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Robust Design Optimization with Bivariate Quality Characteristics

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Abstract

In robust design optimization, if Taguchi quality loss function is employed, its expectation is minimized. When multiple quality characteristics exist, their covariances appear in the expectation and usually require numerical integrations. In this work, we propose an analytical robust design approach without numerical integrations to problems with bivariate quality characteristics. The quality characteristics are assumed to be functions of independent normal random variables with small uncertainties. Because the uncertainties are small, the functions are linearized with good accuracy. Analytical equations are then derived for the expected quality loss. The approach is efficient because no numerical integrations are needed. It is applied to the robust synthesis of a four-bar linkage.

Key words: robust design, optimization, uncertainty, mechanism synthesis

1. Introduction

Robust design (Phadke 1995) ensures that a product functions properly in the presence of uncertainties. As a result, the performances of the product are not sensitive to noises. The key product performances are referred to as quality characteristics, which are those response parameters that significantly affect product quality and customer satisfaction. The actual values of quality characteristics fluctuate around their designed nominal values, as well as around their targeted values, due to noises such as random dimensions of parts, stochastic loading, and varying usage conditions. Such fluctuations or variations lead to quality losses. During robust design, the variations in quality characteristics are minimized. The deviation from the actual quality characteristics to their targets is also minimized. The minimization is achieved through optimization by adjusting the nominal values of design variables without eliminating the sources of variation.

Robust design can be performed through either statistical experiments (Phadke 1995) or model-based optimization (Gu et al. 2000; Chen et al. 1996; Allen et al. 2006; Chen et al. 1997; Hernandez et al. 2001; McAllister and Simpson 2003; Mourelatos and Liang 2006; Youn and Xi 2009; Giassi et al. 2004; Han and Kwak 2004; Lee et al. 2009). Instead of experimentally estimating the quality characteristics, the latter method computationally evaluates quality characteristics with computational models. For example, the quality loss function associated with quality characteristics can be computationally calculated by the performance moment integration (PMI) method (Youn et al. 2005). This kind of robust design is called analytical robust design optimization. The methodologies of analytical robust design optimization are reviewed in (Beyer and Sendhoff 2007; Zang et al. 2005; Murphy et al. 2005; Park et al. 2006; Du and Chen 2000; Huang and Du 2007). This work focuses on analytical robust design optimization.

Quality characteristics usually appear in the objective function of a robust design model. The objective is to maximize the robustness associated with the quality characteristics of a product, process, or component. Various robustness metrics have been used in robust design literature. The most common metric is the Taguchi quality loss function (Phadke 1995). This metric reflects quality losses when the mean quality characteristics are off their targets and when the actual quality characteristics vary around their means. The Taguchi quality loss function is defined by a quadratic function. The quality loss function increases slowly when the quality characteristics deviate from its target in the vicinity of the target; it increases dramatically when the deviation is large. Because the quality loss is random, its expectation is used to measure robustness and is minimized. The expectation consists of two terms. The first is the square of the mean deviation (the deviation of the mean quality characteristic from its target) while the second is the variance of the quality characteristic. By minimizing the expected quality loss function, we can simultaneously bring the mean quality characteristics to its target and minimize the variation in the quality characteristic.

Most of the robust design methodologies handle only a single quality characteristic. With a single quality characteristic and its quality loss function, the optimization model involves only a single objective. If designers are interested in a trade-off between the standard deviation and the mean deviation of a quality characteristic, they may formulate the problem with a multi-objective optimization model (Li et al. 2006; McAllister et al. 2004; Messac and Ismail-Yahaya 2002; Hu et al. 2011; Li and Azarm 2008). The two objectives are the standard deviation and mean deviation of the quality characteristic. The common way of converting the two objectives into a single objective function is the weighted-sum method. The advantage of this method is that designers could give a higher weight to either the standard deviation or mean deviation of the quality characteristics.

When multiple quality characteristics are involved, the weighted-sum method can still be used. There are, however, two drawbacks. It is hard to assign weights to the means and standard deviations of multiple quality characteristics. The dependency between quality characteristics is lost due to the simple weighted sum. The quality characteristics are generally dependent, and ignoring such dependency may lead to erroneous or non-optimal solutions. If quality loss functions are used, the typical method is to sum up the individual expected quality loss functions. The advantage of doing so is that only a single objective function is involved. However, the dependency between the quality characteristics is also lost in the simple summation.

To account for the dependency, we must use the expectation of the total quality loss function, which is the sum of individual quality loss functions. This treatment leads to multivariate quality loss functions (Mao and Danzart 2008; Pignatiello 1993; Govindaluri and Cho 2007; Kovach et al. 2009; Wu and Chyu 2004a; Jeang et al. 2008). One type of multivariate quality loss function is a quadratic quality loss function (Wu and Chyu 2004b). The current method (Wu and Chyu 2004b) can account for the joint losses of all pairs of quality characteristics. In other words, the multivariate quality loss function considers joint losses up to the second order. As a result, if a multivariate quality loss function is applied to a design problem with bivariate quality characteristics, it can accurately reflect the true quality loss for the bivariate quality characteristics. However, the equation for the expected quality loss function (Wu and Chyu 2004b) is just an approximation. The purpose of this work is to develop an accurate and efficient model for the expectation of a bivariate quality loss function. We focus on small uncertainties in random input variables so that analytical equations can be derived.

In the next section, we discuss the bivariate robust optimization model. We then in Section 3 develop a bivariate robustness analysis model based on the First Order Second Moment method. A four-bar linkage synthesis problem is used as an example in Section 4. Conclusions are made in Section 5.

2. Model of Bivariate Robust Design Optimization

There are three components in a general optimization model, including design variables, objectives, and constraints. We first discuss the three components of a bivariate robust design problem and then give the optimization model with the three components.

2.1 Design variables and parameter

We discuss the most general robust design model where there are three types of variables and parameters. They include deterministic design variables $\mathbf{d} = (d_1, d_2, \ldots, d_{nd})$, random design variables $\mathbf{X} = (X_1, X_2, \ldots, X_{nx})$, and input parameters $\mathbf{P} = (P_1, P_2, \ldots, P_{np})$. Design variables are those variables whose distribution parameters can be controlled and changed during the optimization process, for examples, the part dimensions. Design variables can be deterministic, such as the number of teeth of a gear; or random, such as the diameter of the gear. We assume that the means $\boldsymbol{\mu}_{\mathbf{X}} = (\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_{nx}})$ of $\mathbf{X}$ are changeable during optimization. Therefore, the actual variables to be determined are $\mathbf{d}$ and $\boldsymbol{\mu}_X$. Input parameters $\mathbf{P}$ are known parameters and are kept constant during optimization. Examples of input parameters include the environmental temperature, wind speed, and material properties.

In general, input parameters may also be random. Therefore we have random variables $\mathbf{Z} = (\mathbf{X}, \mathbf{P})$. Because $\mathbf{Z}$ is the input vector for the quality characteristics, we call $\mathbf{Z}$ random input variables. We assume that all the input random variables are independently and normally distributed.

2.2 Objective function

In this work, we are interested in problems involving two quality characteristics, Q_1 and Q_2. They are functions of design variables and input parameters and are given by

$$Q_i = f_i(\mathbf{d}, \mathbf{X}, \mathbf{P}) \ (i = 1, 2) \tag{1}$$

Because Q_1 and Q_2 share the same input variables and are therefore statistically dependent.

There are three types of quality characteristics: the nominal-the-best, the smaller-the-better, and the larger-the-better. For a quality characteristic of the first type, there is a defined target value to be achieved. For a quality characteristic of the second type, the ideal target value is zero. For a quality characteristic of the third type, it is preferred to maximize the quality characteristics. In this work, we focus on the nominal-the-best type.

In this section, we review the modeling methodology for multiple quality characteristics (Wu and Chyu 2004b). For nominal-the-best quality characteristics Q_i $(i = 1, 2)$, the Taguchi quality loss function can be approximated with a second order Taylor series expansion at the targets $\mathbf{m} = (m_1, m_2)$, where the derivatives $L'(\mathbf{m}) = \left(\frac{\partial L}{\partial Q_1}, \frac{\partial L}{\partial Q_2} \right)\Big|_{\mathbf{m}} = \mathbf{0}$. Then the quality loss function is given by

$$L \approx \frac{1}{2} \sum_{i=1}^{2} \frac{\partial^2 L}{\partial Q_i^2}\Bigg|_{\mathbf{m}} (Q_i - m_i)^2 + \frac{\partial^2 L}{\partial Q_1 \partial Q_2}\Bigg|_{\mathbf{m}} (Q_1 - m_1)(Q_2 - m_2) \tag{2}$$

or

$$L \approx \sum_{i=1}^{2} q_i (Q_i - m_i)^2 + q_{12}(Q_1 - m_1)(Q_2 - m_2) \tag{3}$$

where q_i and q_{ij} are constants and are determined by the quality losses at the lower specification limits $m_i - \Delta_i^-$ and upper specification limits $m_i + \Delta_i^+$.

Suppose the quality loss is A_i^+ when $Q_i = m_i + \Delta_i$ $(i = 1, 2)$ and Q_j $(j \neq i)$ is on its target m_j. From Eq. (3),

$$A_i^+ = q_i^+ (\Delta_i^+)^2$$

Therefore

$$q_i^+ = \frac{A_i^+}{(\Delta_i^+)^2}$$

If the quality loss is A_i^- when $Q_i = m_i - \Delta_i^-$ $(.i = 1, 2.)$ and Q_j $(j \neq i)$ is on its target m_j. From Eq. (3),

$$A_i^- = q_i^-(\Delta_i^-)^2$$

Then

$$q_i^- = \frac{A_i^-}{(\Delta_i^-)^2}$$

Assume that the joint quality loss is A_{12}^{++} when both Q_1 and Q_2 are at their upper specification limits, and then

$$A_{12}^{++} = A_1^+ + A_2^+ + q_{12}^{++}\Delta_1^+\Delta_2^+$$

$$q_{12}^{++} = \frac{A_1^+ + A_2^+ - A_{12}^{++}}{\Delta_1^+\Delta_2^+}$$

When Q_1 is at its upper specification limit and Q_2 is at its lower specification limit, the quality loss is A_{12}^{+-}.

$$A_{12}^{+-} = A_1^+ + A_2^- + q_{12}^{+-}\Delta_1^+(-\Delta_2^-)$$

Then

$$q_{12}^{+-} = -\frac{A_1^+ + A_2^- - A_{12}^{+-}}{\Delta_1^+\Delta_2^-}$$

Similarly, we have

$$q_{12}^{-+} = -\frac{A_1^- + A_2^+ - A_{12}^{-+}}{\Delta_1^-\Delta_2^+}$$

and

$$q_{12}^{--} = \frac{A_1^- + A_2^- - A_{12}^{--}}{\Delta_1^-\Delta_2^-}$$

where A_{12}^{-+} is the joint quality loss when Q_1 is at its lower specification limit and Q_2 is at its upper specification limit, and A_{12}^{--} is the joint quality loss when Y_1 and Y_2 are at their lower specification limits.

The coefficients in Eq. (3) are therefore given by

$$q_i = \begin{cases} q_i^+ & \text{if } Y_i \ge m_i \\ q_i^- & \text{if } Y_i < m_i \end{cases}$$

and

$$q_{12} = \begin{cases} q_{12}^{++} & \text{if } Y_1 \ge m_1,\ Y_2 \ge m_2 \\ q_{12}^{+-} & \text{if } Y_1 \ge m_1,\ Y_2 < m_2 \\ q_{12}^{-+} & \text{if } Y_1 < m_1,\ Y_2 \ge m_2 \\ q_{12}^{--} & \text{if } Y_1 < m_1,\ Y_2 < m_2 \end{cases}$$

The expected quality loss function is thus given by

$$E_L = \sum_{i=1}^{2} E[q_i(Q_i - m_i)]^2 + E[q_{12}(Q_1 - m_1)(Q_2 - m_2)] \tag{4}$$

2.3 Constraints

Suppose that a constraint function is $g(\mathbf{d}, \mathbf{X}, \mathbf{P})$ and that $g(\mathbf{d}, \mathbf{X}, \mathbf{P}) \le 0$ is required. Because the constraint is random, we may not satisfy the requirement in an absolute sense. Instead, we can meet the requirement up to a desired level of probability or reliability R_e. Then the constraint is formulated as

$$\Pr\{g(\mathbf{d}, \mathbf{X}, \mathbf{P}) \le 0\} \ge R_e \tag{5}$$

The above treatment is referred to as design feasibility robustness (Du and Chen 2000; Parkinson et al. 1993). With the constraint, we can obtain a feasible design with the probability of R_e.

2.4 Optimization model

Given the above three optimization components, the optimization model with bivariate quality characteristics is formulated as

$$\begin{cases} \min\limits_{(\mathbf{d}, \boldsymbol{\mu}_\mathbf{X})} \; E_L(\mathbf{d}, \mathbf{X}, \mathbf{P}) = \sum_{i=1}^{2} E[q_i(Q_i - m_i)]^2 + E[q_{12}(Q_1 - m_1)(Q_2 - m_2)] \\ \text{subject to} \\ \quad \Pr\{g_j(\mathbf{d}, \mathbf{X}, \mathbf{P}) \le 0\} \ge R_{ej} \;\; (j = 1, 2, \ldots, n_g) \end{cases} \tag{6}$$

To perform the optimization, we need to calculate the objective function E_L. In this work, we develop an accurate model for E_L. Details are given in the following section.

3. Robustness Analysis for Bivariate Quality Characteristics

We now develop a model for the expected bivariate quality loss function. We are interested in small uncertainties in the input random variables $\mathbf{Z} = (\mathbf{X}, \mathbf{P})$, which are independently and normally distributed. This situation is commonly encountered in mechanism analysis and synthesis. The primary source of uncertainty comes from mechanism dimension tolerances. Because the tolerances are relatively small compared to the nominal dimensions, the standard deviations of the input random variables are also small. For this situation, a quality characteristic can be approximated with the first order Taylor expansion series with respect to **Z** with good accuracy. The approximation is given by

$$Q_i = f_i(\mathbf{d}, \mathbf{X}, \mathbf{P}) = f_i(\mathbf{d}, \mathbf{Z}) \approx a_{i0} + \sum_{j=1}^{nz} a_{ij} Z_j \;\; (i = 1, 2) \tag{7}$$

where $a_{i0} = f_i(\mathbf{d}, \mu_{\mathbf{Z}}) - \sum_{j=1}^{nz} \left.\frac{\partial f_i}{\partial Z_i}\right|_{\boldsymbol{\mu}_{\mathbf{Z}}} \mu_{Z_j}$ and $a_{ij} = \left.\frac{\partial f_i}{\partial Z_i}\right|_{\boldsymbol{\mu}_{\mathbf{Z}}}$. $\boldsymbol{\mu}_{\mathbf{Z}}$ is the vector of the means of **Z**, and $nz = nx + np$. We now calculate the expected quality loss function of Q_1 and Q_2 based on the above approximation.

It should be noted that the first order approximation in Eq. (7) may not be accurate for some extreme situations where the second derivatives of $f_i(\cdot)$ at $\boldsymbol{\mu}_{\mathbf{Z}}$ are large.

3.1 Expected Quality Loss Function

To make the derivation easy, we define new variables $Y_i \;\; (i = 1, 2)$ by

$$Y_i = Q_i - m_i \;\; (i = 1, 2) \tag{8}$$

With Eq. (7), the mean and standard deviation of Y_i are

$$\mu_{Y_i} = a_{i0} + \sum_{j=1}^{nz} a_{ij} \mu_{Z_j} - m_i \tag{9}$$

and

$$\sigma_{Y_i} = \sqrt{\sum_{j=1}^{nz} a_{ij}^2 \sigma_{Z_j}^2} \tag{10}$$

According to Eq. (4), the expected quality loss function is

$$E(L) = E\left(q_1 Y_1^2 + q_2 Y_2^2 + q_{12} Y_1 Y_2\right) = \\ = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(q_1 y_1^2 + q_2 y_2^2 + q_{12} y_1 y_2\right) f_{Y_1 Y_2}(y_1, y_2) dy_1 dy_2 \\ = E_1^+ + E_1^- + E_2^+ + E_2^- + E_{12}^{++} + E_{12}^{--} + E_{12}^{+-} + E_{12}^{-+} \tag{11}$$

where $f_{Y_1}(\cdot)$ and $f_{Y_2}(\cdot)$ are probability density functions (PDF) of Y_1 and Y_2, respectively, and $f_{Y_1 Y_2}(\cdot,\cdot) = f_{Y_1}(\cdot) f_{Y_2}(\cdot)$ is the joint PDF of Y_1 and Y_2. The expectations on the right-hand side of Eq. (11) are as follows:

$$E_i^+ = q_i^+ \int_0^{\infty} y^2 f_{Y_i}(y) dy, \quad i = 1, 2 \tag{12}$$

$$E_i^- = q_i^- \int_{-\infty}^{0} y^2 f_{Y_i}(y) dy, \quad i = 1, 2 \tag{13}$$

$$E_{12}^{++} = q_{12}^{++} \int_0^{\infty}\int_0^{\infty} y_1 y_2 f_{Y_1 Y_2}(y_1, y_2) dy_1 dy_2 \tag{14}$$

$$E_{12}^{--} = q_{12}^{--} \int_{-\infty}^{0}\int_{-\infty}^{0} y_1 y_2 f_{Y_1 Y_2}(y_1, y_2) dy_1 dy_2 \tag{15}$$

$$E_{12}^{+-} = q_{12}^{+-} \int_0^{\infty}\int_{-\infty}^{0} y_1 y_2 f_{Y_1 Y_2}(y_1, y_2) dy_1 dy_2 \tag{16}$$

$$E_{12}^{-+} = q_{12}^{-+} \int_{-\infty}^{0}\int_0^{\infty} y_1 y_2 f_{Y_1 Y_2}(y_1 y_2) dy_1 dy_2 \tag{17}$$

Next, we discuss how to obtain these expectations analytically.

3.2 E_i^+ and E_i^- $(i = 1, 2)$

The derivation of expectations involves truncated normal distributions. Suppose a normally distributed variable is $W \sim N\left(\mu_W, \sigma_W\right)$ and is truncated with $w_l \le W \le w_u$. Let $\tilde{W}$ denote such a truncated variable. According to (Patel and Read 1996), the mean of $\tilde{W}$ is

$$\mu_{\tilde{W}} = \mu_W + \frac{\phi(v_l) - \phi(v_u)}{\Phi(v_u) - \Phi(v_l)} \sigma_W \tag{18}$$

where

$$v_l = \frac{w_l - \mu_W}{\sigma_W} \tag{19}$$

$$v_u = \frac{w_u - \mu_W}{\sigma_W} \tag{20}$$

and $\phi(\cdot)$ and $\Phi(\cdot)$ are the PDF and cumulative distribution function (CDF) of a standard normal distribution.

The variance of $\tilde{W}$ is

$$\sigma_{\tilde{W}}^2 = \left[1 + \frac{v_l\phi(v_l) - v_u\phi(v_u)}{\Phi(v_u) - \Phi(v_l)}\right]\sigma_W^2 - \left(\mu_{\tilde{W}} - \mu_W\right)^2 \tag{21}$$

With the above equations, we now evaluate E_i^+ and E_i^-.

$Y_i \geq 0$ can be considered as a left-truncated variable of Y_i with the truncation point at 0. We denote the truncated variable by $\tilde{Y}_i^+$.

$$E_i^+ = q_i^+ \int_0^\infty y^2 f_{Y_i}(y)\,dy = q_i^+ H_i^+ \int_0^\infty y^2 \frac{f_{Y_i}(y)}{H_i^+} dy = q_i^+ H_i^+ E\left[(\tilde{Y}_i^+)^2\right] \tag{22}$$

where

$$H_i^+ = \Pr\{Y_i \geq 0\} = 1 - \Phi\left(-\frac{\mu_{Y_i}}{\sigma_{Y_i}}\right) \tag{23}$$

Then

$$E_i^+ = q_i^+ H_i^+ \left(\mu_{\tilde{Y}_i^+}^2 + \sigma_{\tilde{Y}_i^+}^2\right) \tag{24}$$

To use Eqs. (18) and (21), we set $\tilde{W} = \tilde{Y}_i$, $w_l = 0$, $v_l = -\frac{\mu_{Y_i}}{\sigma_{Y_i}}$, $w_u = \infty$, and $v_u = \infty$. Using $\phi(\infty) = 0$ and $\Phi(\infty) = 1$, we obtain

$$\mu_{\tilde{Y}_i^+} = \mu_{Y_i} + \sigma_{Y_i} \frac{\phi(v_l)}{\Phi(-v_l)} \tag{25}$$

and

$$\sigma_{\tilde{Y}_i^+}^2 = \left[1 + \frac{v_l\phi(v_l)}{\Phi(-v_l)}\right]\sigma_{Y_i}^2 - \left(\mu_{\tilde{Y}_i^+} - \mu_Y\right)^2 \tag{26}$$

Then

$$E_i^+ = q_i^+ H_i^+ \left[\mu_{Y_i}^2 + \sigma_{Y_i}^2 + \sigma_{Y_i}\left(2\mu_{Y_i} + v_l\sigma_{Y_i}\right)\frac{\phi(v_l)}{\Phi(-v_l)}\right] \tag{27}$$

For E_i^-, $Y < 0$ is a right-truncated variable and is denoted by $\tilde{Y}_i^-$.

$$E_i^- = q_i^- \int_{-\infty}^0 y^2 f_{Y_i}(y)\,dy = q_i^- H_i^- \int_{-\infty}^0 y^2 \frac{f_{Y_i}(y)}{H_i^-} dy = q_i^- H_i^- E\left[(\tilde{Y}_i^-)^2\right] \tag{28}$$

where

$$H_i^- = \Pr\{Y_i < 0\} = \Phi\left(-\frac{\mu_{Y_i}}{\sigma_{Y_i}}\right) \tag{29}$$

We set $\tilde{W} = \tilde{Y}_i$, $w_u = 0$, $v_u = -\frac{\mu_{Y_i}}{\sigma_{Y_i}}$, $w_l = -\infty$, and $v_l = -\infty$. Using $\phi(-\infty) = 0$, $\Phi(-\infty) = 0$, and Eqs. (18) and (21), we obtain

$$\mu_{\tilde{Y}_i^-} = \mu_{Y_i} - \sigma_{Y_i} \frac{\phi(v_u)}{\Phi(v_u)} \tag{30}$$

and

$$\sigma^2_{\tilde{Y}_i^-} = \left[1 - \frac{v_l \phi(v_u)}{\Phi(v_u)}\right] \sigma^2_{Y_i} - \left(\mu_{\tilde{Y}_i^-} - \mu_Y\right)^2 \tag{31}$$

Then

$$E_i^- = q_i^- H_i^- \left[\mu^2_{Y_i} + \sigma^2_{Y_i} - \sigma_{Y_i}\left(2\mu_{Y_i} + v_u \sigma_{Y_i}\right) \frac{\phi(v_u)}{\Phi(v_u)}\right] \tag{32}$$

3.3 E^{++}

As indicated in Eq. (14), E^{++} is for $Y_1 \geq 0$ and $Y_2 \geq 0$. We denote these truncated random variables by $\tilde{\mathbf{Y}}^{++} = \left(\tilde{Y}_1^+, \tilde{Y}_2^+\right) \sim N\left(\boldsymbol{\mu}_{\tilde{\mathbf{Y}}^{++}}, \sum_{\tilde{\mathbf{Y}}^{++}}, 0, \infty, 0, \infty\right)$, where the mean vector is $\boldsymbol{\mu}_{\tilde{\mathbf{Y}}^{++}} = \left(\mu_{\tilde{Y}_1^+}, \mu_{\tilde{Y}_2^+}\right)$ and the covariance matrix is $\sum_{\tilde{\mathbf{Y}}^{++}} = \begin{pmatrix} \sigma^2_{\tilde{Y}_1} & \sigma_{\tilde{Y}_1\tilde{Y}_2} \\ \sigma_{\tilde{Y}_1\tilde{Y}_2} & \sigma^2_{\tilde{Y}_2} \end{pmatrix}$.

Ref. (Rosenbaum 1961) has derived the first two moments for a right-truncated bivariate distribution from a standard bivariate normal distribution. To use its results, we further transform Y_1 and Y_2 into standard normal variables U_1 and U_2. Then the corresponding truncated variables are

$$\tilde{U}_i^+ = \frac{\tilde{Y}_i^+ - \mu_{Y_i}}{\sigma_{Y_i}} \quad (i = 1, 2) \tag{33}$$

or

$$\tilde{Y}_i^+ = \mu_{Y_i} + \sigma_{Y_i} \tilde{U}_i^+ \tag{34}$$

The truncation points are

$$U_1^+ = h = -\frac{\mu_{Y_1}}{\sigma_{Y_1}} \tag{35}$$

and

$$U_2^+ = k = -\frac{\mu_{Y_2}}{\sigma_{Y_2}} \tag{36}$$

$$\begin{aligned}
E_{12}^{++} &= q_{12}^{++}\int_0^\infty\int_0^\infty y_1 y_2 f_{Y_1Y_2}(y_1,y_2)dy_1dy_2 \\
&= q_{12}^{++}H_{12}^{++}\int_0^\infty\int_0^\infty y_1 y_2 \frac{f_{Y_1Y_2}(y_1,y_2)}{H_{12}^{++}}dy_1dy_2 \\
&= q_{12}^{++}H_{12}^{++}E\left(\tilde{Y}_1^+,\tilde{Y}_2^+\right) \\
&= q_{12}^{++}H_{12}^{++}\left[\mathrm{COV}\left(\tilde{Y}_1^+,\tilde{Y}_2^+\right)+\mu_{\tilde{Y}_1^+}\mu_{\tilde{Y}_2^+}\right] \\
&= q_{12}^{++}H_{12}^{++}\left[\sigma_{Y_1}\sigma_{Y_2}\mathrm{COV}\left(\tilde{U}_1^+,\tilde{U}_2^+\right)+\left(\mu_{Y_1}+\sigma_{Y_1}\mu_{\tilde{U}_1^+}\right)\left(\mu_{Y_2}+\sigma_{Y_2}\mu_{\tilde{U}_2^+}\right)\right]
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
H_{12}^{++} &= \Pr\left(Y_1>0,Y_2>0\right) \\
&= \Phi(h)+\Phi(k)-\Phi_2(h,k,\rho)-1
\end{aligned} \tag{38}$$

$\Phi_2\left(h,k,\rho\right)$ is the standard bivariate normal CDF with the coefficient of correlation ρ.

The covariance $\mathrm{COV}\left(\tilde{U}_1^+,\tilde{U}_2^+\right)$ and means $\mu_{\tilde{U}_i^+}$ $(i=1,2)$ are to be determined. Using the results in (Rosenbaum 1961), we have

$$\mu_{\tilde{U}_1^+} = \frac{1}{H_{12}^{++}}\left[\phi(h)\Phi\left(-\frac{k-\rho h}{\sqrt{1-\rho^2}}\right)+\rho\phi(k)\Phi\left(-\frac{h-\rho k}{\sqrt{1-\rho^2}}\right)\right] \tag{39}$$

$$\mu_{\tilde{U}_2^+} = \frac{1}{H_{12}^{++}}\left[\rho\phi(h)\Phi\left(-\frac{k-\rho h}{\sqrt{1-\rho^2}}\right)+\phi(k)\Phi\left(-\frac{h-\rho k}{\sqrt{1-\rho^2}}\right)\right] \tag{40}$$

$$\mathrm{COV}\left(\tilde{U}_1^+,\tilde{U}_2^+\right) = \frac{1}{H_{12}^{++}}\begin{bmatrix}\rho L^{++}+\rho h\phi(h)\Phi\left(-\frac{k-\rho h}{\sqrt{1-\rho^2}}\right)+\rho k\phi(k)\Phi\left(-\frac{h-\rho h}{\sqrt{1-\rho^2}}\right) \\ +\sqrt{1-\rho^2}\phi(h)\Phi\left(\frac{k-\rho h}{\sqrt{1-\rho^2}}\right)\end{bmatrix} \tag{41}$$

where

$$\rho = \frac{\sigma_{Y_1Y_2}}{\sigma_{Y_1}\sigma_{Y_2}} \tag{42}$$

3.4 E_{12}^{--}

As indicated in Eq. (15), E_{12}^{--} is for $Y_1 < 0$ and $Y_2 < 0$ or for truncated variables $\tilde{\mathbf{Y}}^{--} = \left(\tilde{Y}_1^-, \tilde{Y}_2^-\right) \sim N\left(\boldsymbol{\mu}_{\tilde{\mathbf{Y}}^{--}}, \sum_{\tilde{\mathbf{Y}}^{--}}, -\infty, 0, -\infty, 0\right)$, where $\boldsymbol{\mu}_{\tilde{\mathbf{Y}}^{--}} = \left(\mu_{\tilde{Y}_1^-}, \mu_{\tilde{Y}_2^-}\right)$ and $\sum_{\tilde{\mathbf{Y}}^{--}} = \begin{pmatrix} \sigma_{\tilde{Y}_1^-}^2 & \sigma_{\tilde{Y}_1^-\tilde{Y}_2^-} \\ \sigma_{\tilde{Y}_1^-\tilde{Y}_2^-} & \sigma_{\tilde{Y}_2^-}^2 \end{pmatrix}$. We also transform $\tilde{Y}_1^-$ and $\tilde{Y}_2^-$ into $\tilde{U}_1^-$ and $\tilde{U}_2^-$. Using the same principles of (Rosenbaum 1961) , we can derive an equation for E^{--}. The detailed derivations are omitted, and the results are given below.

$$E_{12}^{--} = q_{12}^{--} H_{12}^{--} \left[\sigma_{Y_1}\sigma_{Y_2} \mathrm{COV}\left(\tilde{U}_1^-, \tilde{U}_2^-\right) + \left(\mu_{Y_1} + \sigma_{Y_1}\mu_{\tilde{U}_1^-}\right)\left(\mu_{Y_2} + \sigma_{Y_2}\mu_{\tilde{U}_2^-}\right)\right] \tag{43}$$

where

$$H_{12}^{--} = \Pr\left(Y_1 < 0, Y_2 < 0\right) = \Phi_2(h, k, \rho) \tag{44}$$

$$\mu_{\tilde{U}_1^-} = \frac{1}{H_{12}^{--}}\left[-\phi(h)\Phi\left(\frac{k-\rho h}{\sqrt{1-\rho^2}}\right) - \rho\phi(k)\Phi\left(\frac{h-\rho k}{\sqrt{1-\rho^2}}\right)\right] \tag{45}$$

$$\mu_{\tilde{U}_2^-} = \frac{1}{H_{12}^{--}}\left[\rho\phi(h)\Phi\left(-\frac{k-\rho h}{\sqrt{1-\rho^2}}\right) - \phi(k)\Phi\left(\frac{h-\rho k}{\sqrt{1-\rho^2}}\right)\right] \tag{46}$$

$$\mathrm{COV}\left(\tilde{U}_1^-, \tilde{U}_2^-\right) = \frac{1}{H_{12}^{--}}\left[\begin{array}{l} \rho H_{12}^{--} - \rho h\phi(h)\Phi\left(\frac{k-\rho h}{\sqrt{1-\rho^2}}\right) - \rho k\phi(k)\Phi\left(\frac{h-\rho h}{\sqrt{1-\rho^2}}\right) \\ +\sqrt{1-\rho^2}\phi(h)\Phi\left(\frac{k-\rho h}{\sqrt{1-\rho^2}}\right) \end{array}\right] \tag{47}$$

3.5 E_{12}^{+-}

Eq. (16) indicates that E_{12}^{+-} involves $Y_1 \geq 0$ and $Y_2 < 0$, which are truncated variables $\tilde{\mathbf{Y}}^{+-} = \left(\tilde{Y}_1^+, \tilde{Y}_2^-\right) \sim N\left(\boldsymbol{\mu}_{\tilde{\mathbf{Y}}^{+-}}, \sum_{\tilde{\mathbf{Y}}^{+-}}, 0, \infty, -\infty, 0\right)$, where $\boldsymbol{\mu}_{\tilde{\mathbf{Y}}^{+-}} = (\mu_{\tilde{Y}_1^+}, \mu_{\tilde{Y}_2^-})$ and $\sum_{\tilde{\mathbf{Y}}^{--}} = \begin{pmatrix} \sigma_{\tilde{Y}_1^+}^2 & \sigma_{\tilde{Y}_1^+\tilde{Y}_2^-} \\ \sigma_{\tilde{Y}_1^+\tilde{Y}_2^-} & \sigma_{\tilde{Y}_2^-}^2 \end{pmatrix}$. We also transform $\tilde{Y}_1^+$ and $\tilde{Y}_2^-$ into $\tilde{U}_1^+$ and $\tilde{U}_2^-$. Following the principle of (Rosenbaum 1961), we have

$$E_{12}^{+-} = q_{12}^{+-} H_{12}^{+-} \left[\sigma_{Y_1}\sigma_{Y_2} \mathrm{COV}(\tilde{U}_1^+, \tilde{U}_2^-) + (\mu_{Y_1} + \sigma_{Y_1}\mu_{\tilde{U}_1^+})(\mu_{Y_2} + \sigma_{Y_2}\mu_{\tilde{U}_2^-})\right] \tag{48}$$

where

$$H_{12}^{+-} = \Pr(Y_1 \geq 0, Y_2 < 0) = \Phi(k) - \Phi_2(h,k,\rho) \quad (49)$$

$$\mu_{\tilde{U}_1^+} = \frac{1}{H_{12}^{+-}}\left[\phi(h)\Phi\left(\frac{k-\rho h}{\sqrt{1-\rho^2}}\right) - \rho\phi(k)\Phi\left(-\frac{h-\rho k}{\sqrt{1-\rho^2}}\right)\right] \quad (50)$$

$$\mu_{\tilde{U}_2^-} = \frac{1}{H_{12}^{+-}}\left[\rho\phi(h)\Phi\left(\frac{k-\rho h}{\sqrt{1-\rho^2}}\right) - \phi(k)\Phi\left(-\frac{h-\rho k}{\sqrt{1-\rho^2}}\right)\right] \quad (51)$$

$$\mathrm{COV}\left(\tilde{U}_1^+, \tilde{U}_2^-\right) = \frac{1}{H_{12}^{+-}}\begin{bmatrix} \rho L^{+-} + \rho h\phi(h)\Phi\left(\frac{k-\rho h}{\sqrt{1-\rho^2}}\right) - \rho k\phi(k)\Phi\left(-\frac{h-\rho h}{\sqrt{1-\rho^2}}\right) \\ -\sqrt{1-\rho^2}\phi(h)\Phi\left(\frac{k-\rho h}{\sqrt{1-\rho^2}}\right) \end{bmatrix} \quad (52)$$

3.6 E_{12}^{-+}

Eq. (17) shows that E_{12}^{-+} is for truncated variables $Y_1 < 0$ and $Y_2 \geq 0$, donated by $\tilde{\mathbf{Y}}^{-+} = (\tilde{Y}_1^-, \tilde{Y}_2^+) \sim N(\boldsymbol{\mu}_{\tilde{\mathbf{Y}}^{-+}}, \sum_{\tilde{\mathbf{Y}}^{-+}}, -\infty, 0, 0, \infty)$, where $\boldsymbol{\mu}_{\tilde{\mathbf{Y}}^{-+}} = \left(\mu_{\tilde{Y}_1^-}, \mu_{\tilde{Y}_2^+}\right)$ and $\sum_{\tilde{\mathbf{Y}}^{--}} = \begin{pmatrix} \sigma_{\tilde{Y}_1^-}^2 & \sigma_{\tilde{Y}_1^-\tilde{Y}_2^+} \\ \sigma_{\tilde{Y}_1^-\tilde{Y}_2^+} & \sigma_{\tilde{Y}_2^+}^2 \end{pmatrix}$. We also transform $\tilde{Y}_1^-$ and $\tilde{Y}_2^+$ into $\tilde{U}_1^-$ and $\tilde{U}_2^+$. Following the principle of (Rosenbaum 1961), we have

$$E_{12}^{-+} = q_{12}^{-+} H_{12}^{-+}\left[\sigma_{Y_1}\sigma_{Y_2}\mathrm{COV}\left(\tilde{U}_1^-, \tilde{U}_2^+\right) + \left(\mu_{Y_1} + \sigma_{Y_1}\mu_{\tilde{U}_1^-}\right)\left(\mu_{Y_2} + \sigma_{Y_2}\mu_{\tilde{U}_2^+}\right)\right] \quad (53)$$

where

$$H_{12}^{-+} = \Pr(Y_1 < 0, Y_2 \geq 0) = \Phi(h) - \Phi_2(h,k,\rho) \quad (54)$$

$$\mu_{\tilde{U}_1^-} = \frac{1}{H_{12}^{-+}}\left[-\phi(h)\Phi\left(-\frac{k-\rho h}{\sqrt{1-\rho^2}}\right) + \rho\phi(k)\Phi\left(\frac{h-\rho k}{\sqrt{1-\rho^2}}\right)\right] \quad (55)$$

$$\mu_{\tilde{U}_2^+} = \frac{1}{H_{12}^{-+}}\left[-\rho\phi(h)\Phi\left(-\frac{k-\rho h}{\sqrt{1-\rho^2}}\right) + \phi(k)\Phi\left(\frac{h-\rho k}{\sqrt{1-\rho^2}}\right)\right] \quad (56)$$

$$\mathrm{COV}\left(\tilde{U}_1^+, \tilde{U}_2^-\right) = \frac{1}{H_{12}^{-+}}\begin{bmatrix} \rho H_{12}^{-+} - \rho h\phi(h)\Phi\left(-\frac{k-\rho h}{\sqrt{1-\rho^2}}\right) + \rho k\phi(k)\Phi\left(\frac{h-\rho h}{\sqrt{1-\rho^2}}\right) \\ -\sqrt{1-\rho^2}\phi(h)\Phi\left(\frac{k-\rho h}{\sqrt{1-\rho^2}}\right) \end{bmatrix} \quad (57)$$

3.7 Robust design optimization

As discussed above, with the First Order Second Moment Method (FOSM), the expected quality loss function can be analytically evaluated without any numerical iterative processes or integrations. We can also use FOSM to treat the constraints. As reported in robust design literature, a constraint $\Pr\{g_i(\mathbf{d},\mathbf{Z}) \le 0\} \ge R_{ei}$ $(i = 1, 2, \ldots, n_g)$ can be approximated as

$$\mu_{g_i} + \beta_i \sigma_{g_i} \le 0 \ (i = 1, 2, \ldots, n_g) \tag{58}$$

where μ_{g_i} is the mean of $g_i(\cdot)$ and is given by

$$\mu_{g_i} \approx g_i(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{Z}}) \tag{59}$$

and σ_{g_i} is the standard deviation of $g_i(\cdot)$ and is given by

$$\sigma_{g_i} = \sqrt{\sum_{j=1}^{nz} \left(\left. \frac{\partial g_i}{\partial Z_j} \right|_{(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{Z}})} \right)^2 \sigma_{Z_j}^2} \tag{60}$$

The reliability index β_i is determined by the required reliability R_{ei} and is given by

$$\beta_i = \Phi^{-1}(R_{ei}) \tag{61}$$

The robust design optimization can then be performed by solving the following model:

$$\begin{cases} \min\limits_{(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}})} E_L = E_1^+ + E_1^- + E_2^+ + E_2^- + E_{12}^{++} + E_{12}^{--} + E_{12}^{+-} + E_{12}^{-+} \\ \text{subject to} \\ \quad \mu_{g_i} + \beta_i \sigma_{g_i} \le 0 \ (i = 1, 2, \ldots, n_g) \end{cases} \tag{62}$$

Once the quality characteristics Q_1 and Q_2 are approximated with the first order Taylor expansion, the expected quality loss function in the objective function can be evaluated analytically with the equations in Sections 3.2 through 3.6. The reliability constraints can also be evaluated analytically with Eqs. (58) and (61). This treatment allows for quick solutions to robust design optimization.

4 Example - Robust Design Optimization for Four-Bar Linkage Synthesis

We now apply the proposed method to the robust design of a four-bar linkage (Zhang and Du 2011; Du et al. 2009) as shown in Fig. 1.

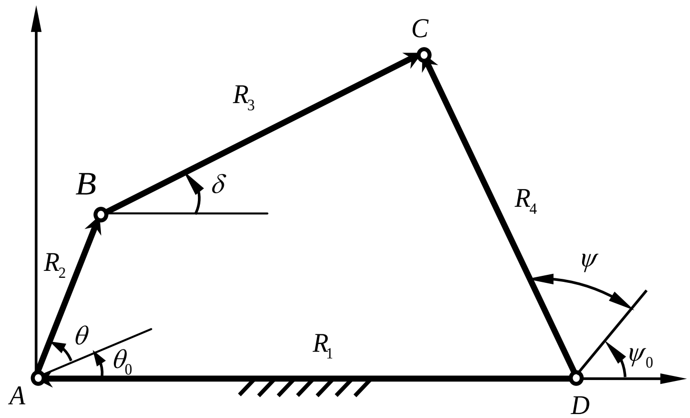

Figure 1 Four-bar linkage

4.1 Design requirements

The crank-rocker mechanism intends to fulfill the following motion requirements:

(1) When the crank angle is $\theta = 30^\circ$, the rocker output angle is $\psi = 90^\circ$.

(2) When the crank angle is $\theta = 150^\circ$, the rocker output angle is $\psi = 110^\circ$.

(3) Link *AB* is a crank, and link CD is a rocker.

(4) The minimal transmission angle is $\lambda = 40^\circ$.

(5) The constraints for the crank existence and transmission angles should be satisfied at the probability level of $\beta = 4$. This probability level means that the probability of failure is 3.17×10^{-5} or that the reliability is $R = 1 - 3.17\times10^{-5}$ according to Eq. (61).

(6) The lower and upper limit specifications of the output motion are $\psi_i \in [m_i - 0.8^\circ, m_i + 0.5^\circ]$, where $m_1 = 90^\circ$ and $m_2 = 110^\circ$.

(7) The quality characteristics are the motion output angles ψ, which are defined by

$$Q_i = f_i(\mathbf{d}, \mathbf{X}, \mathbf{P}) = \psi(\mathbf{d}, \mathbf{X}, \mathbf{P}; \theta)$$

where $i = 1, 2$; and $\theta = 30^\circ$ for $i = 1$, and $\theta = 150^\circ$ for $i = 2$.

(8) The quality losses are specified as follows: $A_1^+ = A_2^+ = \$1000$, $A_1^- = A_2^- = \$2200$, $A_{12}^{++} = \$2200$, $A_{12}^{--} = \$2800$, $A_{12}^{+-} = \$2500$, and $A_{12}^{-+} = \$2500$. We assume that negative motion errors will result in higher quality losses than positive motion errors. Therefore, A_i^- is larger than A_i^+ and that A_i^{--} is larger than $A_i^{++}(i = 1, 2)$.

The design variables are the initial crank angle θ_0, initial output angle ψ_0, and link lengths R_2, R_3, and R_4. The first two variables are deterministic, and therefore the deterministic design variables are $\mathbf{d} = (\theta_0, \psi_0)$, and the rest are random design variables, or $\mathbf{X} = (R_2, R_3, R_4)$. The distance between revolute joints *A* and *D* is R_1, whose distribution is known. The input parameter vector is therefore $\mathbf{P} = (R_1)$. The vector of all the random variables is $\mathbf{Z} = (\mathbf{X}, \mathbf{P}) = (R_2, R_3, R_4, R_1)$. Details of all the variables are summarized in Table 1.

Table 1 Design variables and parameters

Variable	Distribution	Mean (mm)	Standard deviation (mm)
Random input variable R_1	Normal	1000.0	1.0
Random design variable R_2	Normal	μ_{R_2}	1/3
Random design variable R_3	Normal	μ_{R_3}	1/3
Random design variable R_4	Normal	μ_{R_4}	1/3
Deterministic design variable θ_0	–	–	
Deterministic design variable ψ_0	–	–	

The standard deviations of R_2, R_3 and R_4 are determined by the tolerances of the dimensions with the 3-sigma rule. The tolerances of the lengths are ±1.0 mm; then the standard deviations are one third of the tolerances, resulting in $\sigma_{R_2} = \sigma_{R_3} = \sigma_{R_4} = 1/3$ mm. σ_{R_1} is higher because the uncertainty in R_1 comes not only from manufacturing imprecision but also from installation imprecision.

4.2 Mechanism analysis equations

We now derive equations for the quality characteristics and constraint functions. The loop-closure equations of the four-bar mechanism are given by

$$\begin{bmatrix} R_2 \cos(\theta + \theta_0) + R_3 \cos\delta - R_1 - R_4 \cos(\psi + \psi_0) \\ R_2 \sin(\theta + \theta_0) + R_3 \sin\delta - R_4 \sin(\psi + \psi_0) \end{bmatrix} = \mathbf{0}$$

There are two unknowns, ψ and δ, in the above equations. Solving the equations yields following equations

$$\psi = \arctan\left(\frac{E}{D}\right) + \arccos\left(\frac{A}{B}\right) - \psi_0$$

where $D = R_1 - R_2 \cos(\theta + \theta_0)$, $E = -R_2 \sin(\theta + \theta_0)$, $A = D^2 + E^2 + R_4^2 - R_3^2$, and $B = -2R_4\sqrt{D^2 + E^2}$.

$$\delta = \arctan\left[\frac{E + R_4 \sin(\theta + \theta_0)}{D + R_4 \cos(\theta + \theta_0)}\right]$$

The quality characteristics Y are then given by

$$Q_1 = \psi\Big|_{\theta=30^\circ}$$

and

$$Q_2 = \psi\big|_{\theta=150^\circ}$$

The derivatives of the quality characteristics are given by

$$\frac{\partial Q_i}{\partial Z_1} = \frac{\partial Q_i}{\partial R_2} = \frac{\cos(\theta+\theta_0-\delta)}{R_4\sin(\delta-\psi-\psi_0)}$$

$$\frac{\partial Q_i}{\partial Z_2} = \frac{\partial Q_i}{\partial R_3} = \frac{1}{R_4\sin(\delta-\psi-\psi_0)}$$

$$\frac{\partial Q_i}{\partial Z_3} = \frac{\partial Q_i}{\partial R_4} = -\frac{\cos(\delta-\psi-\psi_0)}{R_4\sin(\delta-\psi-\psi_0)}$$

$$\frac{\partial Q_i}{\partial Z_4} = \frac{\partial Q_i}{\partial R_1} = -\frac{\cos\delta}{R_4\sin(\delta-\psi-\psi_0)}$$

Using the above equations and the proposed methodology, the expected quality loss function can be calculated analytically without any numerical integrations or iterative processes.

We now discuss the constraint functions. Because link *AB* should be a crank, the following three constraint functions (Grashof's theorem) should be satisfied:

$$g_1 = R_2 + R_3 - (R_1 + R_4) \le 0$$

$$g_2 = R_2 + R_4 - (R_1 + R_3) \le 0$$

$$g_3 = R_2 + R_1 - (R_3 + R_4) \le 0$$

We can also derive constraint functions for the transmission requirement. The two constraint functions are given by

$$g_4 = R_3^2 + R_4^2 - (R_1 - R_2)^2 - 2R_3R_4\cos\lambda \le 0$$

$$g_5 = -[R_3^2 + R_4^2 - (R_1 + R_2)^2] - 2R_3R_4\cos\lambda \le 0$$

It is easy to derive the derivatives of the five constraint functions with respect to the random variables $\mathbf{Z} = (\mathbf{X}, \mathbf{P}) = (R_1, R_2, R_3, R_1)$. Then the means and standard deviations of the constraint functions are analytically available. Thus the reliability constraints in the robust design optimization model can be easily calculated.

4.3 Optimization models

To show the advantages of the proposed method, we compare its results with those from deterministic optimal synthesis and robust optimal synthesis without considering the dependency between the two quality characteristics.

The deterministic optimization model is given by

$$\begin{cases} \min\limits_{(\mathbf{d},\mathbf{X})} f = (\psi_1 - m_1)^2 + (\psi_2 - m_2)^2 \\ \text{subject to} \\ \quad \mathrm{g}_j \le 0 \ (j = 1, 2, \ldots, 5) \end{cases}$$

where no uncertainties are considered, and the mean values are used.

The robust design model without dependency consideration is formulated as

$$\begin{cases} \min\limits_{(\mathbf{d},\boldsymbol{\mu}_\mathbf{X})} E_L = E_1^+ + E_1^- + E_2^+ + E_2^- \\ \text{subject to} \\ \quad \mu_{g_j} + \beta_j \sigma_{g_j} \le 0 \ (j = 1, 2, \ldots, 5) \end{cases}$$

In the above model, no joint quality losses are included.

The proposed method considers the joint quality losses, and the optimization model is built as

$$\begin{cases} \min\limits_{(\mathbf{d},\boldsymbol{\mu}_\mathbf{X})} E_L = E_1^+ + E_1^- + E_2^+ + E_2^- + E_{12}^{++} + E_{12}^{--} + E_{12}^{+-} + E_{12}^{-+} \\ \text{subject to} \\ \quad \mu_{g_j} + \beta_j \sigma_{g_j} \le 0 \ (j = 1, 2, \ldots, 5) \end{cases}$$

4.4 Results and discussions

Sequential Quadratic Programming was used to solve the three optimization models. The starting point of the three problems was $\mathbf{d} = (0^\circ, 0^\circ)$ and $\boldsymbol{\mu}_\mathbf{X} = (500, 1200, 600)$ mm. The lower and upper bounds of $\mathbf{d}$ are $(0^\circ, 0^\circ)$ and $(360^\circ, 360^\circ)$, respectively. The lower and upper bounds of $\boldsymbol{\mu}_\mathbf{X}$ are $(200, 400, 400)$ mm and $(1000, 1500, 1500)$ mm, respectively. The optimal points from the three methods are given in Table 2.

Table 2 Optimal design points

Design variables	Deterministic optimization	Robust design without dependency consideration	Robust design with dependency consideration
μ_{R_2}	229.77 mm	349.38 mm	220.60 mm
μ_{R_3}	915.78 mm	751.36 mm	631.70 mm
μ_{R_4}	770.51 mm	1005.04 mm	1144.03 mm
θ_0	350.11°	0.0°	82.52°
ψ_0	0.21°	0.48°	0.82°

As shown in Table 3, the three methods produced the desired average motion as the means of the output angles are almost on their targets. The deterministic optimization has the largest expected quality loss, \$34.17, because of the largest standard deviations of the motion output. The robust design without dependency consideration produced an expected quality loss of \$14, which is a huge improvement with respect to the deterministic optimization. But it is not the best. With the incorporation of the dependency between the two output angles, the proposed bivariate robust design optimization further improved the robustness. It produced the lowest expected quality loss \$4.99. The two robust design methodologies generated very similar standard deviations of the two output angles, but their coefficients of correlation are quite different – one is 0.6564, and the other is 0.3920. The different coefficients of correlation are also a contributing factor for different quality losses. The coefficient of correlation was computed with the following equations

$$\rho = \frac{COV(Y_1, Y_2)}{\sigma_{Y_1}\sigma_{Y_2}} = \frac{\sum_{j=1}^{nz} a_{1j}a_{2j}\sigma_{Z_j}^2}{\sigma_{Y_1}\sigma_{Y_2}} \tag{63}$$

where a_{ij} $(i = 1, 2;\ j = 1, 2, \cdots, nz)$ are given in Eq. (7), and σ_{Y_i} $(i = 1, 2)$ are given in Eq. (10).

Table 3 Optimal points

Design variables	Deterministic optimization	Robust design without dependency consideration	Robust design with dependency consideration
μ_{ψ_1}	89.9999°	89.9974°	89.9946°
μ_{ψ_2}	109.9999°	109.9970°	109.9938°
σ_{ψ_1}	0.0757°	0.0452°	0.0308°
σ_{ψ_2}	0.0672°	0.0471°	0.0244°
ρ	0.8620	0.6564	0.3920
E_L	\$34.17	\$14.0	\$4.99
E_L (MCS)	\$34.08	\$13.97	\$4.99

The expected quality losses E_L in Table 3 were calculated by the proposed methodology. They were confirmed by Monte Carlo simulation (MCS) with a sample size of 10^6 as shown in the last row of Table 3. The MCS solutions indicate that the proposed bivariate robust analysis is accurate.

The contours of the joint PDF of the two output angles are plotted in Fig. 2. The contours are at a probability level of 0.5. The figure clearly shows that the proposed method is much more robust because its PDF contour is much smaller than those of the other two methods.

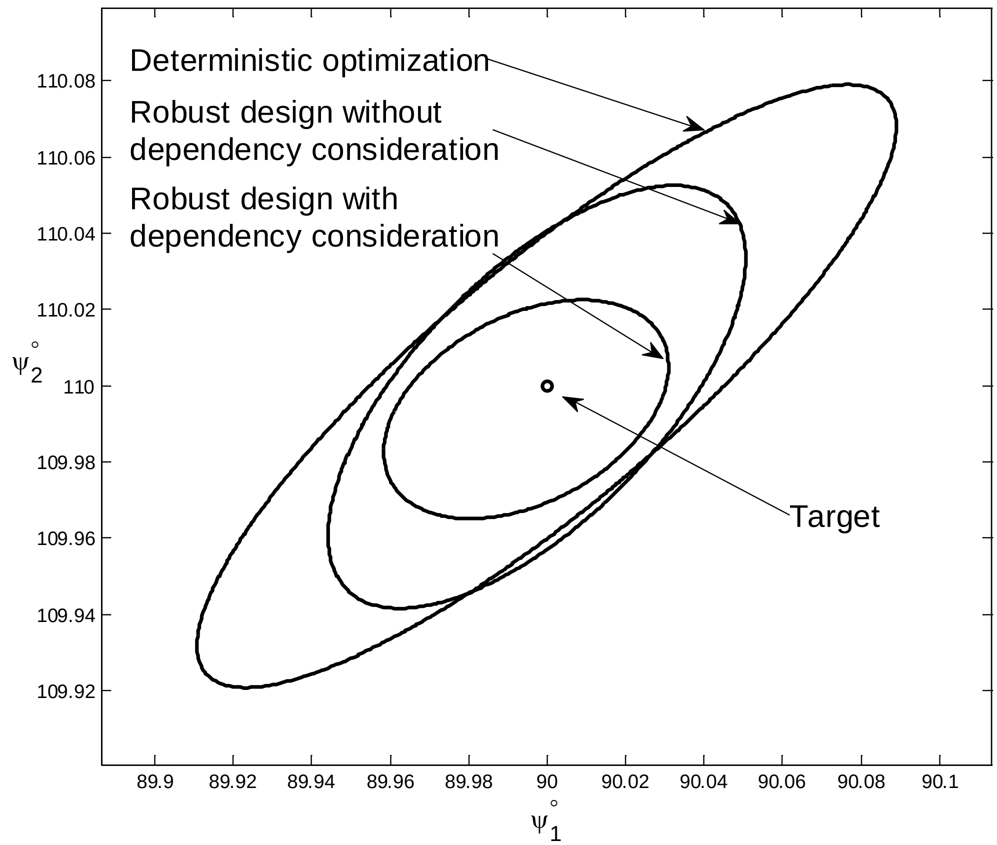

Figure 2 PDF contours

5 Conclusions

Quality characteristics are generally dependent because they may share common random input variables. Considering their dependency may produce better design results with lower quality losses. This can be achieved by using an expected total quality loss function, where the dependency between quality characteristics is automatically accounted for with the expectation operation.

If the input random variables are normally distributed with small standard deviations, for bivariate quality characteristics, their expected quality loss function can be derived as shown in this work. The analytical derivations are based on the First Order Second Moment method. It requires the function value of the quality characteristics and its derivatives at the means of random input variables. No iterative processes or numerical integrations are needed. In this work, only the nominal-the-best type of quality characteristics is addressed. The results can be easily extended to the-smaller-the-better type of quality characteristics. We can simply set the target of a quality characteristic to zero for this type. The extension to the-large-the-better type is possible, but it needs a further investigation.

The proposed bivariate robust design method can be directly applied to the existing multivariate quality loss function (Wu and Chyu 2004b) with more than two quality characteristics. The reason is that a joint quality loss considered is only up to the second order. For example, if there are three quality characteristics Q_1, Q_2, and Q_3, the joint quality losses are between Q_1 and Q_2; Q_1 and Q_3; and Q_2 and Q_3. If all the random variables are normally distributed with small standard deviations, the proposed method will also be accurate for the expected multivariate quality loss function.

When uncertainties are large, the function of a quality characteristic will no longer be close to linear in the vicinity of the means of random input variables. Then analytical algorithms will not be available, and the proposed method will produce large errors. For this situation, efficient numerical integrations should be employed to evaluate the expectation of bivariate or multivariate quality loss functions. The most popular methods include the point estimation methods or the dimension reduction methods (Huang and Du 2005; Lee et al. 2008; Li and Chen 2006; Liping et al. 2007; Rahman 2008; Rahman 2009; Rahman and Xu 2004; Wang et al. 2008).

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