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Reliability Analysis for Multidisciplinary Systems with Random and Interval variables

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Tremendous efforts have been devoted to developing efficient approaches to reliability analysis for multidisciplinary systems. Most of the approaches are only capable of dealing with random variables modeled by probability distributions. Both random and interval variables, however, may exist in multidisciplinary systems. Their propagation through coupled subsystems makes reliability analysis computationally expensive. In this work, a unified reliability analysis framework is proposed to deal with both random and interval variables in multidisciplinary systems. The framework is an extension of the existent unified uncertainty analysis framework for single-disciplinary problems. The new framework involves probabilistic analysis (PA) and interval analysis (IA). Both PA and IA are decoupled from each other and are performed sequentially. The First Order Reliability Method (FORM) is used for PA. Three supporting algorithms are developed. The effectiveness of the algorithms is demonstrated with a mathematical example and an engineering application.

c	=	limit state
F_X	=	cumulative distribution function of X
f_X	=	joint probability function of X
G	=	response
$G_{\max}$	=	maximum value of G
$G_{\min}$	=	minimum value of G
g	=	limit state function
h	=	equality constraint
Pr	=	probability
p_f	=	probability of failure
p_f^L	=	lower bound of probability of failure
p_f^U	=	upper bound of probability of failure
R	=	reliability
$\mathbf{U}$	=	vector of standard normal random variables transformed from $\mathbf{X}$
$\mathbf{u}^*$	=	Most Probable Point
$\mathbf{W}_i$	=	vector of interval input variables of the i-th discipline
$\mathbf{w}^L$	=	vector of lower bounds of $\mathbf{W}$
$\mathbf{w}^U$	=	vector of upper bounds of $\mathbf{W}$

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$\mathbf{X}$	=	vector of random variables
$\mathbf{X}_i$	=	vector of random input variables of the *i*-th discipline
$\mathbf{Y}_{ij}$	=	vector of coupling variables from the *i*-th discipline to the *j*-th discipline
$\mathbf{Z}_i$	=	vector of outputs from the *i*-th discipline
β	=	reliability index
Φ	=	cumulative distribution function of a standard normal variable
Φ^{-1}	=	inverse function of Φ

I. Introduction

Compared with single-disciplinary reliability analysis, multidisciplinary reliability analysis is much more complicated. The subsystems (disciplines) of a multidisciplinary system are often highly coupled with each other. The output of one subsystem may be the input to other subsystems, and vice versa. Uncertainty in one discipline can then be propagated to other disciplines through the interdisciplinary interfaces. A large number of uncertain variables may also be involved in a multidisciplinary system.

Due to these complexities, computationally efficient reliability analysis becomes essential. Several multidisciplinary reliability analysis methods have been reported [1-11]. Sues et al. [1] use response surface models to replace the computationally expensive simulation models in reliability analysis for multidisciplinary design optimization (MDO). A multi-stage and parallel implementation strategy is developed to integrate reliability analysis and the MDO framework [2]. The reliability analysis methods in [3, 4] employ a concurrent subspace optimization framework; and in a similar manner, the collaborative reliability analysis [5] concurrently performs reliability analysis and multidisciplinary analysis (MDA). On the contrary, Ahn et al. [6] employ a sequential approach to reliability analysis with MDA. They also develop a strategy to associate single-level reliability-based design with the bi-level integrated system synthesis; and the sequential single loops of reliability analysis and optimization are conducted based on the approximated functions [7]. To avoid the tremendous computational burden caused by the direct integration of reliability-based design (RBD) with MDO, a method of Sequential Optimization and Reliability Assessment (SORA) for MDO is developed in [8]. The SORA decouples reliability analysis from MDO.

On the other hand, Analytical Target Cascading (ATC) is formulated for design optimization under uncertainty for hierarchically decomposed multilevel systems [9]. The advanced mean value (AMV) based technique and a bottom-to-top coordination are used. ATC is also reported in [10], where reliability-based MDO is decomposed into several individual RBD problems at the subsystem level, and then the SORA is used to solve the individual RBD problems. The study in [11] focuses on the tradeoff between system performances and the probabilities of failure of subsystems. The study employs an all-in-one approach to the coupling analysis, where the First Order Reliability Method (FORM) and multi-objective optimization are integrated. A methodology of non-deterministic design optimization for hierarchically coupled structural systems is proposed in [12], where parameter uncertainties are considered with deterministic multilevel decomposition formulations.

All of the aforementioned methods deal with only random variables with probability distributions. In many engineering applications, however, information or knowledge might not be sufficient to build probability distributions. Intervals are usually suitable for those uncertain variables, about which we may have too limited information to fit distributions. Examples of using intervals in multidisciplinary systems are given in [13, 14].

Random variables and interval variables may present in a system simultaneously. A framework of Unified Reliability Analysis is developed to quantify the effect of random and intervals variables [15]. In this work, we extend the strategy in [15] to reliability analysis for multidisciplinary systems when both random and interval variables are involved. In Section II, the Unified Reliability Analysis (URA) framework for a single disciplinary system is briefly reviewed. A multidisciplinary system model with random and interval input variables is also provided therein. In Section III, three algorithms, which support the extension of the URA to multidisciplinary systems, are presented. These algorithms are demonstrated by a mathematical example and an aircraft wing design application in Section IV. Conclusions are given in Section V.

II. Modeling and Methodology

A. Reliability Analysis

For a single-disciplinary system where only random variables $\mathbf{X}$ are involved, reliability is defined by

$$R = \Pr\{G = g(\mathbf{X}) \geq 0\} \tag{1}$$

where $\Pr\{\cdot\}$ denotes a probability, G is a response, and $\mathbf{X} = \left(X_1, X_2, \cdots, X_{n_X}\right)$ is a vector of random variables, and g is a limit-state function [16]. In this paper, we assume that X_i (i=1,2,...,n_X) are independent.

If the joint *probability density function* (PDF) of **X** is $f_{\mathbf{X}}$, the probability of failure p_f, which is $1-R$, can be calculated by

$$p_f = \Pr\{G = g(\mathbf{X}) < 0\} = \int_{g(\mathbf{X})<0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{2}$$

The limit state function $g(\mathbf{X})$ is usually a nonlinear function of **X**; the integration boundary, $g(\mathbf{X}) = 0$, therefore, is nonlinear. The probability integration in Eq. (2) is also multidimensional. There is rarely a close-form solution to Eq. (2). Numerical integration methods are also computationally expensive when the dimension is high. To this end, the efficient First Order Reliability Method (FORM) is widely used to obtain an approximate solution to Eq. (2).

The FORM is performed with the following three steps.

Step 1: Transform random variables **X** into standard normal random variables **U**. The *i*-th random variable X_i is transformed by

$$u_i = \Phi^{-1}\left[F_{X_i}(x_i)\right] \tag{3}$$

where F_{X_i} is the *cumulative distribution function* (CDF) of X_i, and Φ^{-1} is the inverse CDF of a standard normal distribution.

Step 2: Search the Most Probable Point (MPP). The MPP $\mathbf{u}^*$ is located by

$$\begin{cases} \min_{\mathbf{u}} \|\mathbf{u}\| \\ s.t. \quad g(\mathbf{u}) = 0 \end{cases} \tag{4}$$

in which $\|\cdot\|$ stands for the norm (length) of a vector. Geometrically, the MPP is the shortest distance point from the limit state $g(\mathbf{U}) = 0$ to the origin in the **U**-space. The minimum distance $\beta = \|\mathbf{u}^*\|$ is called a *reliability index*.

Step 3: Compute the probability of failure. p_f is obtained by

$$p_f = \Phi(-\beta) \tag{5}$$

where Φ is the CDF of a standard normal distribution.

The most computation-intensive work of the FORM is the MPP search. The following recursive algorithm [17] is commonly used for the MPP search,

$$\begin{cases} \beta^{(k)} = \beta^{(k-1)} + \dfrac{g(\mathbf{u}^{(k-1)})}{\left\|\nabla g(\mathbf{u}^{(k-1)})\right\|} \\ \mathbf{u}^{(k)} = -\beta^{(k)} \dfrac{\nabla g(\mathbf{u}^{(k-1)})}{\left\|\nabla g(\mathbf{u}^{(k-1)})\right\|} \end{cases} \tag{6}$$

where $\nabla g(\mathbf{u}^{(k)})$ is the gradient of g at $\mathbf{u}^{(k)}$, $\left\|\nabla g(\mathbf{u}^{(k)})\right\|$ is its magnitude, and k is the iteration counter.

B. Unified Reliability Analysis Framework

The purpose of this work is to establish a Unified Reliability Analysis (URA) framework that can handle both random and interval variables in multidisciplinary systems. For this purpose, we employ the URA framework that has been developed for single-disciplinary systems [15]. The framework is illustrated in Fig.1. The input to the framework are random variables **X** characterized by probability distributions and interval variables **W** represented by their bounds $[\mathbf{w}^L, \mathbf{w}^U]$. It is obvious that the uncertain output (response) $G = g(\mathbf{X}, \mathbf{W})$ is also characterized by two bounds of its probability distributions [15]. Thus the reliability of the system is also bounded within its maximum and minimum values.

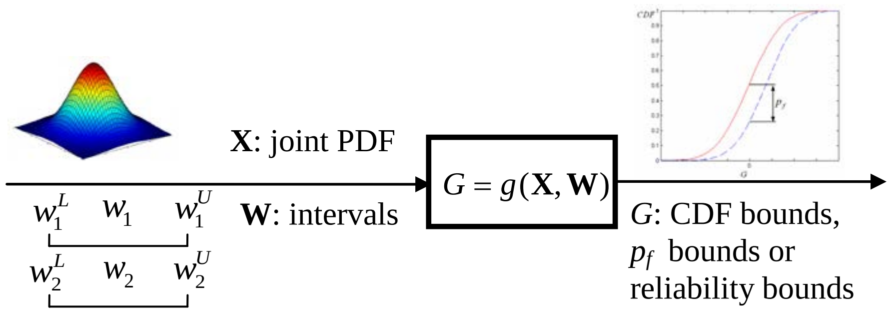

Fig. 1 Unified reliability analysis framework.

Reliability analysis calls the limit-state function $G = g(\mathbf{X}, \mathbf{W})$ a number of times; so does multidisciplinary analysis (MDA), which is responsible for solving for the linking variables between subsystems. Different computational algorithms integrate reliability analysis and MDA in different ways; and their efficiency and applicability are also different. In Section III, we develop three computational algorithms that support the URA framework.

C. FORM-Based URA

Let $\Delta_{\mathbf{w}}$ denote the set of intervals **W** and $g(\mathbf{X}, \mathbf{W}) < 0$ denote a failure event. The lower and upper bounds of the probability of failure, p_f^L and p_f^U, can then be calculated by

$$p_f^L = \Pr\left\{ G_{\max} = \max_{\mathbf{W}} g(\mathbf{X},\mathbf{W}) < 0 \mid \mathbf{W} \in \Delta_{\mathbf{W}} \right\} \tag{7}$$

and

$$p_f^U = \Pr\left\{ G_{\min} = \min_{\mathbf{W}} g(\mathbf{X},\mathbf{W}) < 0 \mid \mathbf{W} \in \Delta_{\mathbf{W}} \right\} \tag{8}$$

respectively [15]. $G_{\max}$ and $G_{\min}$ are the global maximum and minimum values of G over $\Delta_{\mathbf{w}}$, respectively.

According to Eqs. (7) and (8), the procedure to calculate p_f^L and p_f^U consists of two loops: one is interval analysis (IA) for searching $G_{\min}$ and $G_{\max}$, and the other is probability analysis (PA) for calculating probabilities $\Pr\left\{G_{\min} < 0 \middle| \mathbf{W} \in \Delta_{\mathbf{w}}\right\}$ and $\Pr\left\{G_{\max} < 0 \middle| \mathbf{W} \in \Delta_{\mathbf{w}}\right\}$. If the FORM is used for PA, the Most Probable Point (MPP) needs to be identified by solving the following model

$$\begin{cases} \min_{\mathbf{u}} & \|\mathbf{u}\| \\ s.t. & g(\mathbf{u},\mathbf{w}) = 0 \end{cases} \tag{9}$$

where $\mathbf{w}$ is treated as a constant vector. For IA, an optimization problem can be formulated for $G_{\max}$:

$$\begin{cases} \max_{\mathbf{w}} & g(\mathbf{u},\mathbf{w}) \\ s.t. & \mathbf{w} \in \Delta_{\mathbf{w}} \end{cases} \tag{10}$$

where $\mathbf{u}$ is treated as a constant vector. For $G_{\min}$, Eq. (10) becomes a minimization problem.

To solve for $\mathbf{u}$ in PA in Eq. (9), $\mathbf{w}$ should be given; and to solve for $\mathbf{w}$ in IA in Eq. (10), $\mathbf{u}$ should be given. This indicates that both PA and IA are fully coupled. To reduce the computational cost, a FORM-based URA (FORM-URA) framework is proposed [15]. Under this framework, PA and IA are decoupled and are performed sequentially. This FORM-URA framework for the calculation of p_f^L is illustrated in Fig.2. PA is performed followed by IA. After PA, the KKT conditions of IA are checked at the solution of PA. If the KKT conditions are satisfied, IA will be skipped. Skipping the IA loop saves the computational time dramatically.

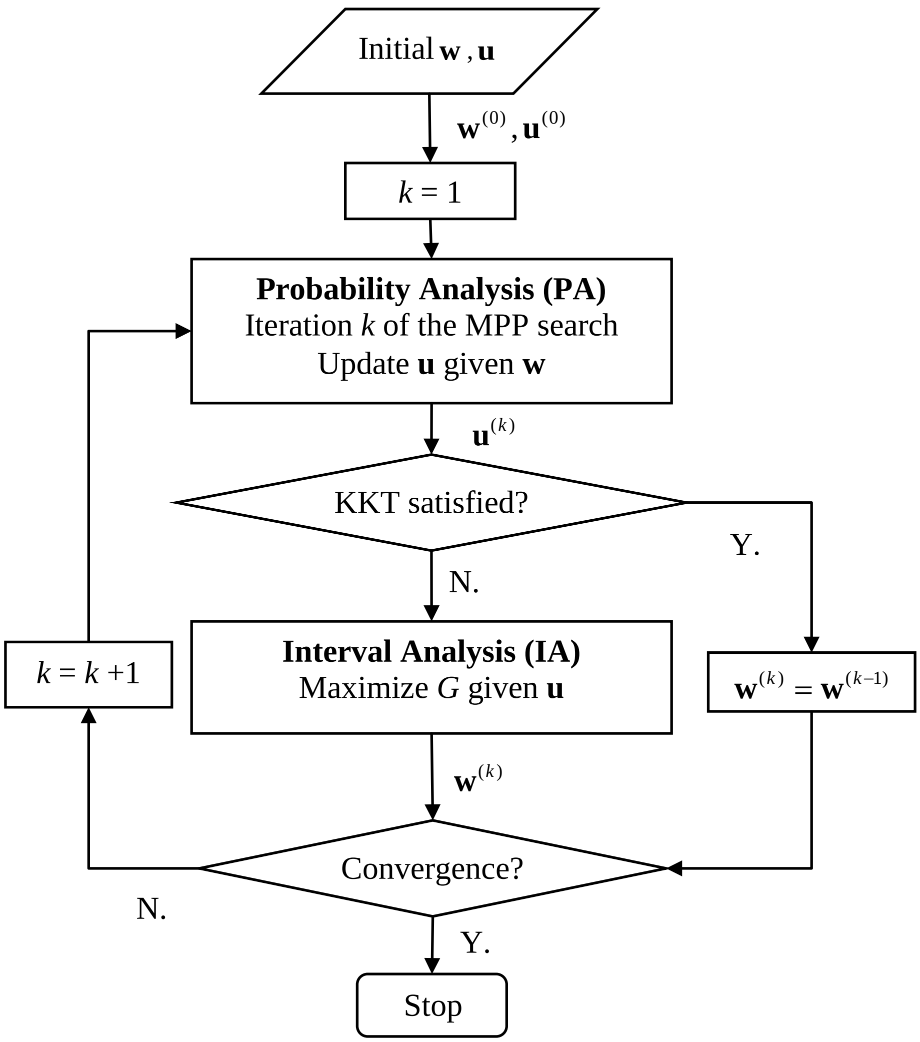

Fig. 2 Flowchart of the FORM-URA method.

The efficiency and robustness of an MPP search algorithm are very important for the FORM-URA method. The efficient MPP search algorithm HLRF [18, 19] is therefore used. It is known, however, that the HLRF algorithm may not converge for a nonlinear function. If this happens, the improved version of HLRF algorithm [20], denoted by iHLRF, will take over the PA process. iHLRF is computationally efficient and guarantees to converge to a local MPP. On the other hand, as indicated in Eq. (10), IA is formulated as a bound-constrained optimization problem. Then most of nonlinear optimization algorithms can be used for IA. Both FORM and optimization are capable of handling black-box performance functions, and hence so is FORM-URA.

D. Multidisciplinary Systems (MDA) Analysis with Random and Interval Variables

To integrate URA with MDA, we need to understand the relationships among random variables, interval variables, and coupling variables in a multidisciplinary system. A three-discipline system in Fig.3 illustrates such relationships. The notations are given below.

$\mathbf{X}_s$: sharing random input variables,

$\mathbf{X}_i$: local random input variables of discipline *i*,

$\mathbf{W}_s$: sharing interval input variables,

$\mathbf{W}_i$: local interval input variables of discipline *i*, and

$\mathbf{Z}_i$: outputs of discipline *i*;

$\mathbf{Y}_{ij}$: coupling (linking) variables from discipline *i* to discipline *j*.

Multidisciplinary analysis (MDA) is responsible for solving for output $\mathbf{Z}_i$ given all the input variables. Since $\mathbf{Z}_i$ depends on coupling variables, MDA must first solve for coupling variables $\mathbf{Y}_{ij}$. A set of coupling variables from the *i*-th discipline is formulated as

$$\mathbf{Y}_{i\bullet} = (\mathbf{Y}_{ij}, j = 1, 2, \cdots, n; j \neq i) = \mathbf{Y}_{i\bullet}(\mathbf{X}_s, \mathbf{X}_i, \mathbf{W}_s, \mathbf{W}_i, \mathbf{Y}_{\bullet i}) \tag{11}$$

where *n* is the number of disciplines, and $\mathbf{Y}_{i\bullet}$ represents dependent variables on the left-hand side and also the functional expressions of the dependent variables. $\mathbf{Y}_{\bullet i}$ is the vector of coupling variables, which are the inputs to discipline *i* and the outputs from other disciplines. In Eq. (11), $\mathbf{Y}_{\bullet i} = (\mathbf{Y}_{ji}, j = 1, 2, \cdots, n; j \neq i)$.

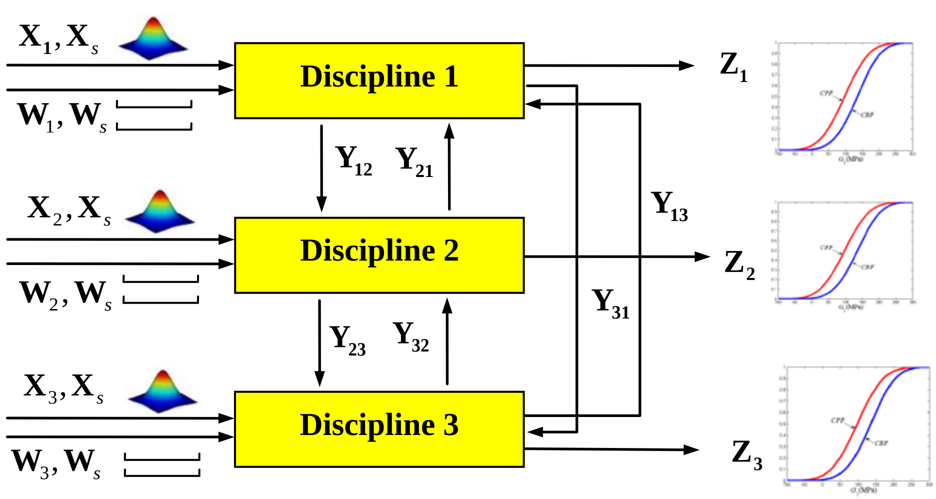

Fig. 3 Multidisciplinary system with random and interval variables.

The system of simultaneous equations in Eq. (11) determines the system consistency over the interfaces among coupled disciplines. Solving those equations is the task of multidisciplinary analysis (MDA). Expanding Eq. (11) over all disciplines, we obtain

$$\begin{cases} \mathbf{Y}_{12} = \mathbf{Y}_{12}(\mathbf{X}_1, \mathbf{X}_s, \mathbf{W}_1, \mathbf{W}_s, \mathbf{Y}_{\bullet 1}) \\ \mathbf{Y}_{13} = \mathbf{Y}_{13}(\mathbf{X}_1, \mathbf{X}_s, \mathbf{W}_1, \mathbf{W}_s, \mathbf{Y}_{\bullet 1}) \\ \mathbf{Y}_{21} = \mathbf{Y}_{21}(\mathbf{X}_2, \mathbf{X}_s, \mathbf{W}_2, \mathbf{W}_s, \mathbf{Y}_{\bullet 2}) \\ \mathbf{Y}_{23} = \mathbf{Y}_{23}(\mathbf{X}_2, \mathbf{X}_s, \mathbf{W}_2, \mathbf{W}_s, \mathbf{Y}_{\bullet 2}) \\ \mathbf{Y}_{31} = \mathbf{Y}_{31}(\mathbf{X}_3, \mathbf{X}_s, \mathbf{W}_3, \mathbf{W}_s, \mathbf{Y}_{\bullet 3}) \\ \mathbf{Y}_{32} = \mathbf{Y}_{32}(\mathbf{X}_3, \mathbf{X}_s, \mathbf{W}_3, \mathbf{W}_s, \mathbf{Y}_{\bullet 3}) \end{cases} \tag{12}$$

Suppose G_i is one of the outputs $\mathbf{Z}_i$ from discipline i and the corresponding function is g_i; then the function is given by

$$G_i = g_i(\mathbf{X}_s, \mathbf{X}_i, \mathbf{W}_s, \mathbf{W}_i, \mathbf{Y}_{\bullet i}) \tag{13}$$

If a failure event is defined by $G_i < 0$, then the task of reliability analysis is to find the probability $\Pr\{G_i < 0\}$. As describe in Sec.II.C, we need to quantify the lower and upper bounds of the probability of failure: $\Pr\left\{G_i^{\max} < 0\right\}$ and $\Pr\left\{G_i^{\min} < 0\right\}$. Solving the probability bounds is computationally expensive because it needs to perform coupled PA, IA, and MDO. Hence, efficient algorithms are desired. Next, we propose three algorithms based on different strategies.

III. Algorithms

This work is to extend the existent Unified Reliability Analysis (URA) framework [15] to multidisciplinary analysis (MDA). For this purpose, we propose three algorithms to integrate URA with MDA. In all the three algorithms, PA and IA are decoupled and are conducted sequentially. PA is performed first while the interval variables are fixed, and then IA is performed while the random variables are fixed. The process of one PA followed by the next IA is referred to as a *cycle*. After the first cycle, PA and IA are performed again in the second cycle. This process repeats cycle by cycle till convergence.

As summarized in Fig. 4, the three algorithms call MDA in different manners. In the first algorithm, which is called the Sequential Double Loops (SDL) algorithm, MDA is called within both the PA and IA loops. Therefore, both PA and IA involve a double-loop procedure. The second algorithm is called the Sequential Single Loops (SSL) algorithm. This algorithm treats MDA as equality constraints in both PA and IA and therefore eliminates the MDA loop. Each of PA and IA then forms a single loop. The last algorithm is called the Sequential Single-Single Loops (SSSL) algorithm. This algorithm performs the PA loop by calling the MPP search and MDA sequentially. The PA loop then becomes a sequential single loop. IA is the same as in the second algorithm and still forms a single loop. The details of the three algorithms are given in the following subsections.

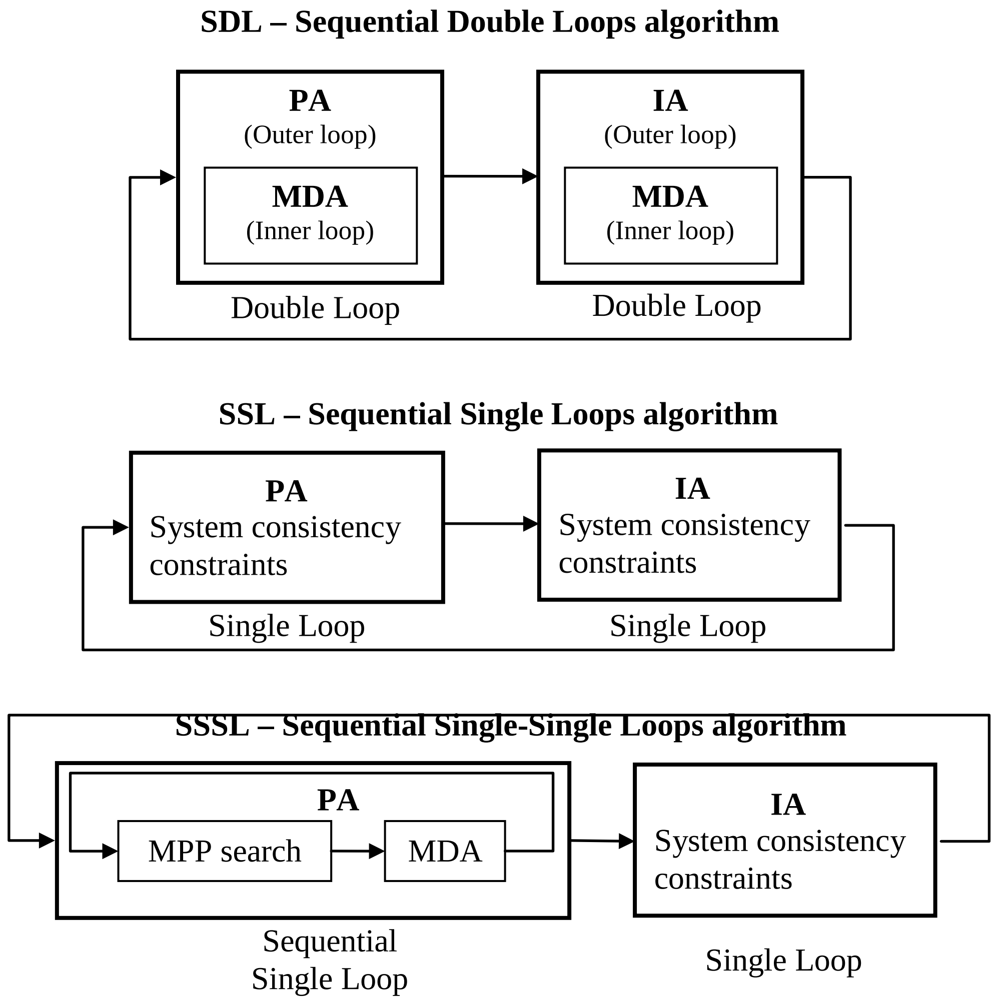

Fig. 4 Outline of proposed algorithms.

A. Sequential Double Loops (SDL) Algorithm

In this algorithm, both PA and IA involve a double-loop procedure. Within the PA and IA loops, MDA is called repeatedly at each iteration. MDA is therefore an inner loop for maintaining the system consistency. The PA and IA double loops are performed sequentially. In this work, the FORM is used for PA, and optimization is used for IA. The flowchart of this algorithm for searching the lower bound of the probability of failure is given in Fig.5.

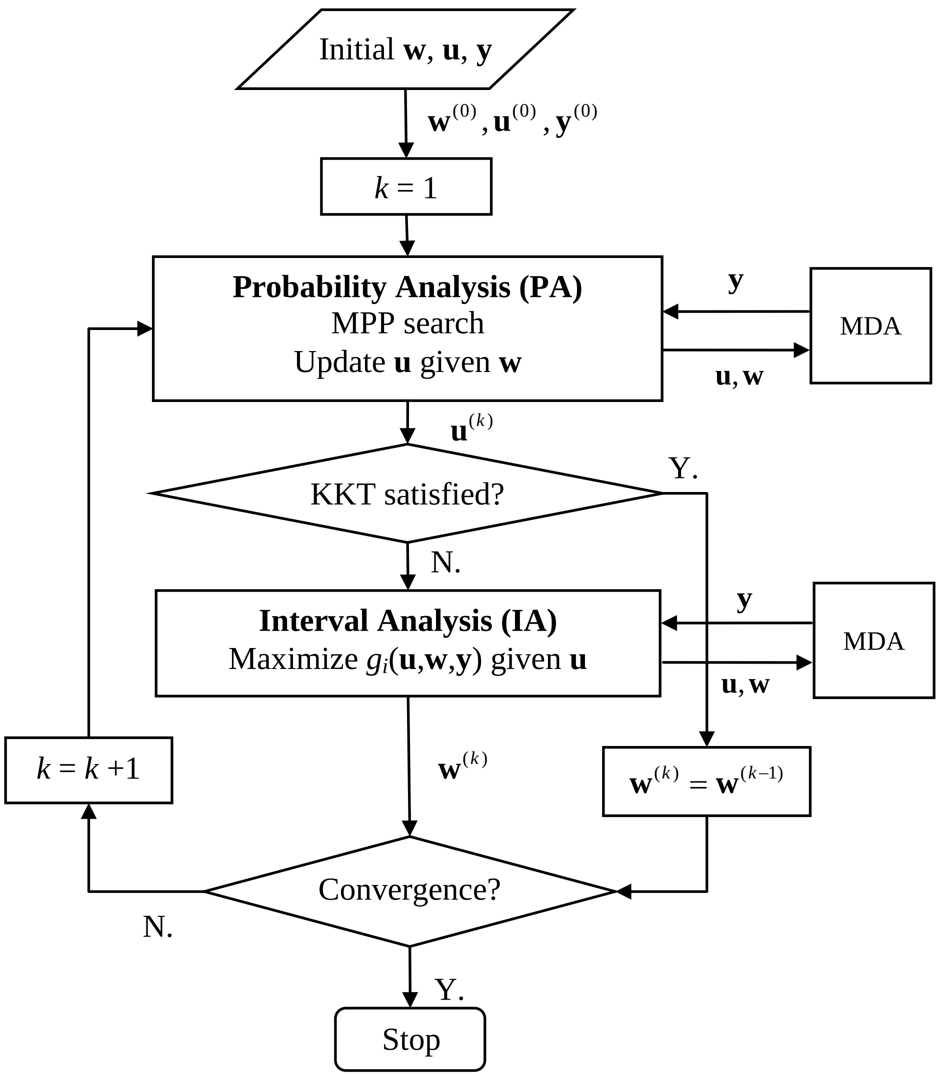

Fig. 5 SDL algorithm for the lower bound of p_f

Specifically, in PA the MPP search is the outer loop, which is modeled as an optimization problem and takes only random variables as its design variables. The interval and coupling variables are treated as constant. Their values are from previous cycle. Suppose the current cycle of the overall reliability analysis is cycle k. The optimization problem is then expressed by

$$\begin{cases} \min\limits_{\mathbf{u}} & \|\mathbf{u}\| \\ s.t. & g_i(\mathbf{u}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i}) = 0 \\ & \mathbf{y}_{\bullet i} \text{ is solved by MDA} \end{cases} \tag{14}$$

In the above model, g_i is a limit-state function in the i-th subsystem. Design variables $\mathbf{u}$ consist of not only the random input variables to the i-th subsystem but also all the random input variables of other subsystems; namely, $\mathbf{u} = (\mathbf{u}_s, \mathbf{u}_1, \cdots, \mathbf{u}_n)$. In Eq. (14), all the interval variables $\mathbf{w}^{(k-1)} = \left(\mathbf{w}_s^{k-1}, \mathbf{w}_1^{k-1}, \mathbf{w}_2^{k-1}, \cdots, \mathbf{w}_n^{k-1}\right)$ are fixed, and they are from the IA in the last cycle. The MDA inner loop is responsible for solving for coupling variables $\mathbf{y}_{\bullet i}$. Since in this work, the FORM is employed for PA, the HLRF algorithm in Eq. (6) is used for the MPP search. With the interval and coupling variables, we modify the HLRF algorithm as follows.

$$\begin{cases} \beta^{(j)} = \beta^{(j-1)} + \dfrac{g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i})}{\left\|\nabla g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i})\right\|} \\ \mathbf{u}^{(j)} = -\beta^{(j)} \dfrac{\nabla g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i})}{\left\|\nabla g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i})\right\|} \end{cases} \tag{15}$$

where j is the iteration counter of the PA loop, interval variables $\mathbf{w}^{(k-1)}$ are kept constant and are from the previous cycle of the overall reliability analysis. The coupling variables $\mathbf{y}_{\bullet i}$ are obtained from the following inner MDA loop.

$$\mathbf{y}_{q\bullet} = (\mathbf{y}_{qm}, q = 1,2,\cdots,n; m = 1,2,\cdots,n; m \neq q) = \mathbf{Y}_{q\bullet}(\mathbf{u}_s^{(j)}, \mathbf{u}_q^{(j)}, \mathbf{w}_s^{(k-1)}, \mathbf{w}_q^{(k-1)}, \mathbf{y}_{\bullet q}) \tag{16}$$

After PA, IA is performed. The outer loop is an optimization problem for the maximum or minimum value of g_i. The design variables are interval variables while the values of random variables have been obtained from PA. Within the optimization loop is the MDA inner loop, which solves for coupling variables $\mathbf{y}_{\bullet i}$. For the lower bound of p_f, IA is a maximization problem with the following formulation

$$\begin{cases} \max\limits_{\mathbf{w}} & g_i(\mathbf{u}^{(k)}, \mathbf{w}, \mathbf{y}_{\bullet i}) \\ s.t. & \mathbf{w} \in \Delta_{\mathbf{w}} \\ & \mathbf{y}_{\bullet i} \text{ is solved by MDA} \end{cases} \tag{17}$$

where design variables are $\mathbf{w} = \left(\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n, \mathbf{w}_s\right)$. Random variables $\mathbf{u}^{(k)}$ are obtained from the MPP search and are kept constant herein. Coupling variables $\mathbf{y}_{\bullet i}$ are solved in the following inner MDA loop.

$$\mathbf{y}_{q\bullet} = (\mathbf{y}_{qm}, q = 1,2,\cdots,n; m = 1,2,\cdots,n; m \neq q) = \mathbf{Y}_{q\bullet}(\mathbf{u}_s^{(k)}, \mathbf{u}_q^{(k)}, \mathbf{w}_s, \mathbf{w}_q, \mathbf{y}_{\bullet q}) \tag{18}$$

This algorithm integrates both PA and IA with MDA in a straightforward manner. In other words, the algorithm involves the direct combination of PA and MDA and the direct combination of IA and MDA. Because of the direct combination, the algorithm is more robust than the other two algorithms that are presented next. However, it may require higher number of MDA calls than the other two algorithms. For instance, at the *j*-th iteration of the MPP search in Eq. (14), MDA is performed whenever the MPP is updated. After $\mathbf{u}^{(j)}$ is obtained, MDA is called to get $\mathbf{y}_{\bullet i}$ in order to calculate $g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i})$. Besides, as shown in Eq. (15), MDA is also needed when the finite difference method is used to calculate the partial derivatives of g_i. The equation of the derivatives of g_i with respect to a particular random variable u_q (the q-th element of $\mathbf{u}$) is given by

$$\frac{\partial g_i}{\partial u_q} = \frac{g_i(\mathbf{u}', \mathbf{w}^{(k-1)}, \mathbf{y}'_{\bullet i}) - g_i(\mathbf{u}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i})}{\Delta} \tag{19}$$

where $\mathbf{u}' = \left(u_1, u_2, \cdots, u_q + \Delta, \cdots, u_{nu}\right)$, nu is length of $\mathbf{u}$, and Δ is a step size. $\mathbf{y}'_{\bullet i}$ is the new values of coupling variables associated with the random variables $\mathbf{u}'$. MDA must be called again to obtain $\mathbf{y}'_{\bullet i}$.

The SDL algorithm suits the systems where the disciplinary analyses and MDA are computationally cheaper. In this work, PA is performed with the FORM; however, other methods can also be used, for example, the Second Order Reliability Method (SORM) and the saddlepoint approximation method [21]. Although nonlinear optimization is used for IA as described above, the efficient interval arithmetic can also be used.

B. Sequential Single Loops (SSL) Algorithm

As described above, when MDA is expensive, the first algorithm (the SDL algorithm) may not be efficient. To alleviate the computational demand from MDA, we propose this second algorithm. Because this algorithm uses a single-loop strategy, it is called the Sequential Single Loops (SSL) algorithm. As shown in Fig.5, the algorithm reformulates the optimization problems of both PA and IA by including the interdisciplinary equilibrium (consistency) as part of constraints. These constraints are the simultaneous equations in MDA and given by

$$\mathbf{h}(\mathbf{u}, \mathbf{w}, \mathbf{y}) = \mathbf{Y}_{i\bullet} - \mathbf{Y}_{i\bullet}(\mathbf{u}, \mathbf{w}, \mathbf{y}_{\bullet i}) = 0, \ i = 1, 2, \cdots, n \tag{20}$$

where **y** contains all the coupling variables.

The optimization model for the PA loop in the *k*-th cycle of the overall reliability analysis is then formulated as

$$\begin{cases} \min\limits_{\mathbf{u},\mathbf{y}} & \|\mathbf{u}\| \\ s.t. & g_i(\mathbf{u},\mathbf{w}^{(k-1)},\mathbf{y}_{\bullet i}) = 0 \\ & \mathbf{h}(\mathbf{u},\mathbf{w}^{(k-1)},\mathbf{y}) = 0 \end{cases} \tag{21}$$

where g_i is a limit-state function of subsystem *i*, interval variables $\mathbf{w}^{(k-1)}$ are given from the IA loop in the last cycle, and random variables **u** and coupling variables **y** are regarded as design variables. Since MDA is part of the constraints, the MDA loop is no longer required.

Then IA is performed. The optimization for the minimum probability of failure in the IA loop (see Eq.(7)) is modeled by

$$\begin{cases} \max\limits_{\mathbf{w},\mathbf{y}} & g_i(\mathbf{u}^{(k)},\mathbf{w},\mathbf{y}_{\bullet i}) \\ s.t. & \mathbf{w} \in \Delta_{\mathbf{w}} \\ & \mathbf{h}(\mathbf{u}^{(k)},\mathbf{w},\mathbf{y}) = 0 \end{cases} \tag{22}$$

where random variables $\mathbf{u}^{(k)}$ is obtained from the PA loop and are constant herein. Interval variables **w** and coupling variables **y** are taken as design variables. Same as in PA, the inner MDA loop is no longer needed.

The entire reliability analysis procedure is depicted in Fig.6. As shown in the figure, the two single loops of PA and IA are performed sequentially.

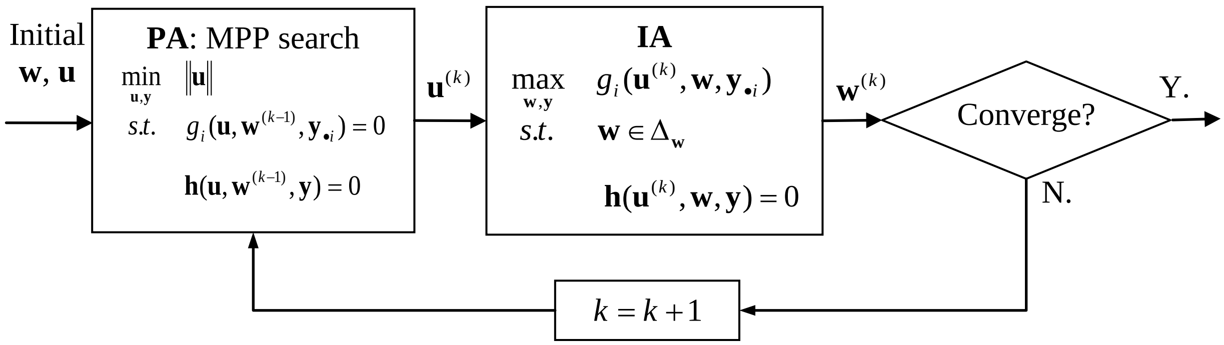

Fig. 6 SSL algorithm for the lower bound of p_f.

Different from the first algorithm, the SSL algorithm does not call MDA directly. The task of MDA is implicitly embedded as equality constraints in the PA and IA loops. Solving these equality constraints requires calling disciplinary analyses. The algorithm is therefore suitable for the situation where it is easy to perform disciplinary analyses concurrently. It is efficient for the systems that contain fewer coupling variables. However, when the number of coupling variables is large, this algorithm will contain a large number of design variables because the coupling variables are part of design variables. This might diminish the efficiency of the SSL algorithm. The other disadvantage of the algorithm is the inclusion of equality constraints for the system consistency. Equality constraints make optimization hard to converge [22]. Since additional constraints are added to the MPP search in PA, the MPP search algorithms such as HLRF algorithm are no longer applicable.

C. Sequential Single-Single Loops (SSSL) Algorithm

In the first algorithm, the SDL algorithm, an efficient MPP search method can be used for PA, while in the second algorithm, the SSL algorithm, only nonlinear optimization can be used for PA. Nonlinear optimization is usually not as efficient as specialized MPP search algorithms. To take advantage of the MPP search algorithms, we combine both of the above two algorithms. The combination comes from the PA loop of the SDL algorithm and the IA loop of the SSL algorithm. An MPP search algorithm can then be used for PA. To save computational resources further, for PA, we change the double-loop procedure to a sequential single loop procedure where the MPP search and MDA are performed sequentially. The same double loop procedure for IA is used as in the SSL algorithm. The algorithm is illustrated in Fig. 7.

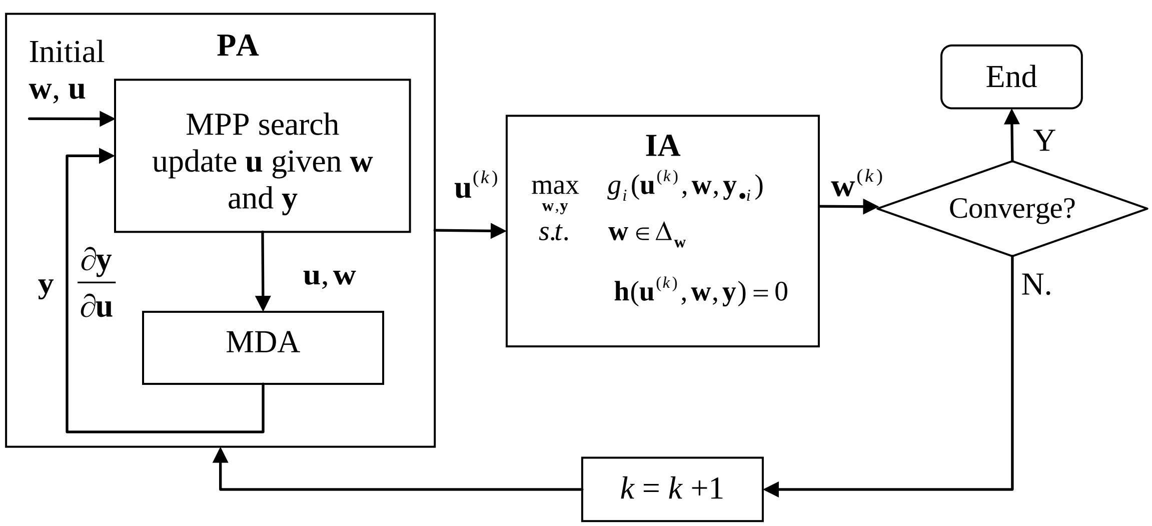

Fig. 7 SSSL algorithm for the lower bound of p_f **.**

In PA, only random variables are solved in the MPP search, and the coupling variables are fixed to the values that are obtained from the last PA iteration. Then the

MDA loop is executed to update the coupling variables. The MPP search and MDA are performed in a sequential manner till convergence is reached. In IA, the system consistency is part of constraints. Interval and coupling variables are solved simultaneously given the random variables from the PA loop. IA includes system consistency constraints and involves a single-loop procedure. The overall reliability analysis is performed sequentially with the sequential single-loop PA and the single-loop IA.

Since in PA the MPP search and MDA are performed sequentially, the MPP search algorithm for the single loop PA in Eq. (15) cannot be used directly. We modify the MPP search algorithm as follows.

$$\begin{cases} \beta^{(j)} = \beta^{(j-1)} + \dfrac{g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i}^{(q-1)})}{\left\| \nabla g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i}^{(q-1)}) \right\|} \\ \mathbf{u}^{(j)} = -\beta^{(j)} \dfrac{\nabla g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i}^{(q-1)})}{\left\| \nabla g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i}^{(q-1)}) \right\|} \end{cases} \tag{23}$$

The above equation is for the *j*-th iteration of the MPP search in the *q*-th iteration of the PA loop and the *k*-th cycle of the overall reliability analysis. The interval variables $\mathbf{w}^{(k-1)}$ are from the previous cycle (cycle *k*-1) of the overall reliability analysis and are kept constant. The coupling variables $\mathbf{y}_{\bullet i}^{(q-1)}$ are from the last iteration (iteration *q*-1) of the PA loop and are also kept constant. The solution is the MPP $\mathbf{u}^{(q)}$.

After the MPP loop is completed, MDA is performed. The coupling variables $\mathbf{y}_{\bullet i}^{q}$ are obtained from the following model.

$$\mathbf{y}_{p\bullet} = (\mathbf{y}_{pm}, p = 1,2,\cdots,n;\ m = 1,2,\cdots,n;\ m \neq p) = \mathbf{Y}_{p\bullet}(\mathbf{u}_s^{(q)}, \mathbf{u}_p^{(q)}, \mathbf{w}_s^{(k-1)}, \mathbf{w}_p^{(k-1)}, \mathbf{y}_{\bullet p}) \tag{24}$$

If analytical derivatives are not available for the gradient $\nabla g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i}^{(q-1)})$ in Eq. (23), the finite difference method in Eq. (19) can be used to estimate the gradients $\partial g_i / \partial u_p$, where u_p is the p-th element of **u**. The equation is given by

$$\frac{\partial g_i}{\partial u_p} = \frac{g_i(\mathbf{u}^{(j-1)}{}', \mathbf{w}^{(k-1)}, \mathbf{y'}_{\bullet i}) - g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i})}{\Delta} \tag{25}$$

where $\mathbf{y'}_{\bullet i}$ is the new values of coupling variables associated with the new random variable $\mathbf{u}' = (u_1, u_2, \cdots, u_p + \Delta, \cdots, u_{nu})$.

It is noted that the coupling variables $\mathbf{y'}_{\bullet i}$ are not constant. They are functions of **u**. $\mathbf{y'}_{\bullet i}$ should therefore be re-calculated. However, $\mathbf{y'}_{\bullet i}$ cannot be obtained from the MPP search because it is solved by MDA. A first order Taylor's series expansion is used to estimate $\mathbf{y'}_{\bullet i}$, and the equation is given by

$$\mathbf{y'}_{\bullet i} = \mathbf{y}_{\bullet i} + \frac{\partial \mathbf{y}_{\bullet i}}{\partial u_p} \Delta \tag{26}$$

where $\frac{\partial \mathbf{y}_{\bullet i}}{\partial u_p}$ is obtained from the MDA loop in the previous iteration (iteration j-1) of PA and is kept constant in the MPP search.

This algorithm is suitable for problems where PA is relatively expensive and IA is relatively cheap. One may also choose this method when the number of random variables is large and the number of interval variables is small.

IV. Example

Two examples are presented for demonstration. The first one is a mathematical problem with two subsystems. In this problem, the probabilistic constraints are simple and the number of variables is small. As a result, this problem effectively shows the formulations and procedures of the three algorithms. The second example is an aircraft wing design problem involving more complicated probabilistic constraints and more coupling variables and random variables. It indicates the potential use of the present method to real engineering applications.

The convergence criteria for the overall reliability analysis and PA in the two examples are:

1) The difference between the norms of the sharing random variables of two consecutive MPPs is less than 10^{-6}. This difference is measured in the standard normal space.

2) The difference between the norms of the local random variables of two consecutive MPPs is less than 10^{-6}. This difference is measured in the standard normal space.

3) The difference between the reliability indexes of two consecutive MPPs is less than 10^{-6}.

Sequential Quadratic Programming (SQP) is used for optimization in IA. It is also used in PA whenever optimization is needed. The termination tolerances on the function values, on design variables, and on the constraint violation are all set to 10^{-6}.

A. Example 1 – A Mathematical Problem

In this example the system consists of two subsystems. Two local interval variables and one sharing interval variable are introduced to the original problem in [23] where only random variables are involved. The new problem is illustrated in Fig. 8 and is formulated as follows.

Subsystem 1:

$$G_1 = (X_s + 0.5W_s)^2 + 2W_1 + X_1 + W_1 e^{-Y_{21}} - 7.65$$

$$Y_{12} = (X_s + 0.5W_s)^2 + 2W_1 - X_1 + 2\sqrt{Y_{21}}$$

Subsystem 2:

$$G_2 = \sqrt{X_s + 0.5W_s} + W_2 + 0.4X_2(X_s + W_s) + 0.2Y_{12} - 9.3$$

$$Y_{21} = (X_s + 0.5W_s)W_2 + W_2^2 + X_2 + Y_{12}$$

W_s is a sharing interval variable; W_1 and W_2 are local interval variables; X_s is a sharing random variable; and X_1 and X_2 are local random variables. $X_s \sim N(0.2, 1)$, $X_1 \sim N(1.4885, 0.1)$, and $X_2 \sim N(3.3227, 0.1)$, where $N(\cdot,\cdot)$ stands for a normal distribution, and its first and second parameters are mean and standard deviation, respectively. $W_s \in [2.065, 2.075]$, $W_1 \in [0.7714, 0.7814]$, and $W_2 \in [0.14, 0.16]$. The probabilities of failure are defined by $p_f = \Pr\{G_i < 0\}$ $(i = 1, 2)$.

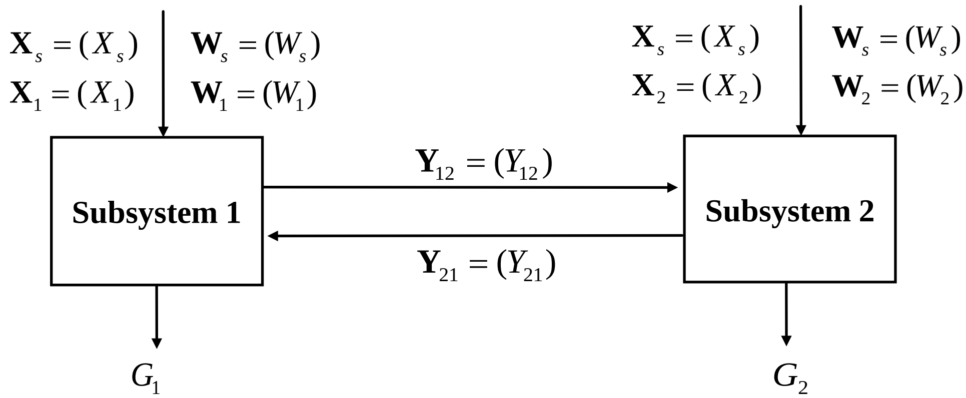

Fig. 8 Mathematical example.

To demonstrate the procedure of each algorithm, we provide the equations of the lower bound of p_f for G_1 at the *k*-th cycle as follows. (Recall that a cycle consists of a sequential process of PA and IA; in other word, it is one iteration of the overall reliability analysis.)

1. SDL algorithm

1) PA loop

The MPP search is modeled by

$$\begin{cases} \min\limits_{\mathbf{u}=(u_s,u_1,u_2)} \sqrt{u_s^2+u_1^2+u_2^2} \\ s.t. \quad G_1=(\mu_s+u_s\sigma_s+0.5w_s^{(k-1)})^2+2w_1^{(k-1)}+\mu_1+u_1\sigma_1+w_1^{(k-1)}e^{-y_{21}}-7.65=0 \end{cases}$$

where the design variables are $\mathbf{u}=(u_s,u_1,u_2)$; and $w_s^{(k-1)}$ and $w_1^{(k-1)}$ are the interval variables from the (*k*-1)-th cycle. y_{21} is the coupling variable from MDA and is solved by

$$\begin{cases} y_{12}=(\mu_s+u_s\sigma_s+0.5w_s^{(k-1)})^2+2w_1^{(k-1)}-(\mu_1+u_1\sigma_1)+2\sqrt{y_{21}} \\ y_{21}=(\mu_s+u_s\sigma_s+0.5w_s^{(k-1)})w_2^{(k-1)}+(w_2^{(k-1)})^2+\mu_2+u_2\sigma_2+y_{12} \end{cases}$$

where interval variable $w_2^{(k-1)}$ is from the (*k*-1)-th cycle.

The above MPP search and MDA are nested and form a single-loop PA. The solution of the PA loop is the MPP $\mathbf{u}^{*,(k)} = \left(u_s^{*,(k)}, u_1^{*,(k)}, u_2^{*,(k)}\right)$. It is noted that in the above equations all the random variables are transformed into standard normal variables.

2) IA loop

The optimization model is given by

$$\begin{cases} \max\limits_{\mathbf{w}=(w_s,w_1,w_2)} G_1 = (\mu_s + u_s^{*,(k)}\sigma_s + 0.5w_s)^2 + 2w_1 + \mu_1 + u_1^{*,(k)}\sigma_1 + w_1 e^{-y_{21}} - 7.65 \\ s.t. \quad w_s \in [2.065,\ 2.075],\ w_1 \in [0.7714,\ 0.7814],\ w_2 \in [0.14,\ 0.16] \end{cases}$$

where the design variables are $\mathbf{w} = \left(w_s, w_1, w_2\right)$, and y_{21} is the coupling variable obtained by the following MDA.

$$\begin{cases} y_{12} = (\mu_s + u_s^{*,(k)}\sigma_s + 0.5w_s)^2 + 2w_1 - (\mu_1 + u_1^{*,(k)}\sigma_1) + 2\sqrt{y_{21}} \\ y_{21} = (\mu_s + u_s^{*,(k)}\sigma_s + 0.5w_s)w_2 + w_2^2 + \mu_2 + u_2^{*,(k)}\sigma_2 + y_{12} \end{cases}$$

The above MDA and the optimization problem are nested; and they form a single-loop IA. The solution of the IA loop is the interval variables $\mathbf{w}^{(k)} = \left(w_s^{(k)}, w_1^{(k)}, w_2^{(k)}\right)$.

2. SSL algorithm

1) PA loop

The MPP search and MDA are formulated together as a single-loop procedure. The formulation is given below.

$$\begin{cases} \min\limits_{\mathbf{u},\mathbf{y}} \sqrt{u_s^2 + u_1^2 + u_2^2} \\ s.t. \quad G_1 = (\mu_s + u_s\sigma_s + 0.5w_s^{(k-1)})^2 + 2w_1^{(k-1)} + \mu_1 + u_1\sigma_1 + w_1^{(k-1)}e^{-y_{21}} - 7.65 = 0 \\ \qquad h_1 = y_{12} - \left[(\mu_s + u_s\sigma_s + 0.5w_s^{(k-1)})^2 + 2w_1^{(k-1)} - (\mu_1 + u_1\sigma_1) + 2\sqrt{y_{21}}\right] = 0 \\ \qquad h_2 = y_{21} - \left[(\mu_s + u_s\sigma_s + 0.5w_s^{(k-1)})w_2^{(k-1)} + (w_2^{(k-1)})^2 + \mu_2 + u_2\sigma_2 + y_{12}\right] = 0 \end{cases}$$

where the design variables are $\mathbf{u}=(u_s,u_1,u_2)$ and $\mathbf{y}=\left(y_{12},y_{21}\right)$.

2) IA loop

$$\begin{cases}\max\limits_{\mathbf{w},\mathbf{y}} \quad G_1=(\mu_s+u_s^{*,(k)}\sigma_s+0.5w_s)^2+2w_1+\mu_1+u_1^{*,(k)}\sigma_1+w_1e^{-y_{21}}-7.65 \\ s.t. \quad h_1=y_{12}-\left[(\mu_s+u_s^{*,(k)}\sigma_s+0.5w_s)^2+2w_1-(\mu_1+u_1^{*,(k)}\sigma_1)+2\sqrt{y_{21}}\right]=0 \\ \qquad h_2=y_{21}-\left[(\mu_s+u_s^{*,(k)}\sigma_s+0.5w_s)w_2+w_2^2+\mu_2+u_2^{*,(k)}\sigma_2+y_{12}\right]=0 \\ \qquad w_s\in[2.065,\ 2.075],\ w_1\in[0.7714,\ 0.7814],\ w_2\in[0.14,\ 0.16]\end{cases}$$

where the design variables are $\mathbf{w}=\left(w_s,w_1,w_2\right)$ and $\mathbf{y}=\left(y_{12},y_{21}\right)$.

3. SSSL algorithm

1) PA loop

The MPP search and MDA are conducted sequentially. In the *j*-th iteration of PA, the MPP search is formulated as

$$\begin{cases}\min\limits_{\mathbf{u}} \quad \sqrt{u_s^2+u_1^2+u_2^2} \\ s.t. \quad G_1=(\mu_s+u_s\sigma_s+0.5w_s^{(k-1)})^2+2w_1^{(k-1)}+\mu_1+u_1\sigma_1+w_1^{(k-1)}e^{-y_{21}^{(j-1)}}-7.65=0\end{cases}$$

where the design variables are $\mathbf{u}=(u_s,u_1,u_2)$. The interval variables $w_s^{(k-1)}$ and $w_1^{(k-1)}$ are from the (*k*-1)-th cycle of the overall reliability analysis. (Recall the current cycle is the *k*-th cycle.) The coupling variable $y_{21}^{(j-1)}$ is obtained from the previous MDA in the (*j*-1)-th iteration. After the MPP search, MDA is performed to solve the coupling variable $y_{21}^{(j)}$ and is formulated as

$$\begin{cases}y_{12}=(\mu_s+u_s\sigma_s+0.5w_s^{(k-1)})^2+2w_1^{(k-1)}-(\mu_1+u_1\sigma_1)+2\sqrt{y_{21}} \\ y_{21}=(\mu_s+u_s\sigma_s+0.5w_s^{(k-1)})w_2^{(k-1)}+(w_2^{(k-1)})^2+\mu_2+u_2\sigma_2+y_{12}\end{cases}$$

2) IA loop

The IA loop is the same as in the SSL algorithm.

Table 1 shows the comparison of the three algorithms with the reliability analysis results for probabilistic constraints G_1 and G_2. The comparison is made with the same convergence criteria applied to each algorithm. Monte Carlo Simulation (MCS), as a sampling-based verification method, is also conducted. Latin Hypercube sampling is used to draw the samples of the interval variables. The result from MCS and a 95% confidence interval of the solution are also listed in Table 1. The computational cost of all the methods is measured by the number of function evaluations(Funcall in Table 1), which are the numbers of analyses at the subsystem level. For example, for $p_f^{\max}$ of G_1, (1221, 1105) are the numbers of analyses in subsystems 1 and 2, respectively.

Table 1 Bounds of p_f

Constraints		SDL	SSL	SSSL	MCS	95% confidence interval
G_1	$p_f^{\max}$	0.1823	0.1823	0.1823	0.1823	[0.1815, 0.1831]
	Funcall	(1221, 1105)	(2540, 2540)	(506, 410)	10^6	
	$p_f^{\min}$	0.1797	0.1797	0.1797	0.1806	[0.1798, 0.1814]
	Funcall	(1231, 1115)	(2084, 2084)	(506, 410)	10^6	
G_2	$p_f^{\max}$	0.1124	0.1124	0.1124	0.1129	[0.1123, 0.1135]
	Funcall	(13185, 14621)	(380, 380)	(785, 977)	10^6	
	$p_f^{\min}$	0.1092	0.1092	0.1092	0.1093	[0.1087, 0.1099]
	Funcall	(7810, 8650)	(380, 380)	(785, 977)	10^6	

It is noted that the results obtained from the SDL, SSL and SSSL algorithms are identical. The results are also very close to the MCS solutions. All the algorithms therefore converge to an accurate solution. For this problem, the SSSL method is most efficient for G_1 and also efficient for G_2. The SSL method is most efficient for G_2 but least efficient for G_2.

B. Example 2 - Aircraft Wing Design

A wing design problem for a light aircraft [24] involves aerodynamic design and structural design. Aerodynamic design is responsible for selecting the external shape of the wing while structural design determines the structural size. The two disciplines are coupled with each other. A structural model is depicted in Fig. 9 [24], and the coupled subsystems are illustrated in Fig.10. The symbols in Fig. 10 are explained in Tables 2 and 3. In this example, aerodynamic model is built based on the lifting line theory, and the structural model is developed with the beam theory [24].

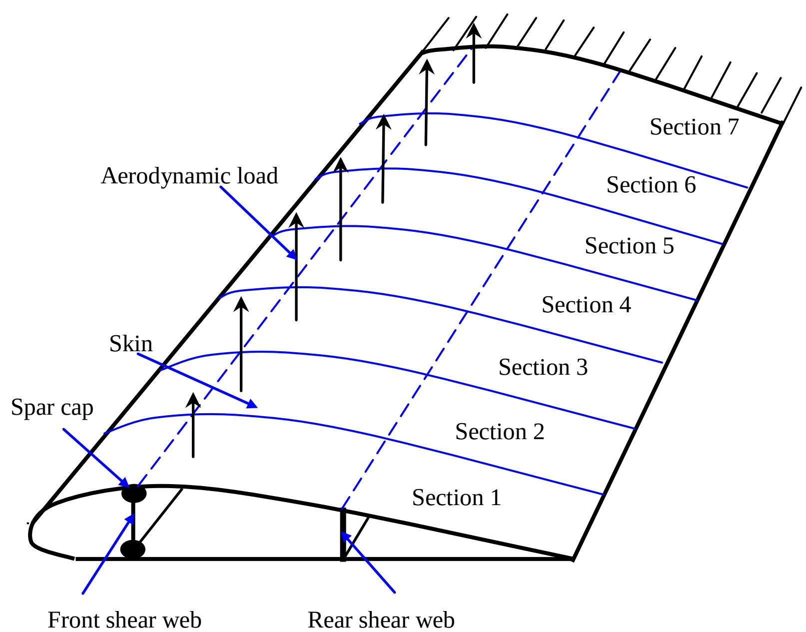

Fig. 9 The wing structure model.

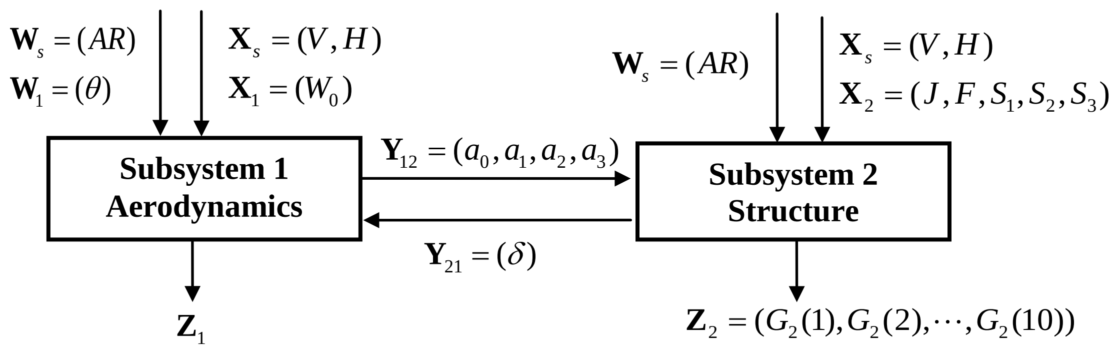

Fig. 10 Coupled aerodynamic and structural subsystems

The lifting-line model predicts the wing's aerodynamic characteristics, including the lift distributions, lift coefficients, and induced drag. The reason to choose this model is that the flight speed of the light aircraft in this paper is in the subsonic region and that the wing is unswept with a large aspect ratio. The lifting line model is able to predict lift distributions, lift coefficients, and induced drag with satisfactory accuracy [25]. This model also provides a convenient way to study the impact of wing twist and the aspect ratio on the aerodynamic characteristics. The total drag is the sum of induced drag and parasite drag. The parasite drag mainly depends on the wetted area of the aircraft. Since the wetted area in this investigation is almost unchanged with the design variables, the parasite drag coefficient is assumed to be 0.015 based on historical data for simplicity.

The beam model is used to calculate the stress and twist deformation of the wing structure. The beam model is an approximate method for the analysis of typical members of wing box structure. For unswept wings with a high aspect ratio, this model can obtain reasonable accurate results for wing structural analysis and has been widely used for preliminary design of wing structure before the finite element method comes into use. Although the finite element method can provide more accurate results, the beam model is selected in this study for alleviating the computational expense.

The reliability associated with each of the following constraints in Subsystem 2 is to be evaluated. The probabilities of failure are given by

$$\Pr\{G_2(i) \le 0\} = \Pr\{\sigma_i - S_1 \le 0\} \qquad (i = 1, 2, \ldots, 7)$$
$$\Pr\{G_2(8) \le 0\} = \Pr\{\tau_{skin} - S_2 \le 0\}$$
$$\Pr\{G_2(9) \le 0\} = \Pr\{\tau_{wf} - S_3 \le 0\}$$
$$\Pr\{G_2(10) \le 0\} = \Pr\{\tau_{wr} - S_3 \le 0\}$$

where

σ_i $(i = 1, 2, \ldots, 7)$ is the bending stresses in the spar cap for each section,

τ_{skin} is the maximum shear stress in the skin,

τ_{wf} is the shear stress in the web,

τ_{wr} is the shear stress in the rear web, and

S_1, S_2 and S_3 are the bending strength of the material of the spar caps, the shear strength of the skin, and the shear strength of the spar web, respectively.

S_1, S_2 and S_3 are normally distributed, and their distribution parameters are given in Table 2 along with other random variables.

Table 2 Distributions of random variables

Variables	Mean	Standard deviation	Distribution
Flight altitude *H*	3000 m	300 m	Normal
Flight speed *V*	200 km/h	20 km/h	Normal
Take-off weight W_0	700 kg	70 kg	Normal
Shear modulus *J*	2.7×10^{10} N/ mm^2	2.7×10^{9} N/ mm^2	Normal
Gust load factor *F*	4.0	0.4	Normal
Bending strength S_1	450 N/mm^2	45 N/mm^2	Normal
Shear strength of the skin S_2	200 N/mm^2	20N/mm^2	Normal
Shear strength of the web S_3	250 N/mm^2	25 N/mm^2	Normal

The aspect ratio (AR) and twist angle (θ) are interval variables. Their nominal values and widths are provided in Table 3. The angle of attack (α) of this wing is 5.0877 degrees. The areas of spar caps in sections 1 through 7 are 50.0 mm^2, 54.81 mm^2, 122.07 mm^2, 215.23 mm^2, 333.13 mm^2, 472.83 mm^2, 628.42 mm^2, respectively. The thicknesses of the skin, front web, and rear web are all 1.0 mm.

Table 3 Interval and deterministic variables

Design variables	Nominal values	Width	Disciplines
Aspect ratio, AR	5.7823	0.40	Aerodynamics
Twist angle, θ	0.8041 (deg)	0.20 (deg)	Aerodynamics

Table 4 shows the results from the three algorithms for limit-state functions $G_2(1)$ through $G_2(10)$. MCS is also conducted to confirm the accuracy of the results. The 95% confidence intervals of the MCS solutions are also included in Table 4. The results show that the three algorithms produce the same solutions, which are all close to the result from MCS. For this problem, the SSL algorithm requires the least number of disciplinary analyses.

Table 4 Two bounds of p_f obtained by different algorithms

Constraints		SDL	SSL	SSSL	Monte	95% conf. interval
	p_f^{max}	≈ 0	≈ 0	≈ 0	≈ 0	[0, 0]
$G_2(1)$	Funcall	(15759, 16221)	(1966, 1966)	(4838, 5963)	10^4	
	p_f^{min}	≈ 0	≈ 0	≈ 0	≈ 0	[0, 0]
	Funcall	(16048, 16519)	(976, 976)	(4816, 5932)	10^4	
	p_f^{max}	5.614×10^{-3}	5.614×10^{-3}	5.614×10^{-3}	5.363×10^{-3}	[5.340×10^{-3}, 5.688×10^{-3}]
$G_2(2)$	Funcall	(12228, 12586)	(1088, 1088)	(3915, 4509)	10^4	
	p_f^{min}	1.800×10^{-3}	1.800×10^{-3}	1.800×10^{-3}	1.812×10^{-3}	[1.699×10^{-3}, 1.864×10^{-3}]
	Funcall	(10948, 11269)	(1172, 1172)	(3898, 4528)	10^4	
$G_2(3)$	p_f^{max}	5.622×10^{-3}	5.622×10^{-3}	5.622×10^{-3}	5.549×10^{-3}	[5.402×10^{-3}, 5.694×10^{-3}]

Constraints		SDL	SSL	SSSL	Monte	95% conf. interval
	Funcall	(13447, 13841)	(1088, 1088)	(3915, 4509)	10^4	
	p_f^{min}	1.802×10^{-3}	1.802×10^{-3}	1.802×10^{-3}	1.808×10^{-3}	[1.701×10^{-3}, 1.867×10^{-3}]
	Funcall	(10957, 11278)	(1172, 1172)	(3898, 4528)	10^4	
$G_2(4)$	p_f^{max}	5.635×10^{-3}	5.635×10^{-3}	5.635×10^{-3}	5.646×10^{-3}	[5.414×10^{-3}, 5.706×10^{-3}]
	Funcall	(11373, 11706)	(1088, 1088)	(3915, 4518)	10^4	
	p_f^{min}	1.801×10^{-3}	1.801×10^{-3}	1.801×10^{-3}	1.786×10^{-3}	[1.703×10^{-3}, 1.869×10^{-3}]
	Funcall	(12478, 12844)	(1172, 1172)	(3898, 4528)	10^4	
$G_2(5)$	p_f^{max}	5.658×10^{-3}	5.658×10^{-3}	5.658×10^{-3}	5.498×10^{-3}	[5.430×10^{-3}, 5.722×10^{-3}]
	Funcall	(13515, 13911)	(1088, 1088)	(3915, 4518)	10^4	
	p_f^{min}	1.800×10^{-3}	1.800×10^{-3}	1.800×10^{-3}	1.741×10^{-3}	[1.705×10^{-3}, 1.871×10^{-3}]
	Funcall	(11356, 11689)	(1172, 1172)	(3898, 4528)	10^4	
$G_2(6)$	p_f^{max}	5.680×10^{-3}	5.680×10^{-3}	5.680×10^{-3}	5.463×10^{-3}	[5.452×10^{-3}, 5.744×10^{-3}]
	Funcall	(11373, 11706)	(1088, 1088)	(3915, 4527)	10^4	
	p_f^{min}	1.797×10^{-3}	1.797×10^{-3}	1.797×10^{-3}	1.742×10^{-3}	[1.699×10^{-3}, 1.865×10^{-3}]
	Funcall	(12478, 12844)	(1172, 1172)	(3898, 4537)	10^4	
$G_2(7)$	p_f^{max}	5.451×10^{-3}	5.451×10^{-3}	5.451×10^{-3}	5.703×10^{-3}	[5.442×10^{-3}, 5.734×10^{-3}]
	Funcall	(10761, 11076)	(1074, 1074)	(3915, 4536)	10^4	
	p_f^{min}	1.794×10^{-3}	1.794×10^{-3}	1.794×10^{-3}	1.777×10^{-3}	[1.697×10^{-3}, 1.863×10^{-3}]
	Funcall	(12070, 12424)	(1158, 1158)	(3898, 4537)	10^4	
$G_2(8)$	p_f^{max}	≈ 0	≈ 0	≈ 0	≈ 0	[0, 0]
	Funcall	(14229, 14646)	(1591, 1591)	(4833, 5787)	10^4	
	p_f^{min}	≈ 0	≈ 0	≈ 0	≈ 0	[0, 0]
	Funcall	(13804, 14209)	(920, 920)	(4816, 5734)	10^4	
$G_2(9)$	p_f^{max}	≈ 0	≈ 0	≈ 0	≈ 0	[0, 0]
	Funcall	(16303, 16781)	(1256, 1256)	(5445, 6597)	10^4	
	p_f^{min}	≈ 0	≈ 0	≈ 0	≈ 0	[0, 0]
	Funcall	(14739, 15171)	(1273, 1273)	(4833, 5949)	10^4	
$G_2(10)$	p_f^{max}	≈ 0	≈ 0	≈ 0	≈ 0	[0, 0]
	Funcall	(12597, 12966)	(906, 906)	(4527, 5346)	10^4	
	p_f^{min}	≈ 0	≈ 0	≈ 0	≈ 0	[0, 0]
	Funcall	(12580, 12949)	(1778, 1778)	(3898, 4645)	10^4	

V. Conclusion

This paper presents a unified reliability analysis framework for multidisciplinary systems with both random and interval variables. Given random and interval variables as inputs, the output of this framework is the bounds of reliability or of the probability of failure. The framework consists of probabilistic analysis (PA) and interval analysis (IA), both of which require multidisciplinary analysis (MDA). The overall reliability analysis, therefore, involves PA, IA and MDA; and the computation is intensive. To maintain computational efficiency, the framework decouples PA and IA and performs them sequentially. Three algorithms are designed to support the framework. They differ from each other with the way of how the PA and IA loops call the MDA loop. The three algorithms are summarized and compared in Table 5.

Table 5 Summary of the three algorithms

Algorithm	Features	PA and IA methods	When to use it
SDL: Sequential Double Loops	The MDA inner loop is nested with the PA and IA outer loops; PA and IA involve a double-loop procedure.	PA: any reliability analysis methods. IA: nonlinear optimization, interval arithmetic, or other IA methods.	MDA is not computational expensive.
SSL: Sequential Single Loops	MDA is embedded as equality constraints within the PA and IA loop; All the coupling variables are treated as additional design variables in the PA or IA single loop.	PA: FORM with nonlinear optimization for the MPP search IA: nonlinear optimization	The number of coupling variables is small; concurrent subsystem analyses can be performed.
SSSL: Sequential Single-Single Loops	PA involves a sequence of MPP search and MDA and therefore forms a sequential single-loops procedure. IA requires a single-loop procedure as in SSL.	PA: any reliability analysis methods, including any MPP search algorithms. IA: nonlinear optimization	PA is relatively expensive and IA is relatively cheap; concurrent subsystem analyses can be performed; the number of interval variables is small.

As shown in the two examples, the three algorithms are capable of producing identical solutions. But their efficiency differs from problem to problem. The efficiency depends on many factors, such as the number of disciplines, the number of random variables, the number of interval variables, the number of sharing variables, and the efficiency of disciplinary analyses.

As indicated in [22], “in a constrained optimization problem, equality constraints make the search process slow and difficult to converge.” In the SSL and SSSL algorithms, equality constraints for maintaining consistency between subsystems are included. As a result, the two algorithms may make the optimization process harder to converge compared to the SDL algorithm. Therefore, it is important to select a good starting point to help convergence.

A general guideline about selecting a specific algorithm is provided in Table 5. It is, however, only based on the limited number of testing problems and the theoretical derivations of the algorithm. The actual performance of the three algorithms needs a further investigation with more testing problems.

Other algorithm variants can also be developed using the similar strategies of the proposed three algorithms. For example, the IA loop of the SSSL algorithm is a single-loop procedure. It can be changed to a sequential single-loops procedure, where the search of the extreme values of the limit-state function and MDA will be conducted sequentially. All the algorithms discussed in this paper are only for reliability analysis. They can be used in reliability based multidisciplinary design optimization.

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