.56×10^{-5}) mm
$\left(\overline{\sigma}_x, \overline{\sigma}_y\right)$ at 60°		(1.80×10^{-1}, 2.50×10^{-1}) mm	(1.36×10^{-1}, 2.55×10^{-1}) mm
$\left(\delta\sigma_x, \delta\sigma_y\right)$ at 60°		(5.03×10^{-3}, 3.79×10^{-3}) mm	(3.84^{-3}, 2.32^{-3}) mm

Even though $\overline{\sigma}_y = 2.05e^{-1}$ at 10° from robust design is slightly greater than $\overline{\sigma}_y = 2.04e^{-1}$ from deterministic design and $\overline{\sigma}_y = 2.55e^{-1}$ at 60° from robust design is slightly greater than $\overline{\sigma}_y = 2.50e^{-1}$ from deterministic design, the results clearly show that the design obtained from robust synthesis is more robust than that from deterministic synthesis.

5. CONCLUSIONS

When both random and interval variables exist in a mechanism, a mechanism performance will have a family of distributions instead of a single distribution. The mean and standard deviation of the performance will also be intervals. To measure the robustness of the mechanism, we propose to use the average standard deviation and the width of the standard deviation of the mechanism performance. Both of the measures are minimized during robust mechanism synthesis.

To calculate the average standard deviation and the width of the standard deviation of the mechanism performance, we propose a double-loop Monte Carlo simulation (MCS) procedure. The outer loop is for interval variables while the inner loop is for random variables. The overall robust mechanism synthesis is solved by an optimization technique. The two examples have shown that the proposed method generates more robust designs than the deterministic mechanism synthesis. Since optimization and MCS are used, it is easy and flexible to use the proposed method. The method is applicable to all distribution types.

MCS, however, is not efficient in evaluating the mean and standard deviation of a mechanism performance. To find the extreme values of the standard deviation of the performance, each interval variable has to be divided into a number of segments. The double-loop procedure must compute the probabilistic characteristics of the performance for all combinations of interval segments. If the number of the segments of each interval variable is not large enough, the extreme values of the mean and standard deviation of the performance may be missed. As a result, the computational efficiency needs to be improved, especially for a problem involving a large number of interval variables. Our future research is targeted to increase computational efficiency.

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