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An Approximation Approach to General Robustness Assessment for Multidisciplinary Systems

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Nomenclature

g	= performance function
$\mathbf{I}$	= vector of interval variables
$\bar{\mathbf{I}}$	= average of $\mathbf{I}$
$\mathbf{I}^l$	= lower bound of $\mathbf{I}$
$\mathbf{I}^u$	= upper bound of $\mathbf{I}$
I	= interval variable, element of $\mathbf{I}$
N	= number of function evaluations
N_i	= number of analyses in subsystem i
nI	= number of interval variables
nR	= number of random variables
nS	= number of subsystems
$\mathbf{R}$	= vector of random variables
$\overline{\mathbf{R}}$	= mean value of $\mathbf{R}$
R	= random variable, element of $\mathbf{R}$
$\mathbf{U}$	= identity matrix
$\mathbf{Y}_{.i}$	= vector of linking variables, which are inputs to subsystem i and outputs of other subsystems
$\mathbf{Z}_i$	= vector of outputs of subsystem i
$\delta\sigma$	= difference between the maximum and minimum standard deviations
∇^2	= Hessian matrix of g with respect to both $\mathbf{R}$ and $\mathbf{I}$
$\nabla_{\mathbf{I}}$	= gradient of g with respect to $\mathbf{I}$
$\nabla_{\mathbf{R}}$	= gradient of g with respect to $\mathbf{R}$
σ	= standard deviation
$\sigma^{\max}$	= maximum standard deviation
$\sigma^{\min}$	= minimum standard deviation

Abstract

In many engineering applications, both random and interval variables exist. The mixture of both types of variables has been dealt with in robust design for single-disciplinary systems. This work focuses on robustness assessment for multidisciplinary systems with both random and interval variables. To alleviate the intensive computational demand, a Semi-Second-Order Taylor Expansion method is proposed to evaluate robustness. A performance function is approximated with linear terms of random and interval variables, as well as the interaction terms between the two types of variables. Then the maximum and minimum standard deviations of the performance function are computed. With the proposed method, the impact of both random and interval variables on the system robustness can be evaluated efficiently.

1 INTRODUCTION

Robust design reduces variations in product performances by minimizing the sensitivity to uncertainties, rather than by eliminating the source of uncertainty [1,2]. High robustness can be achieved by selecting optimal design variables through nonlinear optimization [3-5], where robustness assessment is the most important step. The task of robustness assessment is to evaluate if the design is robust at a given design point. When uncertainty comes from randomness, robustness is achieved by optimizing the mean performance and minimizing the standard deviation of the performance. The standard deviation must be calculated with robustness assessment during an optimization process. This type of analysis is referred to as *probabilistic robustness assessment*. Common methods of probabilistic robustness assessment are reviewed in [6].

In many engineering applications, some uncertainties are not due to randomness and therefore cannot be described by probability distributions [7-12]. For example, uncertainty may come from limited information or the lack of knowledge. In those cases, the interval presentation may be an alternative way to quantity uncertainty. Situations of interval variables are given in [11-14]. When only interval variables are involved, a design performance is also an interval. It is not appropriate to assign a probability distribution, e.g., a uniform distribution, to an interval variable; doing so may result in misleading or erroneous decisions. Instead, robustness must be measured with the width of a performance. The task of robustness assessment is then to find the width of the performance. This kind of analysis is referred to as *interval robustness assessment*. The width of the performance is minimized during a robust design process.

The robustness assessment methods in the third category handle both random and interval variables. Because of their generality, they are termed as the *general robust design* methods. The possible robustness measures are the average standard deviation of a performance and the difference between the maximum and minimum standard deviations of the performance [15]. The process of finding those standard deviations is referred to as *general robustness assessment*.

A vast amount of literature has accumulated on robust design for single-disciplinary systems. The methods fall into three categories. The most common one is the probabilistic robust design [3-6, 16-18]. It relies on probabilistic robustness assessment when only random variables are involved. The second category is interval robust design [9-12] when only interval variables are involved. It therefore relies on interval robustness assessment. The third category is the general robustness design, which, of course, depends upon general robustness assessment. This category deals with both random variables and interval variables [22].

Robustness assessment and robust design for multidisciplinary systems is much more complicated than its counterpart for single-disciplinary systems. The complexities come from (1) coupling between disciplines (subsystems), (2) the propagation of uncertainty between subsystems, and (3) the huge computational demand. The methods for coping with those complexities still fall into the three categories. In the probabilistic analysis category, several probabilistic uncertainty analysis methods have been developed [23-32]. These methods deal with the efficient integration of robust design and multidisciplinary design optimization (MDO), as well as the efficient integration of robustness assessment and multidisciplinary analysis (MDA). These methods are

applicable for robustness assessment and robust design for multidisciplinary systems when uncertainties are represented by probability distributions. In the interval analysis and design category, uncertainty is modeled by intervals, and efficient uncertainty propagation methods have been developed, including those reported in [33, 34]. The evidence theory with the interval presentation has also been considered in multidisciplinary systems design [35]. Reliability analysis methods [36] and Taylor expansion are normally employed in the methods in categories 1 and 2.

A recently developed method, the unified reliability analysis method [37], falls into the third category (general robustness assessment), where both random variables and interval variables exist. The method is also applicable for robustness assessment. It is, however, computationally expensive because of an iterative process for probabilistic analysis and optimization for interval analysis. To maintain good efficiency with acceptable accuracy, in this work, we use the First Order Second Moment (FOSM) method [38]. The original FOSM is for single-disciplinary systems and has been modified for multidisciplinary analysis [26, 39]. The current methods, however, cannot handle interval variables directly. The objective of this work, therefore, is to develop an efficient general robustness assessment method that extends the FOSM and the probabilistic multidisciplinary analysis [26, 39] to situations where both random and interval variables are present.

The rest of the paper is organized as follows. A general robustness assessment method for single-disciplinary systems is described in Section 2. Then the method is extended to multidisciplinary systems in Section 3. Numerical examples are presented in Section 4 followed by conclusions in Section 5.

2 ROBUSTNESS ASSESSMENT FOR SINGLE-DISCIPLINARY SYSTEMS

In this section, we discuss our proposed general robustness assessment method for single-disciplinary systems. The extension of the method to multidisciplinary systems is presented in the next section. Before the discussion of the method, the notations that we use are explained as follows: A bold letter stands for a vector, a capital letter for an uncertain variable, and a lower case letter for a deterministic variable or a realization of an uncertain variable. A vector is a column vector.

As shown in Fig. 1, a performance Z is a function of random variables $\mathbf{R}$ and interval variables $\mathbf{I}$; namely,

$$Z = g(\mathbf{R}, \mathbf{I}) \tag{1}$$

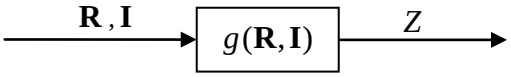


Figure 1. A single-disciplinary system

In the above equation, g is a performance function that specifies the relationship between the performance Z and input variables $\mathbf{R}$ and $\mathbf{I}$. The performance function g could be a black box.

Due to the interval variables $\mathbf{I}$, the standard deviation of Z, σ_Z, is also an interval. In other words, σ_Z is bounded within its maximum value $\sigma_Z^{\max}$ and minimum value $\sigma_Z^{\min}$. The robustness of a design can be measured by the average standard deviation, $\overline{\sigma}_Z$, and the difference, $\delta\sigma_Z$, between $\sigma_Z^{\max}$ and $\sigma_Z^{\min}$ [15]. The equations are given by

$$\overline{\sigma}_Z = \frac{1}{2}\left(\sigma_Z^{\max} + \sigma_Z^{\min}\right) \tag{2}$$

and

$$\delta\sigma_Z = \sigma_Z^{\max} - \sigma_Z^{\min} \tag{3}$$

The standard deviation σ_Z comes from the randomness in random variables **X**. If there were no interval variables **I**, σ_Z would reduce to a point, which could be close to the average standard deviation $\overline{\sigma}_Z$. Therefore, $\overline{\sigma}_Z$ could be used to measure the effect of random variables on *Z*. On the other hand, the standard deviation difference $\delta\sigma_Z$ is due to interval variables **I** and indicates the effect of **I** on *Z* [15]. The task of the general robustness assessment is to find $\overline{\sigma}_Z$ and $\delta\sigma_Z$, or $\sigma_Z^{\max}$ and $\sigma_Z^{\min}$.

$g(\mathbf{R}, \mathbf{I})$ is usually a nonlinear, multidimensional function, and a closed-form expression of σ_Z is rarely available. The most accurate method would be the combination of Monte Carlo simulation (MCS) and nonlinear global optimization. The former is for *probabilistic analysis* (PA), whose task is to find σ_Z due to random variables $\mathbf{R}$; and the latter is for *interval analysis* (IA), whose task is to identify the bounds of σ_Z over interval $\mathbf{I}$. Both PA and IA are nested; and MCS and nonlinear optimization are expensive. As a result, the overall robustness assessment would be computationally intensive.

In this work, we employ the First Order Second Moment (FOSM) method [38], a popular method in structural reliability. The FOSM uses the first order Taylor series expansion to linearize the performance function *g* at the means of random variables. Then

the standard deviation of *g* can be easily obtained. The FOSM is efficient because of the first order approximation. Hence the first order Taylor series expansion could be our first choice due to its high efficiency. The major problem, however, is that no standard deviation bounds can be produced. With the first order approximation, the linearized performance function will contain only the first order terms of $\mathbf{R}$ and $\mathbf{I}$. In calculating the standard deviation, the $\mathbf{I}$ terms are deterministic; their contribution to the standard deviation is therefore zero. Consequently, only a single standard deviation can be generated. The effect of interval variables $\mathbf{I}$ will be omitted due the simple linearization, and hence the error might be large.

To avoid the above problem, we may use the second order Taylor series expansion and then produce a standard deviation bound. But the computation will be intensive because the second derivatives are needed. To have a good balance between accuracy and efficiency, we propose a Semi-Second-Order Taylor Expansion method. The approximated function keeps all the first order terms of **R** (such as $R_1, R_2, \cdots$) and the interaction terms between **R** and **I** (such as $R_1 I_1, R_1 I_2, \cdots, R_2 I_1, R_2 I_2, \cdots$). Neither interaction terms *within* **R** (such as $R_1 R_2, \cdots$) and **I** (such as $I_1 I_2, \cdots$) nor their squared terms (such as $R_1^2, I_1^2, \cdots$) appear in the approximation. The approximated function is given by

$$\Delta g(\mathbf{R}, \mathbf{I}) \approx g(\mathbf{R}, \mathbf{I}) - g(\overline{\mathbf{R}}, \overline{\mathbf{I}}) = \nabla_{\mathbf{R}}^T \Delta\mathbf{R} + \nabla_{\mathbf{I}}^T \Delta\mathbf{I} + \frac{1}{2} \Delta\mathbf{X}^T \nabla^2 \Delta\mathbf{X} \tag{4}$$

In the above equation,

$$\Delta\mathbf{R} = \mathbf{R} - \overline{\mathbf{R}} \tag{5}$$

where $\mathbf{R} = (R_i) = (R_1, R_2, ..., R_{nR})^T$, $i = 1, 2, ..., nR$, in which nR is the number of random variables, and $\bar{\mathbf{R}} = (\bar{R}_i)^T = (\bar{R}_1, \bar{R}_2, ..., \bar{R}_{nR})^T$, in which $\bar{R}_i$ ($i = 1, 2, ..., nR$) is the mean of R_i.

$$\Delta\mathbf{I} = \mathbf{I} - \bar{\mathbf{I}} \tag{6}$$

where $\mathbf{I} = (I_j) = (I_1, I_2, ..., I_{nI})^T$, $j = 1, 2, ..., nI$, in which nI is the number of interval variables; $\bar{\mathbf{I}} = (\bar{I}_j) = (\bar{I}_1, \bar{I}_2, ..., \bar{I}_{nI})^T$, in which $\bar{I}_j$ ($j = 1, 2, ..., nI$) is the average of the *j*-th interval variable I_j and is given by

$$\bar{I}_j = \frac{1}{2}(I_j^l + I_j^u) \tag{7}$$

where I_j^l and I_j^u are the lower and upper bounds of I_j, respectively.

$\nabla_{\mathbf{R}}$ is the gradient of g with respect to $\mathbf{R}$ at $\bar{\mathbf{R}}$ and $\bar{\mathbf{I}}$. It is given by

$$\nabla_{\mathbf{R}} = \left.\frac{\partial g}{\partial \mathbf{R}}\right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} = \left.\left(\frac{\partial g}{\partial R_1}, \frac{\partial g}{\partial R_2}, ..., \frac{\partial g}{\partial R_{nR}}\right)^T\right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} \tag{8}$$

$\nabla_{\mathbf{I}}$ is the gradient of g with respect to $\mathbf{I}$ at $\bar{\mathbf{R}}$ and $\bar{\mathbf{I}}$. It is given by

$$\nabla_{\mathbf{I}} = \left.\frac{\partial g}{\partial \mathbf{I}}\right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} = \left.\left(\frac{\partial g}{\partial I_1}, \frac{\partial g}{\partial I_2}, ..., \frac{\partial g}{\partial I_{nR}}\right)^T\right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} \tag{9}$$

∇^2 in Eq. (4) is the Hessian matrix. Since we need only the interaction terms of $\mathbf{R}$ and $\mathbf{I}$, the Hessian matrix is incomplete and is given by

$$\nabla^2 = \begin{pmatrix} & & & \frac{\partial^2 g}{\partial R_1 \partial I_1} & \cdots & \frac{\partial^2 g}{\partial R_1 \partial I_{nI}} \\ & \mathbf{0} & & \cdots & \cdots & \cdots \\ & & & \frac{\partial^2 g}{\partial R_{nR} \partial I_1} & \cdots & \frac{\partial^2 g}{\partial R_{nR} \partial I_{nI}} \\ \frac{\partial^2 g}{\partial R_1 \partial I_1} & \cdots & \frac{\partial^2 g}{\partial R_1 \partial I_{nI}} & & & \\ \cdots & \cdots & \cdots & & \mathbf{0} & \\ \frac{\partial^2 g}{\partial R_{nR} \partial I_1} & \cdots & \frac{\partial^2 g}{\partial R_{nR} \partial I_{nI}} & & & \end{pmatrix} \tag{10}$$

$\Delta\mathbf{X}$ contains both $\Delta\mathbf{R}$ and $\Delta\mathbf{I}$ and is given by

$$\Delta\mathbf{X} = (\Delta\mathbf{R}^T, \Delta\mathbf{I}^T)^T \tag{11}$$

Because the $\mathbf{I}$ terms are deterministic, their contribution to the standard deviation is zero; only the $\mathbf{R}$ terms are therefore needed to compute σ_Z. We then group the $\mathbf{R}$ and $\mathbf{I}$ terms in Eq. (4) and obtain

$$\Delta g(\mathbf{R}, \mathbf{I}) \approx \mathbf{A}^T \Delta\mathbf{R} + \nabla_{\mathbf{I}}^T \Delta\mathbf{I} \tag{12}$$

where

$$\mathbf{A} = (A_i) = (A_1, A_2, ..., A_{nR})^T \ , \ i = 1, 2, ..., nR \tag{13}$$

in which

$$A_i = \left.\frac{\partial g}{\partial R_i}\right|_{\bar{\mathbf{R}}, \bar{\mathbf{I}}} + \sum_{m=1}^{nI} \left.\frac{\partial^2 g}{\partial R_i \partial I_m}\right|_{\bar{\mathbf{R}}, \bar{\mathbf{I}}} \Delta I_m \tag{14}$$

We now assume that the elements of $\mathbf{R}$ and $\mathbf{I}$ are independent. Let the standard deviations of $\mathbf{R}$ be $\sigma_{\mathbf{R}} = (\sigma_1, \sigma_2, ..., \sigma_{nR})^T$. Then the standard deviations of $Z = g(\mathbf{R}, \mathbf{I})$ is given by

$$\sigma_Z \approx \sqrt{(\mathbf{A}^T)^2 \sigma_{\mathbf{R}}^2} = \sqrt{\sum_{i=1}^{nR} \left(\left. \frac{\partial g}{\partial R_i} \right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} + \sum_{m=1}^{nI} \left. \frac{\partial^2 g}{\partial R_i \partial I_m} \right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} \Delta I_m \right)^2 \sigma_i^2} \tag{15}$$

where $(\mathbf{A}^T)^2$ is a vector whose elements are equal to the squares of the elements of $\mathbf{A}^T$ .This notation is also applied to other vectors throughout the paper.

As shown in Eq. (15), σ_Z is now a function of $\mathbf{I}$. The extreme values of σ_Z could occur at an interior point over the interval $[I_j^l, I_j^u]$ ($j = 1, 2, ..., nI$), or at I_j^l or I_j^u . Finding the maximum and minimum σ_Z can then be formulated as a bound constrained quadratic programming problem. For the minimum standard deviation $\sigma_Z^{\min}$, the optimization model is given by

$$\begin{aligned} &\min_{\mathbf{I}} \sqrt{\sum_{i=1}^{nR} \left(\left. \frac{\partial g}{\partial R_i} \right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} + \sum_{m=1}^{nI} \left. \frac{\partial^2 g}{\partial R_i \partial I_m} \right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} \Delta I_m \right)^2 \sigma_i^2} \\ &s.t.\ \ \mathbf{I}^l \le \mathbf{I} \le \mathbf{I}^u \end{aligned} \tag{16}$$

where $\mathbf{I}^l = \left(I_1^l, I_1^l, \cdots I_{nI}^l\right)$ and $\mathbf{I}^u = \left(I_1^u, I_1^u, \cdots I_{nI}^u\right)$.

For the maximum standard deviation $\sigma_Z^{\max}$, the optimization model becomes

$$\begin{aligned} &\max_{\mathbf{I}} \sqrt{\sum_{i=1}^{nR} \left(\left. \frac{\partial g}{\partial R_i} \right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} + \sum_{m=1}^{nI} \left. \frac{\partial^2 g}{\partial R_i \partial I_m} \right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} \Delta I_m \right)^2 \sigma_i^2} \\ &s.t.\ \ \mathbf{I}^l \le \mathbf{I} \le \mathbf{I}^u \end{aligned} \tag{17}$$

The objective functions in Eqs. (16) or (17) are function of **I** with known coefficients $\left. \frac{\partial g}{\partial R_i} \right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}}$ and $\sum_{m=1}^{nI} \left. \frac{\partial^2 g}{\partial R_i \partial I_m} \right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}}$. Therefore, after these coefficients are found,

solving the optimization problems does not call the original performance function *g*. However, calculating the coefficients before the optimization needs to call *g*. Suppose that the forward finite-difference method is used to numerically evaluate the derivatives and that the step sizes of a random variable and an interval variable are δ_R and δ_I, respectively. The derivatives can then be calculated as follows [40].

$$\left.\frac{\partial g}{\partial R_i}\right|_{\overline{\mathbf{R}},\overline{\mathbf{I}}} = \frac{g_{R_i} - \overline{g}}{\delta_R} \tag{18}$$

$$\frac{\partial^2 g}{\partial R_i \partial I_j} = \frac{g_{R_i I_j} - g_{R_i} - g_{I_j} + \overline{g}}{\delta_R \delta_I} \tag{19}$$

where

$$\overline{g} = g\left(\overline{\mathbf{R}}, \overline{\mathbf{I}}\right) \tag{20}$$

$$g_{R_i} = g\left(\overline{R}_1, \overline{R}_2, \cdots, \overline{R}_{i-1}, \overline{R}_i + \delta_R, \overline{R}_{i+1}, \cdots, \overline{R}_{nR}; \overline{\mathbf{I}}\right) \tag{21}$$

$$g_{I_j} = g\left(\overline{\mathbf{R}}; \overline{I}_1, \overline{I}_2, \cdots, \overline{I}_{j-1}, \overline{I}_j + \delta_I, \overline{I}_{j+1}, \cdots, \overline{I}_{nI}\right) \tag{22}$$

$$g_{R_i I_j} = g\left(\overline{R}_1, \overline{R}_2, \cdots, \overline{R}_{i-1}, \overline{R}_i + \delta_R, \overline{R}_{i+1}, \cdots, \overline{R}_{nR}; \overline{I}_1, \overline{I}_2, \cdots, \overline{I}_{j-1}, \overline{I}_j + \delta_I, \overline{I}_{i+1}, \cdots, \overline{I}_{nI}\right) \tag{23}$$

The numbers of *g* function the calls for $\overline{g}$, g_{R_i} $(i = 1,2,\cdots nR)$, g_{I_j} $(j = 1,2,\cdots nI)$, and $g_{R_i I_j}$ are 1, *nR*, *nI*, and *nR*×*nI* , respectively. Therefore, the total number of function calls is

$$N = nR + nI + nR \times nI + 1 \tag{24}$$

For example, if there are five random variables ($nR = 5$) and two interval variables ($nI = 2$), $N = 5 + 2 + 5 \times 2 + 1 = 18$. This means that *g* will be called 18 times.

3 ROBUSTNESS ASSESSMENT FOR MULTIDISCIPLINARY SYSTEMS

In this section we discuss how to extend the above method to multidisciplinary systems.

3.1 MULTIDISCIPLINARY SYSTEMS

Fig. 2 shows an example of a multidisciplinary system consisting of three disciples (subsystems).

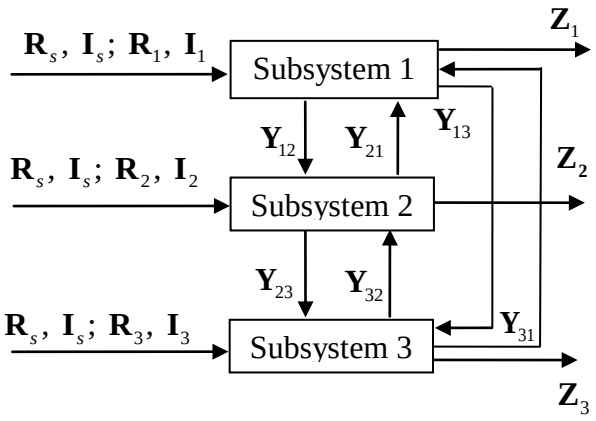


Figure 2. A Multidisciplinary System

The symbols used in Fig. 2 are explained in Table 1.

Table 1. Variables in Fig. 2

Variables	Description
$\mathbf{R}_s$	Random sharing input variables
$\mathbf{I}_s$	Interval sharing input variables
$\mathbf{R}_i$	Random input variables to subsystem *i*
$\mathbf{I}_i$	Interval input variables to subsystem *i*
$\mathbf{Y}_{ij}$	Linking variables, which are outputs of subsystem *i* and inputs to subsystem *j*
$\mathbf{Z}_i$	Outputs of subsystem *i*

As shown in Fig. 2, all the subsystems are coupled through linking variables. The vector of linking variables $\mathbf{Y}_{ij}$ is the output of subsystem i ($i = 1, 2, ..., nS$) and also the input to subsystem j ($j = 1, 2, ..., nS$, $j \neq i$), where nS is the number of subsystems. $\mathbf{Y}_{ij}$ is given by

$$\mathbf{Y}_{ij} = \mathbf{g}_{Y_{ij}}(\mathbf{R}_s, \mathbf{I}_s, \mathbf{R}_i, \mathbf{I}_i, \mathbf{Y}_{\bullet i}) \tag{25}$$

where $\mathbf{g}_{Y_{ij}}$ is a vector of all the functions for all the elements in $\mathbf{Y}_{ij}$. If the size of $\mathbf{Y}_{ij}$ is n_{ij}, then Eq. (25) can be expanded as

$$\begin{cases} Y_{ij,1} = g_{Y_{ij,1}}(\mathbf{R}_s, \mathbf{I}_s, \mathbf{R}_i, \mathbf{I}_i, \mathbf{Y}_{\bullet i}) \\ Y_{ij,1} = g_{Y_{ij,2}}(\mathbf{R}_s, \mathbf{I}_s, \mathbf{R}_i, \mathbf{I}_i, \mathbf{Y}_{\bullet i}) \\ \cdots \\ Y_{ij,n_{ij}} = g_{Y_{ij,n_{ij}}}(\mathbf{R}_s, \mathbf{I}_s, \mathbf{R}_i, \mathbf{I}_i, \mathbf{Y}_{\bullet i}) \end{cases} \tag{26}$$

In the above equations, $\mathbf{Y}_{\bullet i}$ are the linking variables from other subsystems and are given by

$$\mathbf{Y}_{\bullet i} = (\mathbf{Y}^T{}_{1i}, \mathbf{Y}^T{}_{2i}, \cdots, \mathbf{Y}^T{}_{(i-1)i}, \mathbf{Y}^T{}_{(i+1)i}, \cdots, \mathbf{Y}^T{}_{(nS)i})^T \tag{27}$$

The task of multidisciplinary analysis (MDA) is to calculate the output $\mathbf{Z}_i$ ($i = 1, 2, ..., nS$) given all the inputs $\mathbf{R}_s$, $\mathbf{I}_s$, $\mathbf{R}_i$, and $\mathbf{I}_i$. In subsystem i, $\mathbf{Z}_i$ is given by

$$\mathbf{Z}_i = \mathbf{g}_{Z_i}(\mathbf{R}_s, \mathbf{I}_s, \mathbf{R}_i, \mathbf{I}_i, \mathbf{Y}_{\bullet i}) \tag{28}$$

MDA first solves the nonlinear simultaneous equations in Eq. (26). Once the linking variables $\mathbf{Y}_{ij}$ are obtained, the coupling between subsystems is cut off. Then $\mathbf{Z}_i$ can be computed by Eq. (28). Solving the nonlinear simultaneous equation in Eq. (26)

requires calling $\mathbf{g}_{Y_{ij}}$ repeatedly. In this paper, the evaluation of a function in $\mathbf{g}_{Y_{ij}}$ and $\mathbf{g}_{Z_i}$ ($i, j = 1,2,...,nS$, $i \neq j$) is called a *subsystem analysis*.

At first one MDA should be performed to obtain all the linking variables $\overline{\mathbf{Y}}_{ij} = \mathbf{g}_{Y_{ij}}(\overline{\mathbf{R}}_s, \overline{\mathbf{I}}_s, \overline{\mathbf{R}}_i, \overline{\mathbf{I}}_i, \overline{\mathbf{Y}}_{\bullet i})$ at the means of random variables and averages of interval variables. Any MDA method can be used for this purpose. Then two steps at the subsystem level are followed. The first step is to cut off the coupling between subsystems. This step solves the nonlinear simultaneous equations in Eq. (26) by calling subsystem analyses repeatedly. Then in the second step, the output $\mathbf{Z}_i$ is now a function of only random and interval input variables; the linking variables are no longer in $\mathbf{g}_{Z_i}$ ($i = 1,2,...,nS$). We can then apply the single-disciplinary analysis method described in Section 2 to calculate the standard deviation of a system output.

3.2 Step 1 – Eliminating Linking Variables

Eliminating linking variables requires solving the nonlinear simultaneous equations in Eq. (26). It is usually computationally expensive. We also use the Semi-Second-Order Taylor Expansion method to simplify $\mathbf{g}_{Y_{ij}}$ in Eq. (26). $\mathbf{g}_{Y_{ij}}$ is approximated with linear terms of $\mathbf{R}_s$, $\mathbf{R}_i$, and $\mathbf{Y}_{\bullet i}$, as well as the interaction terms between $\mathbf{R}_s$ and $\mathbf{I}_s$, $\mathbf{R}_s$ and $\mathbf{I}_i$, $\mathbf{R}_i$ and $\mathbf{I}_s$, and $\mathbf{R}_i$ and $\mathbf{I}_i$. Similarly to Eq. (12), the approximations of $\mathbf{g}_{Y_{ij}}$ are given by

$$\Delta\mathbf{Y}_{ij} = \mathbf{g}_{Y_{ij}}(\mathbf{R}_s, \mathbf{I}_s, \mathbf{R}_i, \mathbf{I}_i, \mathbf{Y}_{\bullet i}) - \overline{\mathbf{Y}}_{ij} \approx \mathbf{A}_{ij}\Delta\mathbf{R}_s + \mathbf{B}_{ij}\Delta\mathbf{R}_i + \frac{\partial \mathbf{g}_{Y_{ij}}}{\partial \mathbf{Y}_{\bullet i}}\Delta\mathbf{Y}_{\bullet i} + \mathbf{M}_{ij}^{\mathbf{Y}}(\mathbf{I}_s, \mathbf{I}_i) \quad (29)$$

where

$$\frac{\partial \mathbf{g}_{Y_{ij}}}{\partial \mathbf{Y}_{\bullet i}} = \left(\frac{\partial \mathbf{g}_{Y_{ij}}}{\partial \mathbf{Y}_{1i}}, \frac{\partial \mathbf{g}_{Y_{ij}}}{\partial \mathbf{Y}_{2i}}, ..., \frac{\partial \mathbf{g}_{Y_{ij}}}{\partial \mathbf{Y}_{(i-1)i}}, \frac{\partial \mathbf{g}_{Y_{ij}}}{\partial \mathbf{Y}_{(i+1)i}}, ..., \frac{\partial \mathbf{g}_{Y_{ij}}}{\partial \mathbf{Y}_{nS}} \right) \tag{30}$$

in which

$$\frac{\partial \mathbf{g}_{Y_{ij}}}{\partial \mathbf{Y}_{ki}} = \begin{pmatrix} \frac{\partial g_{Y_{ij,1}}}{\partial Y_{ki,1}} & \frac{\partial g_{Y_{ij,1}}}{\partial Y_{ki,2}} & \cdots & \frac{\partial g_{Y_{ij,1}}}{\partial Y_{ki,n_{ki}}} \\ \frac{\partial g_{Y_{ij,2}}}{\partial Y_{ki,1}} & \frac{\partial g_{Y_{ij,2}}}{\partial Y_{ki,2}} & \cdots & \frac{\partial g_{Y_{ij,2}}}{\partial y_{ki,n_{ki}}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_{Y_{ij,n_{ij}}}}{\partial Y_{ki,1}} & \frac{\partial g_{Y_{ij,n_{ij}}}}{\partial Y_{ki,2}} & \cdots & \frac{\partial g_{Y_{ij,n_{ij}}}}{\partial Y_{ki,n_{ki}}} \end{pmatrix} (k = 1, 2, ..., i-1, i+1, ..., nS) \tag{31}$$

$$\Delta \mathbf{Y}_{\bullet i} = \left(\Delta \mathbf{Y}_{1i}^T, \Delta \mathbf{Y}_{2i}^T, ..., \Delta \mathbf{Y}_{(i-1)i}^T, \Delta \mathbf{Y}_{(i+1)i}^T, ..., \Delta \mathbf{Y}_{(nS)i}^T \right)^T \tag{32}$$

$$\mathbf{A}_i = \begin{pmatrix} \frac{\partial g_{Y_{ij,1}}}{\partial R_{s,1}} + \sum_{k=1}^{nSI} \frac{\partial^2 g_{Y_{ij,1}}}{\partial R_{s,1} \partial I_{s,k}} & .... & \frac{\partial g_{Y_{ij,1}}}{\partial R_{s,nSR}} + \sum_{k=1}^{nSI} \frac{\partial^2 g_{Y_{ij,1}}}{\partial R_{s,nSR} \partial I_{s,k}} \\ ... & ... & ... \\ \frac{\partial g_{Y_{ij,n_{ij}}}}{\partial R_{s,1}} + \sum_{k=1}^{nSI} \frac{\partial^2 g_{Y_{ij,n_{ij}}}}{\partial R_{s,1} \partial I_{s,k}} & ... & \frac{\partial g_{Y_{ij,n_{ij}}}}{\partial R_{s,nSR}} + \sum_{k=1}^{nSI} \frac{\partial^2 g_{Y_{ij,n_{ij}}}}{\partial R_{s,nSR} \partial I_{s,k}} \end{pmatrix} \tag{33}$$

where nSI is the number of the sharing interval variables, nSR is the number of the sharing random variables, $R_{s,m}$ is the m-th element of $\mathbf{R}_s$, and $g_{Y_{ij,k}}$ is the k-th element of $\mathbf{g}_{Y_{ij}}$.

$$\mathbf{B}_i = \begin{pmatrix} \frac{\partial g_{Y_{ij,1}}}{\partial R_{i,1}} + \sum_{k=1}^{nI_i} \frac{\partial^2 g_{Y_{ij,1}}}{\partial R_{i,1} \partial I_{i,k}} & .... & \frac{\partial g_{Y_{ij,1}}}{\partial R_{i,nR_i}} + \sum_{k=1}^{nI_i} \frac{\partial^2 g_{Y_{ij,1}}}{\partial R_{i,nR_i} \partial I_{i,k}} \\ ... & ... & ... \\ \frac{\partial g_{Y_{ij,n_{ij}}}}{\partial R_{i,1}} + \sum_{k=1}^{nI_i} \frac{\partial^2 g_{Y_{ij,n_{ij}}}}{\partial R_{i,1} \partial I_{i,k}} & ... & \frac{\partial g_{Y_{ij,n_{ij}}}}{\partial R_{i,nR_i}} + \sum_{k=1}^{nI_i} \frac{\partial^2 g_{Y_{ij,n_{ij}}}}{\partial R_{i,nR_i} \partial I_{i,k}} \end{pmatrix} \tag{34}$$

where nR_i and nI_i are the numbers of the random variables and interval variables in the i-th subsystem, respectively. $R_{i,k}$ is the k-th element of $\mathbf{R}_i$. $\mathbf{M}_{ij}^{\mathbf{Y}}(\mathbf{I}_s, \mathbf{I}_i)$ contains all the first-order terms of $\mathbf{I}_s$ and $\mathbf{I}_i$. It should be noted that all the derivatives are calculated at the means of random variables and averages of interval variables. The derivatives are therefore constants. Assembling all linking variables together yields the following linear simultaneous equations

$$\begin{pmatrix} \mathbf{U}_1 & -\frac{\partial \mathbf{g}_{Y_{1.}}}{\partial \mathbf{Y}_{2.}} & \cdots & -\frac{\partial \mathbf{g}_{Y_{1.}}}{\partial \mathbf{Y}_{nS.}} \\ -\frac{\partial \mathbf{g}_{Y_{2.}}}{\partial \mathbf{Y}_{1.}} & \mathbf{U}_2 & \cdots & -\frac{\partial \mathbf{g}_{Y_{2.}}}{\partial \mathbf{Y}_{nS.}} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{\partial \mathbf{g}_{Y_{nS.}}}{\partial \mathbf{Y}_{1.}} & -\frac{\partial \mathbf{g}_{Y_{nS.}}}{\partial \mathbf{Y}_{2.}} & \cdots & \mathbf{U}_{nS} \end{pmatrix} \Delta\mathbf{Y} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \cdots \\ \mathbf{A}_{nS} \end{pmatrix} \Delta\mathbf{R}_s + \begin{pmatrix} \mathbf{B}_1 & & \mathbf{0} \\ & \mathbf{B}_2 & \\ \mathbf{0} & & \\ & & \mathbf{B}_{nS} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{R}_1 \\ \Delta\mathbf{R}_2 \\ \cdots \\ \Delta\mathbf{R}_{nS} \end{pmatrix} + \mathbf{M} \tag{35}$$

where $\mathbf{U}$ stands for an identity matrix, $\mathbf{g}_{Y_{i.}} = \left(\mathbf{g}_{Y_{i1}}^T, \mathbf{g}_{Y_{i2}}^T, \cdots, \mathbf{g}_{Y_{i(i-1)}}^T, \mathbf{g}_{Y_{i(i+1)}}^T, \cdots, \mathbf{g}_{Y_{i(nS)}}^T\right)^T$,

$\Delta\mathbf{Y}_{i\bullet}^T = \left(\Delta\mathbf{Y}_{i1}^T, \Delta\mathbf{Y}_{i2}^T, \cdots, \Delta\mathbf{Y}_{i(i-1)}^T, \Delta\mathbf{Y}_{i(i+1)}^T, \cdots, \Delta\mathbf{Y}_{inS}^T\right)^T \ (i = 1, 2, \cdots, nS)$,

$$\mathbf{A}_i = \begin{pmatrix} \mathbf{A}_{i1} & & & & & \\ & \cdots & & & \mathbf{0} & \\ & & \mathbf{A}_{i(i-1)} & & & \\ & & & \mathbf{A}_{i(i+1)} & & \\ & \mathbf{0} & & & \cdots & \\ & & & & & \mathbf{A}_{i(nSi)} \end{pmatrix}, \text{ and}$$

$$\mathbf{B}_i = \begin{pmatrix} \mathbf{B}_{i1} & & & & \\ & \ldots & & \mathbf{0} & \\ & & \mathbf{B}_{i(i-1)} & & \\ & & & \mathbf{B}_{i(i+1)} & \\ & \mathbf{0} & & \ldots & \\ & & & & \mathbf{B}_{i(nSi)} \end{pmatrix}, \; i = 1, 2, ..., nS$$

Eq. (35) can be rewritten as

$$\mathbf{C}\Delta\mathbf{Y} = \mathbf{A}\Delta\mathbf{R}_s + \mathbf{B}\Delta\mathbf{R} + \mathbf{M} \tag{36}$$

where $\mathbf{C} = \begin{pmatrix} \mathbf{U_1} & -\frac{\partial \mathbf{g}_{Y_1.}}{\partial \mathbf{Y}_{2.}} & \ldots & -\frac{\partial \mathbf{g}_{Y_1.}}{\partial \mathbf{Y}_{nS.}} \\ -\frac{\partial \mathbf{g}_{Y_2.}}{\partial \mathbf{Y}_{1.}} & \mathbf{U}_2 & \ldots & -\frac{\partial \mathbf{g}_{Y_2.}}{\partial \mathbf{Y}_{nS.}} \\ \ldots & \ldots & \ldots & \ldots \\ -\frac{\partial \mathbf{g}_{Y_{nS}.}}{\partial \mathbf{Y}_{1.}} & -\frac{\partial \mathbf{g}_{Y_{nS}.}}{\partial \mathbf{Y}_{2.}} & \ldots & \mathbf{U}_{nS} \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \ldots \\ \mathbf{A}_{nS} \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \mathbf{B}_1 & & & \mathbf{0} \\ & \mathbf{B}_2 & & \\ \mathbf{0} & & & \\ & & & \mathbf{B}_{nS} \end{pmatrix}$, and

$$\Delta\mathbf{R} = \begin{pmatrix} \Delta\mathbf{R}_1 \\ \Delta\mathbf{R}_2 \\ \ldots \\ \Delta\mathbf{R}_{nS} \end{pmatrix}$$

Solving the system of equations in Eq. (36) yields

$$\Delta\mathbf{Y} = \mathbf{C}^{-1}\mathbf{A}\Delta\mathbf{R}_s + \mathbf{C}^{-1}\mathbf{B}\Delta\mathbf{R} + \mathbf{C}^{-1}\mathbf{M} \tag{37}$$

$\Delta\mathbf{Y}$ is now functions of the input variables.

3.2 Step 2 – Solving Standard Deviations of $\mathbf{Z}$

After $\Delta\mathbf{Y}$ are obtained, all the outputs can be expressed in terms of the input variables. Similarly to Eq. (29), outputs $\Delta\mathbf{Z}_i$ are approximated as

$$\Delta \mathbf{Z}_i \approx \mathbf{D}_i \Delta \mathbf{R}_s + \mathbf{E}_i \Delta \mathbf{R}_i + \frac{\partial \mathbf{g}_{z_i}}{\partial \mathbf{Y}_{\cdot i}} \Delta \mathbf{Y}_{\bullet i} + \mathbf{M}_i^{\mathbf{z}}(\mathbf{I}_s, \mathbf{I}_i) \tag{38}$$

where

$$\mathbf{D}_i = \begin{pmatrix} \frac{\partial g_{z_{i,1}}}{\partial R_{s,1}} + \sum_{k=1}^{nSI} \frac{\partial^2 g_{z_{i,1}}}{\partial R_{s,1} \partial I_{s,k}} & \cdots & \frac{\partial g_{z_{i,1}}}{\partial R_{s,nSR}} + \sum_{k=1}^{nS_I} \frac{\partial^2 g_{z_{i,1}}}{\partial R_{s,nSR} \partial I_{s,k}} \\ \cdots & \cdots & \cdots \\ \frac{\partial g_{z_{i,n_{ij}}}}{\partial R_{s,1}} + \sum_{k=1}^{nSI} \frac{\partial^2 g_{z_{i,n_{ij}}}}{\partial R_{s,1} \partial I_{s,k}} & \cdots & \frac{\partial g_{z_{i,n_{ij}}}}{\partial R_{s,nSR}} + \sum_{k=1}^{nS_I} \frac{\partial^2 g_{z_{i,n_{ij}}}}{\partial R_{s,nSR} \partial I_{s,k}} \end{pmatrix},$$

$$\mathbf{B}_i = \begin{pmatrix} \frac{\partial g_{z_{i,1}}}{\partial R_{i,1}} + \sum_{k=1}^{nI_i} \frac{\partial^2 g_{z_{i,1}}}{\partial R_{i,1} \partial I_{i,k}} & \cdots & \frac{\partial g_{z_{i,1}}}{\partial R_{i,nR_i}} + \sum_{k=1}^{nI_i} \frac{\partial^2 g_{z_{i,1}}}{\partial R_{i,nR_i} \partial I_{i,k}} \\ \cdots & \cdots & \cdots \\ \frac{\partial g_{z_{i,n_{ij}}}}{\partial R_{i,1}} + \sum_{k=1}^{nI_i} \frac{\partial^2 g_{z_{i,n_{ij}}}}{\partial R_{i,1} \partial I_{i,k}} & \cdots & \frac{\partial g_{z_{i,n_{ij}}}}{\partial R_{i,nR_i}} + \sum_{k=1}^{nI_i} \frac{\partial^2 g_{z_{i,n_{ij}}}}{\partial R_{i,nR_i} \partial I_{i,k}} \end{pmatrix}, \text{ and}$$

$$\frac{\partial \mathbf{g}_{z_i}}{\partial \mathbf{Y}_{\bullet i}} = \left(\frac{\partial \mathbf{g}_{z_i}}{\partial \mathbf{Y}_{1i}}, \frac{\partial \mathbf{g}_{z_i}}{\partial \mathbf{Y}_{2i}}, \cdots, \frac{\partial \mathbf{g}_{z_i}}{\partial \mathbf{Y}_{(i-1)i}}, \frac{\partial \mathbf{g}_{z_i}}{\partial \mathbf{Y}_{(i+1)i}}, \cdots, \frac{\partial \mathbf{g}_{z_i}}{\partial \mathbf{Y}_{nS}} \right)$$

in which $\frac{\partial \mathbf{g}_{z_i}}{\partial \mathbf{Y}_{ki}} = \begin{pmatrix} \frac{\partial g_{z_{i,1}}}{\partial Y_{ki,1}} & \frac{\partial g_{z_{i,1}}}{\partial Y_{ki,2}} & \cdots & \frac{\partial g_{z_{i,1}}}{\partial Y_{ki,n_{ki}}} \\ \frac{\partial g_{z_{i,2}}}{\partial Y_{ki,1}} & \frac{\partial g_{z_{i,2}}}{\partial Y_{ki,2}} & \cdots & \frac{\partial g_{z_{i,2}}}{\partial Y_{ki,n_{ki}}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_{z_{i,n_{ij}}}}{\partial Y_{ki,1}} & \frac{\partial g_{z_{i,n_{ij}}}}{\partial Y_{ki,2}} & \cdots & \frac{\partial g_{z_{i,n_{ij}}}}{\partial Y_{ki,n_{ki}}} \end{pmatrix}$, $k = 1, 2, ..., i-1, i+1, ..., nS$

Assembling Eq. (38) for all subsystems and plugging Eq. (37) into Eq. (38), we obtain $\Delta \mathbf{Z}$ in the following matrix form

$$\Delta \mathbf{Z} = \mathbf{D}\Delta \mathbf{R}_s + \mathbf{E}\Delta \mathbf{R} + \mathbf{F}\Delta \mathbf{Y} + \mathbf{M}^Z = [\mathbf{D} + \mathbf{F}(\mathbf{C}^{-1}\mathbf{A})]\Delta \mathbf{R}_s + [\mathbf{E} + \mathbf{F}(\mathbf{C}^{-1}\mathbf{B})]\Delta \mathbf{R} + \mathbf{F}\mathbf{C}^{-1}\mathbf{M}^Y + \mathbf{M}^Z \tag{39}$$

where $\mathbf{F} = \begin{pmatrix} \mathbf{0} & -\frac{\partial \mathbf{g}_{z_1}}{\partial \mathbf{Y}_{2\cdot}} & \cdots & -\frac{\partial \mathbf{g}_{z_1}}{\partial \mathbf{Y}_{nS\cdot}} \\ -\frac{\partial \mathbf{g}_{z_2}}{\partial \mathbf{Y}_{1\cdot}} & \mathbf{0} & \cdots & -\frac{\partial \mathbf{g}_{z_2}}{\partial \mathbf{Y}_{nS\cdot}} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{\partial \mathbf{g}_{z_1(nS)}}{\partial \mathbf{Y}_{1\cdot}} & -\frac{\partial \mathbf{g}_{z_1(nS)}}{\partial \mathbf{Y}_{2\cdot}} & \cdots & \mathbf{0} \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \\ \cdots \\ \mathbf{D}_{nS} \end{pmatrix}$ and $\mathbf{E} = \begin{pmatrix} \mathbf{E}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_2 & \mathbf{0} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0}. & \cdots & \mathbf{E}_{nS} \end{pmatrix}$.

Then the variance of $\mathbf{Z}$ can be expressed as

$$\boldsymbol{\sigma}_z^2 = \mathbf{Q}\boldsymbol{\sigma}_{Rs}^2 + \mathbf{T}\boldsymbol{\sigma}_R^2 \tag{40}$$

where $\boldsymbol{\sigma}_Z = \begin{pmatrix} \boldsymbol{\sigma}_{\mathbf{Z}_1}^2 \\ \boldsymbol{\sigma}_{\mathbf{Z}_2}^2 \\ \cdots \\ \boldsymbol{\sigma}_{\mathbf{Z}_{nS}}^2 \end{pmatrix}$, $\boldsymbol{\sigma}_R = \begin{pmatrix} \sigma_{R_1}^2 \\ \sigma_{R_2}^2 \\ \cdots \\ \sigma_{R_{nS}}^2 \end{pmatrix}$, $\mathbf{Q} = \left[\mathbf{D} + \mathbf{F}(\mathbf{C}^{-1}\mathbf{A})\right]^2$, and $\mathbf{T} = \left[\mathbf{E} + \mathbf{F}(\mathbf{C}^{-1}\mathbf{B})\right]^2$.

$\boldsymbol{\sigma}_Z$ now is a function of $\mathbf{I}_s$ and $\mathbf{I}_i$. The optimization is used again to calculate the maximum standard deviation $\sigma_Z^{\max}$ and the minimum standard deviation $\sigma_Z^{\min}$. Similarly to Eq. (16), the optimization model for $\sigma_Z^{\min}$ is

$$\begin{aligned} &\min_{\mathbf{I}} \ \sigma_Z(\mathbf{I}) \\ &s.t. \ \ \mathbf{I}^l \le \mathbf{I} \le \mathbf{I}^u \end{aligned} \tag{41}$$

where $\mathbf{I} = \left(\mathbf{I}_s, \mathbf{I}_1, \mathbf{I}_2, \cdots, \mathbf{I}_{nS}\right)$, $\mathbf{I}^l = \left(\mathbf{I}_S^l, \mathbf{I}_1^l, \mathbf{I}_2^l, \cdots, \mathbf{I}_{nS}^l\right)$, and $\mathbf{I}^u = \left(\mathbf{I}_S^u, \mathbf{I}_1^u, \mathbf{I}_2^u, \cdots, \mathbf{I}_{nS}^u\right)$. σ_Z is one element of $\boldsymbol{\sigma}_Z$.

The optimization model for $\sigma_Z^{\max}$ is

$$\begin{aligned} &\max_{\mathbf{I}} \ \sigma_Z(\mathbf{I}) \\ &s.t. \ \ \mathbf{I}^l \le \mathbf{I} \le \mathbf{I}^u \end{aligned} \tag{42}$$

$\mathbf{g}_{Y_{ij}}$ and $\mathbf{g}_{z_i}$ ($i, j = 1,2,...,nS$, $i \neq j$) are expanded at the mean values of $\mathbf{R}_s$ and $\mathbf{R}_i$, and the averages of $\mathbf{I}_s$ and $\mathbf{R}_i$. And all the derivatives of $\mathbf{g}_{Y_{ij}}$ and $\mathbf{g}_{z_i}$ discussed above are evaluated locally at the subsystem level. Therefore, only one MDA at the system level should be performed initially. Similarly to Eq. (24), the computational cost of the proposed method is measured by

$$N_i = \left[(nR_s + nR_i + nY_i) + (nR_s + nR_i)(nI_s + nI_i)\right] n_i \tag{43}$$

where N_i is the number of analyses in subsystem *i*, nY_i is the number of linking variables in $\mathbf{Y}_{\cdot i}$ that are the input to subsystem *i*, and n_i is the total number of output in $\mathbf{Y}_{i\cdot}$ and $\mathbf{Z}_i$. The first term $(nR_s + nR_i + nY_i)n_i$ on the right-hand side is for the first derivatives of $\mathbf{g}_{Y_{ij}}$ and $\mathbf{g}_{z_i}$ with respect to $\mathbf{R}_s$, $\mathbf{I}_s$, $\mathbf{R}_i$, $\mathbf{I}_i$, and $\mathbf{Y}_{\cdot i}$, and the second term $N_i = (nR_s + nR_i)(nI_s + nI_i)n_i$ is for the second derivatives of $\mathbf{g}_{Y_{ij}}$ and $\mathbf{g}_{z_i}$ with respect to $\mathbf{R}_i$ and $\mathbf{I}_s$, $\mathbf{R}_i$ and $\mathbf{I}_i$, $\mathbf{R}_s$ and $\mathbf{I}_s$, and $\mathbf{R}_s$ and $\mathbf{I}_i$.

4. EXAMPLES

Two examples are used to for demonstration. The first one is simple, and therefore all the details are presented. It allows the reader to repeat the process easily.

4.1 A Math Example

Two subsystems are involved as shown in Fig. 3 .The details of the problem are given as follows.

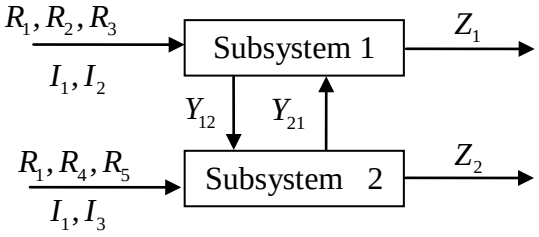


Figure 3. Example 1

Subsystem 1

Input: $\mathbf{R}_s = (R_1)$, $\mathbf{I}_s = (I_1)$, $\mathbf{R}_1 = (R_2, R_3)^T$, $\mathbf{I}_1 = (I_2)$, and $\mathbf{Y}_{\cdot 1} = (Y_{21})$

Output: $\mathbf{Y}_{1\cdot} = (Y_{12})$ and $\mathbf{Z}_1 = (Z_1)$

where $Y_{12} = (R_1 + I_1)^2 + 2(R_2 + I_2) - R_3 + 2\sqrt{Y_{21}}$, and

$$Z_1 = (R_1 + I_1)^2 + 2(R_2 + I_2) + R_3 + (R_2 + I_2)e^{-Y_{21}}$$

Subsystem 2

Input: $\mathbf{R}_s = (R_1)$, $\mathbf{I}_s = (I_1)$, $\mathbf{R}_2 = (R_4, R_5)^T$, $\mathbf{I}_2 = (I_3)$, and $\mathbf{Y}_{\cdot 2} = (Y_{12})$

Output: $\mathbf{Y}_{2\cdot} = (Y_{21})$ and $\mathbf{Z}_2 = (Z_2)$

where $Y_{21} = (R_1 + I_1)(R_4 + I_3) + (R_4 + I_3)^2 + R_5 + Y_{12}$, and

$$Z_2 = \sqrt{R_1 + I_1} + (R_4 + I_3) + 0.4R_5(R_1 + I_1)$$

The distributions of the random variables are given in Table 2. The averages of the interval variables $I_1 \sim I_3$ are 1.0, and their widths are 0.1. The values are given in Table 3.

Table 2. Random Variables

Variable	Mean	Standard Deviation	Distribution
R_1	1.0	01	Normal
R_2	1.0	0.1	Normal
R_3	1.0	0.1	Normal
R_4	1.0	0.1	Normal
R_5	1.0	0.1	Normal

Table 3. Interval Variables

Variable	I^L	I^U
I_1	$1.0-0.05$	$1.0+0.05$
I_2	$1.0-0.05$	$1.0+0.05$
I_3	$1.0-0.05$	$1.0+0.05$

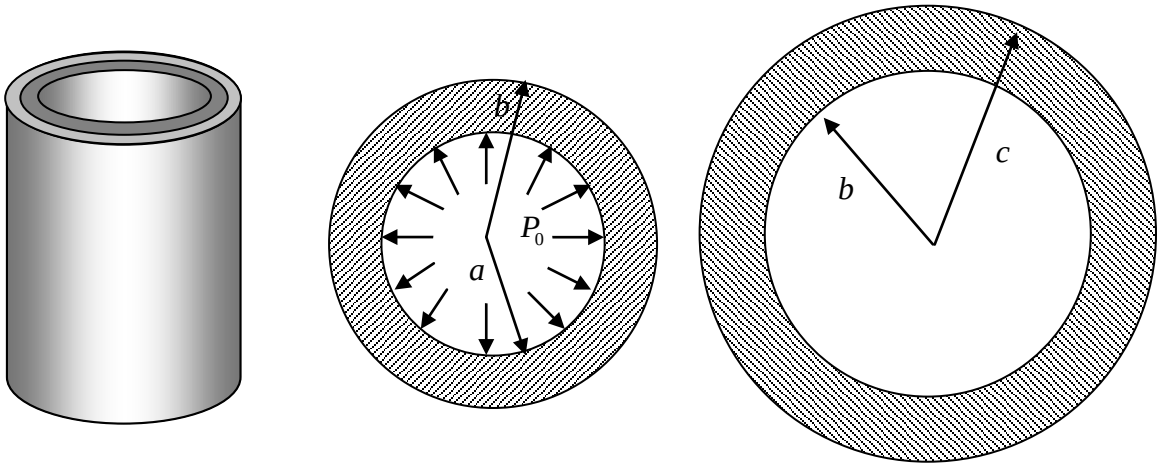
The proposed method is used to estimate the standard deviations of the output of each subsystem. The results are given in Table 4. The number of MDA is 1, and the number of subsystem analyses for each subsystem is 26. Monte Carlo simulation (MCS) is used to confirm the results. For MCS, each of interval variable is divided into 10 smaller intervals, and 500 samples are taken for each random variable. The number of MDA is therefore equal to $10\times10\times10\times500=5\times10^5$. The results from the proposed method match well with those from MCS, while the former method is much more efficient.

Table 4. Robustness Assessment Result

	GRA	MCS
$\sigma_{Z_1}^{\max}$	0.467	0.479
$\sigma_{Z_1}^{\min}$	0.450	0.457
$\sigma_{Z_2}^{\max}$	0.150	0.151
$\sigma_{Z_2}^{\min}$	0.148	0.145

4.2 Cylinder Problem

For a further illustration, we present an engineering application example [29]. The example is a compound cylinders design (Fig. 4). The two cylinders are treated as a multidisciplinary system, where the inner and outer cylinders are considered as subsystems 1 and 2, respectively. The internal and external radii of the inner cylinder are *a* and *b*, respectively, and the internal and external radii of the outer cylinder are *b* and *c*, respectively. The internal pressure is P_0.



Compound cylinder system Subsystem 1: Inner cylinder Subsystem 2: Outer cylinder

Figure 4. Compound cylinder system

The notations used in this example are given in Fig.5.

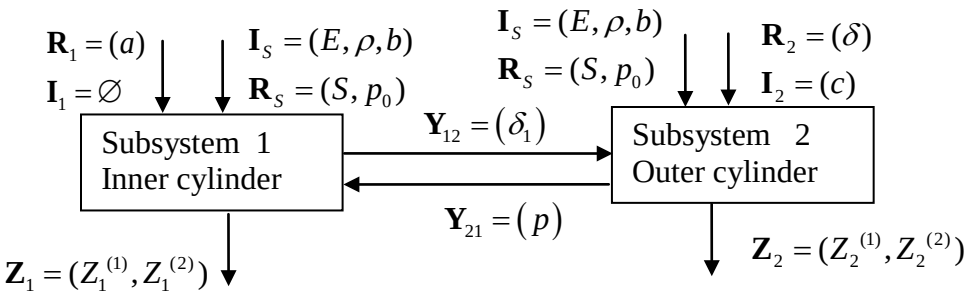


Figure 5. System structure of the compound cylinders

Subsystem 1: the inner cylinder

Input: $\mathbf{R}_S = (S, p_0)^T$, $\mathbf{I}_S = (E, \rho, b)^T$, $\mathbf{R}_1 = (a)$, $\mathbf{I}_1 = \varnothing$,

$\mathbf{Y}_{\cdot 1} = (Y_{21})$ where $\mathbf{Y}_{21} = (p)$, $\delta_2 = \delta - \delta_1$, and

$$p = \frac{\delta_2 E}{b} \Big/ \left(\frac{b^2 + c^2}{c^2 - b^2} + \rho \right) = \frac{(\delta - \delta_1) E}{b} \Big/ \left(\frac{b^2 + c^2}{c^2 - b^2} + \rho \right)$$

Output: $\mathbf{Y}_{1\cdot} = (Y_{12})$, $\mathbf{Z}_1 = (Z_{1,1}, Z_{1,2})^T$, $\mathbf{Y}_{12} = (\delta_1)$,

$$\delta_1 = \frac{pb}{E} \left(\frac{b^2 + a^2}{b^2 - a^2} - \rho \right), Z_{1,1} = \sigma_a - S, \; Z_{1,2} = \sigma_b^{in} - S, \; \sigma_a = \frac{-2pb^2}{b^2 - a^2} + \frac{(a^2 + c^2) p_0}{c^2 - a^2}, \text{ and}$$

$$\sigma_b^{in} = \frac{-p(b^2 + a^2)}{b^2 - a^2} + \frac{a^2 (b^2 + c^2) p_0}{(c^2 - a^2) b^2}$$

Subsystem 2: the outer cylinder

Input: $\mathbf{R}_S = (S, p_0)^T$, $\mathbf{I}_S = (E, \rho, b)^T$, $\mathbf{R}_2 = (\delta)$, $\mathbf{I}_2 = (c)$, $\mathbf{Y}_{\cdot 2} = (Y_{12})$ where

$\mathbf{Y}_{12} = (\delta_1)$

Output: $\mathbf{Y}_{2\bullet} = (Y_{21})$, $\mathbf{Z}_2 = (Z_{2,1}, Z_{2,2})$, $\mathbf{Y}_{21} = (p)$, $Z_{2,1} = \sigma_b^{out} - S$, $Z_{2,2} = \sigma_c - S$,

$$\delta_2 = \delta - \delta_1,\ p = \frac{\delta_2 E}{b} \Bigg/ \left(\frac{b^2 + c^2}{c^2 - b^2} + \rho \right) = \frac{(\delta - \delta_1) E}{b} \Bigg/ \left(\frac{b^2 + c^2}{c^2 - b^2} + \rho \right),$$

$$\sigma_b^{out} = p \left(\frac{b^2 + c^2}{c^2 - b^2} \right) + \frac{a^2 (b^2 + c^2)}{(c^2 - a^2) b^2} p_0, \qquad \text{and } \sigma_c = \frac{2b^2 p}{c^2 - b^2} + \frac{2a^2 p_0}{c^2 - a^2}$$

In the above equations,

E is the modulus of elasticity,

S is the allowable stress,

p is the contact stress at the interface,

δ is the total shrinkage allowance of the two cylinders at the interface,

δ_1 is the radial deformation of the inner cylinder at radius *b*,

δ_2 is the radial deformation of the outer cylinder at radius *b*,

ρ is the Poisson's ratio,

σ_a is the tangential stress of the inner cylinder at the internal radius *a*,

σ_b^{in} is the tangential stress of the inner cylinder at the external radius *b*,

σ_b^{out} is the tangential stress of the outer cylinder at the internal radius *b*, and

σ_c is the tangential stress of the outer cylinder at the external radius *c*.

The distributions of the random variables and interval variables are given in Tables 5 and 6, respectively.

Table 5. Distributions of Random Variables

Variables	Mean	Standard Deviation	Distribution
S	10.0×10^3 psi	1.0×10^3 psi	Normal
p_0	20.0×10^3 psi	2.0×10^3 psi	Normal
a	75 in	0.1 in	Normal
δ	0.004 in	0.0004 in	Normal

Table 6. Distributions of Interval Variables

Variables	I^L	I^U
E	$30\times10^6\times(1-2\%)$ psi	$30\times10^6\times(1+2\%)$ psi
ρ	$0.3\times(1-2\%)$	$0.3\times(1+2\%)$
b	9.95 in	10.05 in
c	14.95 in	15.05 in

The proposed method is used to estimate the standard deviations of the output of each subsystem. The results are given in Table 7. Monte Carlo simulation (MCS) is used to confirm the results. For MCS, 4 smaller intervals are taken for each of the interval variables, and 10^4 samples are taken for each of the random variables. The number of MDA is equal to $5\times5\times5\times5\times10000=6.25\times10^6$. The results obtained by the proposed method match well with those by MCS, while the efficiency of the proposed method is much higher.

Table 7. Comparison of Standard Deviation

	GRA	MCS
$\left(\sigma_{\mathbf{z}_{1,1}}^{\max},\sigma_{\mathbf{z}_{1,2}}^{\max}\right)$(psi)	$\left(3.485\times10^3, 2.444\times10^3\right)$	$\left(3.597\times10^3, 2.570\times10^3\right)$
$\left(\sigma_{\mathbf{z}_{1,1}}^{\min},\sigma_{\mathbf{z}_{1,2}}^{\min}\right)$(psi)	$\left(3.484\times10^3, 2.419\times10^3\right)$	$\left(3.562\times10^3, 2.515\times10^3\right)$
$\left(\sigma_{\mathbf{z}_{2,1}}^{\max},\sigma_{\mathbf{z}_{2,2}}^{\max}\right)$(psi)	$\left(2.40\times10^3, 1.667\times10^3\right)$	$\left(2.544\times10^3, 1.750\times10^3\right)$
$\left(\sigma_{\mathbf{z}_{2,1}}^{\min},\sigma_{\mathbf{z}_{2,2}}^{\min}\right)$(psi)	$\left(2.373\times10^3, 1.667\times10^3\right)$	$\left(2.50\times10^3, 1.724\times10^3\right)$

5. CONCLUDING REMARKS

The general robustness assessment method computes the bound of the standard deviation of an output of multidisciplinary systems given both random and interval input variables. The challenge is to maintain the computational efficiency. The proposed Semi-Second-Order Taylor Expansion method performs well in terms of efficiency. At first one MDA is performed to solve the linking variables at the means of random input variables and the averages of interval input variables. And then two steps are followed. In the first step, each of the linking variables in one subsystem is approximated with the Semi-Second-Order Taylor Expansion. Solving the simplified functions eliminates all the linking variables. And hence in the second step all the outputs are functions of only input variables, without any linking variables. The Semi-Second-Order Taylor Expansion is used again to approximate the output functions, and then their standard deviations are estimated. All the analyses in the two steps are performed locally at subsystem level.

The examples have demonstrated the good performance of the proposed method. The advantages of the proposed method include high efficiency and ease of use. If the derivatives of the performance function are evaluated numerically, the general robustness analysis method can be used for black-box functions. However, it should be noted that the accuracy may not be good in the following situations. (1) When the performance functions are highly nonlinear, the Semi-Second-Order Taylor Expansion may not give accurate approximations. (2) When uncertainties are huge, or in other words, when the standard deviations of random variables and the widths of interval variables are large, the accuracy will not be high. The reason is that the accuracy of the Taylor expansion decreases at a point that is far away from the expansion point. (3) When a performance

function is expanded at a saddlepoint, the proposed method will not be applicable. Detailed explanations of the saddle point situation are given in [6]. The possible ways to increase accuracy include using higher order Taylor expansion or Monte Carlo simulation. Of course, doing so in general will require more computational efforts. The method can be modified for dependent random variables, but it is not applicable for situations where random variables **R** and interval variables **I** are dependent.

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