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Towards Time-Dependent Robustness Metrics

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Abstract

Quality characteristics are often treated as constants in robust design while many of them actually vary over time. It is desirable to define new robustness metrics for time-dependent quality characteristics. This work shows that using the static robustness metrics for time-dependent quality characteristics may lead to erroneous design results. We then propose criteria of new robustness metrics for time-dependent quality characteristics. Instead of using an expected point quality loss over the time period of interest, we use the expectation of the maximal quality loss over the time period to quantify the robustness for time-dependent quality characteristics. Through a four-bar function generator mechanism synthesis, we demonstrate that the new robustness metrics can capture the full information of robustness of a time-dependent quality characteristic over a time interval. The new robustness metrics can then be used as objective functions for time-dependent robust design optimization.

1. Introduction

A quality characteristic (QC) refers to any performance, response, or behavior that significantly affects product quality and customer satisfaction. The value of a QC fluctuates due to uncertainties, and the variation may lead to a quality loss. During robust design, the variation in QCs is minimized. Such minimization is achieved by adjusting the nominal values of design variables without eliminating the sources of uncertainty. Robust design can therefore make QCs insensitive to raw material variation, immune to manufacturing imprecision and inert to the variation in the operating environment. As a result, robust design allows for the use of low grade materials, reduces labor and material cost, and improves reliability and reduces operating cost [1]. Because of its advantages, robust design has been applied across a wide range of industry sectors, such as automotive, aerospace, defense, and telecommunication industries.

The traditional robust design is based on statistical experiments [1]. With the growth of computational power, robust design has been increasingly performed through model-based optimization [2-6]. Instead of experimentally estimating the QCs, designers can computationally evaluate QCs with analysis models, such as finite element simulation. This kind of robust design is referred to as analytical robust design optimization [6]. The methodologies of analytical robust design optimization are reviewed in [4, 6-10]

The objective of analytical robust optimization is to maximize the robustness of a product, process, or component. Various metrics that measure robustness have been proposed in literature. The most common robustness metric is the Taguchi's quality loss function (QLF) [1]. This metric is based on the recognition that it is vital to produce QCs on their targets and that the variation around the mean QCs causes poor quality. The QLF is defined by a quadratic function of the deviation of a QC from its target. The QLF increases slowly when the deviation is small; it increases rapidly when the deviation is large. Given the variation in the QC, the expected QLF is used as the robustness metric. It is equal to the square of the mean deviation (the deviation of the mean QC from its target) plus the variance of the QC. Minimizing the expected QLF reduces both of the terms. As a result, we can bring the mean QC to its target and in the meantime minimize the variation in the QC.

When multiple QCs are involved, the total quality loss is just the summation of individual QLFs. This is possible because the quality losses are expressed in a monetary amount and are additive. If a designer is concerned with the combined quality losses when more than two QCs are away from their targets, a multivariate quadratic loss function [11] could be used. This kind of quality loss function accounts for the joint losses of all pairs of QCs. With the QLF-type robustness metric, the robust design optimization involves a single objective function, which is the expected QLF that is to be minimized.

If designers are interested in a trade-off between the standard deviation and mean deviation of a QC, they may formulate a multi-objective optimization problem [12-14]. The two objectives are the standard deviation and mean deviation of the QC. If the designers would like to give a higher weight to either the standard deviation or mean deviation of the QC, they may use the weighted sum of the two, which also results in a single objective function. Other robustness metrics have also been used in robust design optimization, such as the signal-to-noise ratio [1], percentile difference [15], and worst-case QCs [16].

Most of the above robustness metrics are defined for time-invariant QCs that do not change over time. Some of the metrics could be used for dynamics problems, but they are only applicable for situations where the targets of QCs changes with signals [17-18], instead of changing with time. The dynamic robustness metrics are not directly related to time-dependent QCs.

In many engineering applications, QCs vary over time during the product service life. There are two reasons for the time-dependent QCs. (1) A QC is a function of time-dependent variables, such as the wind load and road conditions. (2) The function of a QC itself is time dependent; in other words, the function contains a time factor. Two examples of time-dependent QCs are provided below.

In a kinematic mechanism synthesis problem, the QC of a mechanism is its motion error, which is the difference between the actual motion and the desired motion. For instance, the position of the slider of a slider-crank mechanism should follow a desired function of the angle of the crank (or time). The actual slider position may deviate from its desired function. The deviation is the motion error. We prefer the motion error to be zero. Because of uncertainties in mechanism dimensions, joints, loading, and installation, the motion error fluctuates at any given instant of time; it also varies over time. The motion error at time instant t_1 may be different from that at time instant t_2. In many cases, the motion error changes dramatically with time, and the QC is therefore strongly time dependent. In this example, the QC, or the motion error, is a function of time, which appears explicitly in the function. For mechanisms with rotational motion, the QC associated with their motion errors is periodic or cyclic over time because the motion repeats cycle by cycle.

A wind turbine is another example where the wind speed changes over time significantly. Observed for a long period of time, for example, several years, the change in the wind speed may be periodic. The strengths of the materials of the turbine components also deteriorate with time. This change indicates a trend of decreasing strengths. The wind speed and strengths are actually stochastic processes, which vary over time randomly. As a result, the QCs of the wind turbine are time dependent. Examples of the time-dependent QCs include the safety margin, servicing life, reliability, and availability.

Time-dependent QCs are commonly encountered in engineering systems such as vehicles, aircraft, and large constructions. It is desirable to investigate new robustness metrics that can truly measure the robustness of designs involving time-dependent QCs.

In this work, we are interested in modeling the robustness of time-dependent robust design problems. The uniqueness of this work consists of four elements. The first is the establishment of the criteria for the time-dependent robustness metrics. The second includes the two proposed time-dependent robustness metrics that describe the true robustness over a time interval. The third is the examination on why the traditional robustness metrics may not work for time-dependent problems and why the time-dependent metrics work. The fourth is a Monte Carlo simulation (MCS) approach for evaluating the time-dependent metrics.

We review the traditional robustness modeling methods in Section 2 and then define time-dependent robustness metrics in Section 3. General formulations are derived in Section 4. Section 5 is dedicated to the development of the MCS approach for evaluating the time-dependent robustness metrics. In Section 6, we use a four-bar mechanism synthesis problem to demonstrate the utility of time-dependent robustness metrics. Conclusions are given in Section 7.

2. Review of Existing Robustness Models

To ensure robustness, for a particular quality loss (QC) Y, we prefer its mean μ_Y to be optimal and its standard deviation σ_Y to be minimal. This is the reason why both μ_Y

and σ_Y appear in most of robustness metrics. Next we review two commonly used robustness metrics.

2.1 Time-invariant problems

There are three types of QCs: Nominal-the-best, smaller-the-better, and larger-the-better. Several examples of the three types are as follows:

Nominal-the-best: The motion error of a mechanism should be zero or close to zero; dimensions of mechanical parts should be at their nominal values. This type of QCs may be time-dependent; for example, the aforementioned motion error is time dependent.

Smaller-the-better: The costs, stress concentration, and wear of a mechanical component should be small. All of these QCs are generally time dependent.

Larger-the-better: The expected life, yield, efficiency of a product should be large. These QCs can also be time dependent

The most commonly used robustness metric is the quality loss function (QLF) [1]. Next we examine the normal-the-best type QC. Let the QC be defined by $Y = g(\mathbf{X})$ with input variables $\mathbf{X} = (X_1, \ldots, X_n)^T$. The QLF is given by

$$L = A(Y - m)^2 \tag{1}$$

where m is the target, and A is a constant, which is determined by a monetary loss when Y is at its lower specification limit (LSL) or upper specification limit (USL). As shown in Fig. 1, when Y is on its target m, the quality loss is zero, and the farther is Y from m, the higher is the quality loss.

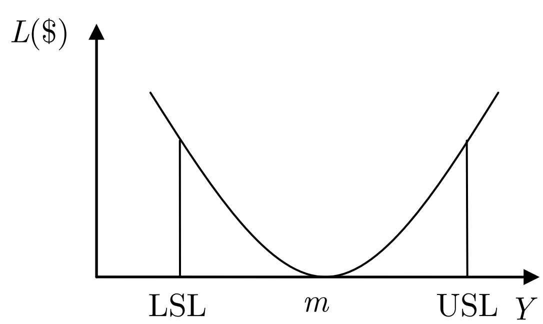

Fig. 1 Quality loss function

During robust design, the expected QLF is minimized. It is computed by [1]

$$E_L = A[(\mu_Y - m)^2 + \sigma_Y^2] \tag{2}$$

Minimizing E_L brings the mean QC to its target m and in the meantime minimizes the variation of the QC, which is measured by σ_Y. The expected QLF serves as a robustness metric in the sense of quality loss.

The other commonly used robustness measure is the weighted sum of the mean deviation $| \mu_Y - m |$ and the standard deviation σ_Y. The weighted sum robustness function is defined by

$$R_w = w_1 | \mu_Y - m | + w_2 \sigma_Y \tag{3}$$

where the weights satisfy $w_1 + w_2 = 1$. Minimizing R_w can also bring the mean QC to its target and minimize the standard deviation of the QC.

Many QCs reported in literature belong to the smaller-the-better type. If we consider the target being zero, the weighted sum robustness metric becomes [6, 19]

$$R_w = w_1 \mu_Y + w_2 \sigma_Y \tag{4}$$

2.2 Time-dependent problems

As discussed above, many engineering problems are time dependent. The general function of a QC is depicted with a solid line in Fig. 2 and is represented by

$$Y(t) = g(\mathbf{X}(t), t) \tag{5}$$

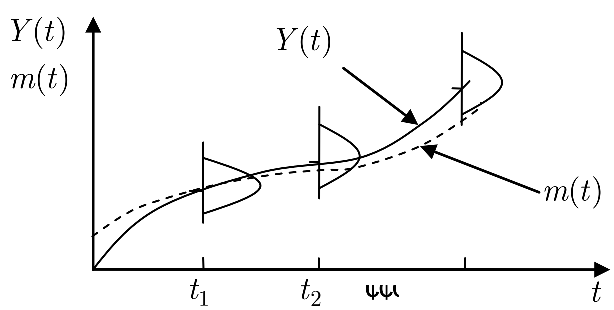

Fig. 2 Time-dependent QC

Eq. (5) indicates that the QC is a function of stochastic processes $\mathbf{Y}(t)$, which are random variables varying over time. The QC may also be an explicit function of time t. As shown in Fig. 2, the target function $m(t)$, represented by the dotted curve, may also vary over time.

When Y is time dependent, the QLF at an instant of time becomes

$$L(t) = A(t)[Y(t) - m(t)]^2 \tag{6}$$

This QLF gives the instantaneous quality loss at a specific instant of time. We call it the *point quality loss function* (P-QLF). The dependence of the P-QLF on time is shown in Fig. 3. The figure indicates that $L(t)$ is a quadratic function at an instant of time with respect to Y, and the quadratic functions of $L(t)$ change over t.

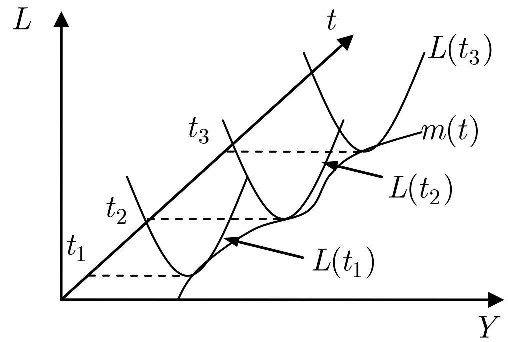

Fig. 3 P-QLF in terms of Y and t

To show the time dependence of $L(t)$ more clearly, we can image drawing a sample curve (trajectory) from the stochastic process $Y(t)$ and let the sample curve be $y(t)$, which is a deterministic function of t. Then the sample curve (realization) of the point QLF becomes a deterministic function of time; namely, $L(t) = A(t)[y(t) - m(t)]^2$. The realization is shown in Fig. 4.

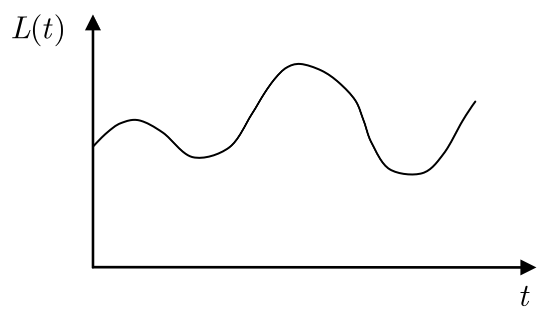

Fig. 4 P-QLF in terms of t

The expected P-QLF can be obtained from Eq. (2) by adding the time factor. It is given by

$$E_L(t) = A(t)\left\{[\mu_Y(t) - m(t)]^2 + \sigma_Y^2(t)\right\} \qquad (7)$$

The expected P-QLF $E_L(t)$ tells us the average quality loss at a particular time instant. Even though from $E_L(t)$ we know how the instantaneous expected quality losses vary over time, we cannot have a complete picture about the expected quality loss over the time interval $[t_0, t_f]$. Specifically, P-QLFs at t_1, t_2, t_3 … are dependent because they share the same input random variables or stochastic processes. As calculating the expected P-QLF $E_L(t)$ does not requiring knowing the dependency, having only the expected P-QLFs, we do not know the dependency. In other words, $E_L(t)$ tells us only the expected quality loss at t regardless if there were strong or weak correlations to the quality losses prior to t. As will be seen next, such correlations are an important factor governing the quality loss over the time interval. As a result, it is not desirable to simply extend the P-QLF to time-dependent problems.

There have been few studies on the time-dependent robust design; among those, the robustness is modeled by the sum of the expected P-QLFs or the weighted sum robustness function at discretized points over the time interval $[t_0, t_f]$ [20-21]. For example, the sum of expected P-QLFs is in the form of

$$\sum_{i=1}^{p} E_L(t_i) = \sum_{i=1}^{p} A(t_i)\left\{[\mu_Y(t_i) - m(t_i)]^2 + \sigma_Y^2(t_i)\right\}, t_0 = t_1 < t_2 < \ldots < t_f \quad (8)$$

Minimizing $\sum_{i=1}^{p} E_L(t_i)$ is numerically equivalent to minimizing the average expected P-QLFs $\bar{E}_L(t_0, t_f)$ if sufficient discretization points are taken. The average expected P-QLFs $\bar{E}_L(t_0, t_f)$ is given by

$$\bar{E}_L(t_0, t_f) = \int_{t_0}^{t_f} A(\tau)\left\{[(\mu_Y(\tau) - m(\tau)]^2 + \sigma_Y^2(\tau)\right\} d\tau \quad (9)$$

over $[t_0, t_f]$.

Because $\bar{E}_L(t_0, t_f)$ cannot account for the auto-dependency of the quality loss over time, it shares the same drawbacks as the expected P-QLF. We therefore need new robustness metrics for time-dependent robust design problems.

3. New Robustness Metrics for Time-Dependent Problems

To truly capture the robustness over the time period $[t_0, t_f]$ where the product is supposed to function, we need to define new robustness metrics. As there are multiple static robustness metrics, there may be multiple time-dependent robustness metrics. To provide a guideline to defining new robustness metrics for time-dependent QCs, we propose the following criteria:

(1) A metric must represent the maximal quality loss over $[t_0, t_f]$ if the robustness is defined in term of a quality loss. This feature comes from the fact that the quality loss is irreversible – once a quality loss occurred; there is no way to go back. For example, if the quality loss $L(t_2)$ at the current instant t_2 is greater than the quality loss $L(t_1)$ at the previous instant t_1 ($t_1 < t_2$), then over time interval

$[t_0, t_2]$, which covers t_1, we should consider the quality loss being $L(t_2)$. It is therefore natural to use the maximal quality loss over a time interval.

(2) The metric should capture the auto-dependency of quality losses over $[t_0, t_f]$. This feature is important because two QCs with the same point quality loss at each instant of time may have totally different quality losses over the entire time interval. The reason is that a general QC is a stochastic process. To fully describe the stochastic process, we should know not only its distributions at all instances of time but also its autocovariances at any pairs of instances of time.

(3) The robustness metric, if expressed in the form of a quality loss, should be a non-decreasing function with respect to time. The reason is that the longer is the product put into service, the worse might be its robustness. The other reason has been mentioned in (1), which is that the quality loss is irreversible.

(4) Minimizing (or maximizing) a robustness metric will lead to optimizing the mean QCs and minimizing the variations of the QCs over $[t_0, t_f]$. This comes from the purpose of robust optimization.

Based on the above criteria, we propose to use the extreme value or the worst-case value of the point quality loss over $[t_0, t_f]$ to form robustness metrics for time-dependent QCs. We call the worst-case quality loss an *interval* quality loss because it is defined over a time interval. Then the interval quality loss function (I-QLF) over $[t_0, t_f]$ is given by

$$L(t_0, t_f) = \max_{\tau}\left\{A(\tau)[Y(\tau) - m(\tau)]^2\right\}, t_0 \le \tau \le t_f \tag{10}$$

It is the maximum quality loss with respect to time over $[t_0, t_f]$. Fig. 5 shows the realizations (sample curves) of both the proposed I-QLF and the traditional P-QLF. If we look at a specific instant t_1 and interval $[t_0, t_1]$, the P-QLF is L_1 while the I-QLF over $[t_0, t_1]$ is L_2. It is obvious that $L_2 > L_1$ because L_2 is the maximum quality loss over $[t_0, t_1]$.

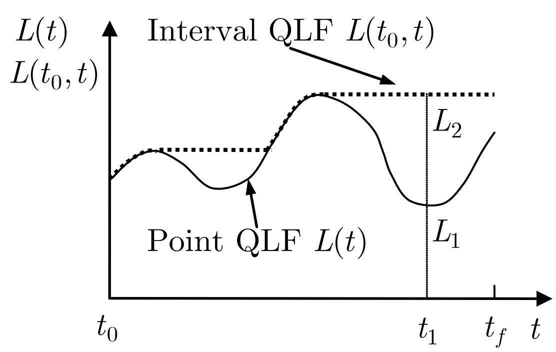

Fig. 5 P-QLF and I-QLF

The time-dependent robustness metric is then the expectation of the I-QLF $Z = L(t_0, t_f)$, given by

$$E_L(t_0, t_f) = \int_0^{\infty} z f_Z(z) dz \tag{11}$$

where $f_Z(z)$ is the probability density function (PDF) of Z. The expectation should be minimized during robust design.

For the weighted sum robustness function in Eq. (3), we can also similarly define its interval counterpart as

$$R_w(t_0, t_f) = w_1 \mu_G + w_2 \sigma_G \tag{12}$$

where

$$G = \max_{\tau} \mid Y(\tau) - m(\tau) \mid, \;\; t_0 \leq \tau \leq t_f \tag{13}$$

which is the maximal absolute deviation of the QC from its target over $[t_0, t_f]$.

We now use the I-QLF as an example to explain why the interval robustness metrics work and why the point robustness metrics do not, with the following reasons:

(1) The I-QLF $L(t_0, t_f)$ is the largest quality loss over the time interval $[t_0, t_f]$; it is the true quality loss over the time interval.

(2) The expectation, $E_L(t_0, t_f)$, of the I-QLF, can accommodate the auto-dependency of the quality loss over the time interval. This is illustrated in Fig. 6, where two QCs have the same expected point quality losses (marked as P-QLF) at any instant of time over $[t_0, t_f]$. But their expected interval quality losses (marked as I-QLF) are quite different because they have different coefficients of autocorrelation $\rho_1(t_1, t_2)$ and $\rho_2(t_1, t_2)$, where $t_0 \leq t_1 < t_2 \leq t_f$.

(3) The expected I-QLF is non-decreasing over time, but the expected P-QLF is not, as indicated in Fig. 6.

(4) Because the expected I-QLF truly represents the robustness over $[t_0, t_f]$, minimizing this expectation will result in a true robust design over $[t_0, t_f]$.

Fig. 6 is only for a demonstration purpose, for which the two QCs are assumed to be two Gaussian stochastic processes with the same mean function $\mu(t) = 0.99\sin(1.5t - 0.1) - \sin t$ and the same standard deviation function $\sigma(t) = 0.1\mu(t)$. Because both QCs have the same mean and standard deviation functions at each instance of time, they have the same distribution (a normal distribution). Their expected P-QLFs are therefore also the same. Their functions of coefficient of autocorrelation are different. $\rho_1(t_1, t_2) = \exp[-(t_2 - t_1)^2 / \lambda_1^2]$ ($\lambda_1 = 1/3000$) and $\rho_2(t_1, t_2) = \exp[-(t_2 - t_1)^2 / \lambda_2^2]$ ($\lambda_2 = 1/2$). The different autocorrelation structures result in different expected interval quality losses.

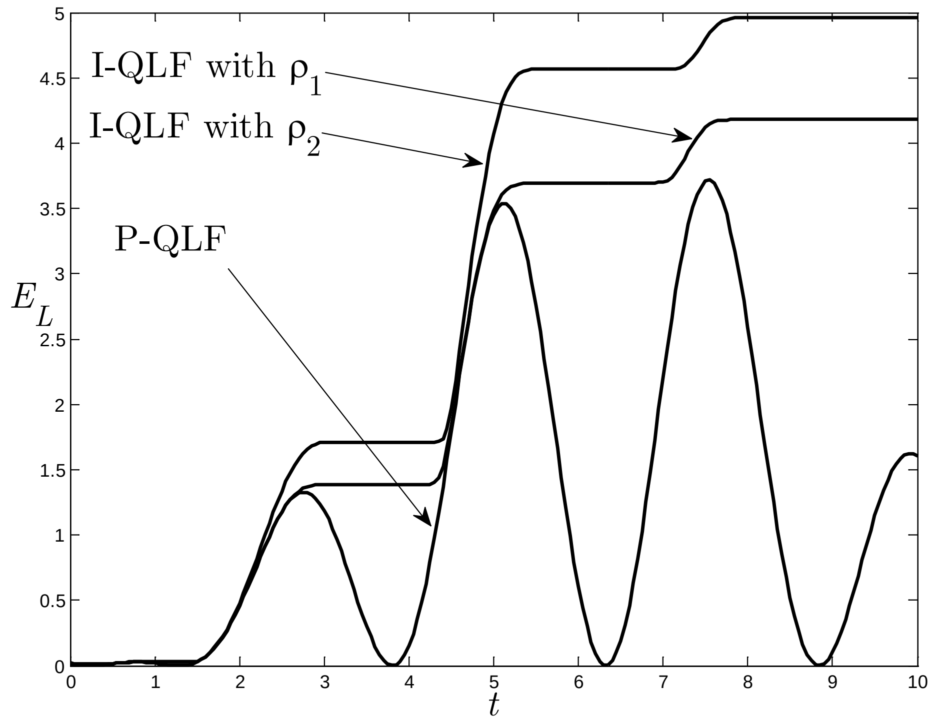

Fig.6 Two QCs with same $E_L(t)$ but different $E_L(t_0, t)$

4. Expectation of I-QLF

Because we will minimize the expected I-QLF during the robust optimization, it is necessary to evaluate the expected I-QLF. This is the task of time-dependent robustness analysis. Define

$$H(t) = \sqrt{A(t)}\left[Y(t) - m(t)\right] \tag{14}$$

Also define $Z(t_0, t_f)$ to be the maximal P-QLF over $[t_0, t_f]$. We then have

$$\begin{aligned} Z(t_0, t_f) &= \max_{\tau}\{A(\tau)[Y(\mathbf{X}(\tau), \tau) - m(\tau)]^2\} \\ &= \max_{\tau} H^2(\tau)(t_0 \le \tau \le t_f) \end{aligned} \tag{15}$$

The expectation of the I-QLF can be calculated by Eq. (11). To calculate the expectation, we need to know the PDF or cumulative distribution function (CDF) of Z. This is a challenging task because the extreme value of the nonlinear function $H^2(\tau) = A(\tau)[Y(\mathbf{X}(\tau), \tau) - m(\tau)]^2$ is involved. Next we develop a way to obtain the expectation from only $H(\tau)$ instead of $H^2(\tau)$.

The CDF of Z is given by

$$F_Z(z) = \Pr\left\{\max_\tau H^2(\tau) < z,\ t_0 \le \tau \le t_f\right\} \tag{16}$$

where $\Pr\{\cdot\}$ denotes a probability.

The CDF can be derived as

$$F_Z(z) = \Pr\{\max_\tau \left|H(\mathbf{X}(\tau),\tau)\right| < \sqrt{z},\ \ t_0 \le \tau \le t_f\} \tag{17}$$

Let $W = \max_\tau \left|H(\mathbf{X}(\tau),\tau)\right|$ and its PDF be $f_W(w)$. Then from Eq. (17) we have

$$F_Z(z) = F_W(\sqrt{z}) = F_W(w) \tag{18}$$

where $w = \sqrt{z}$.

If the CDF $F_W(w)$ is available, the PDF of W is given by

$$f_W(w) = \frac{dF_W(w)}{dw} \tag{19}$$

And then the expectation of Z is computed by

$$E_z(t_0, t_f) = \int_0^\infty z f_z(z) dz = \int_0^\infty w^2 f_W(w) dw \tag{20}$$

The key is then the evaluation of the CDF $F_W(w)$. Because $W = \max_\tau \left|H(\mathbf{X}(\tau),\tau)\right|$, we can obtain $F_W(w)$ from the CDF of the function $H(\cdot)$.

$$F_W(w) = \Pr\{W < w\} = \Pr\{-w < \max_\tau \left|H(\mathbf{X}(\tau),\tau)\right| < w\} \tag{21}$$

or

$$F_W(w) = 1 - \Pr\left\{\left[\max_\tau H(\mathbf{X}(\tau),\tau) > w\right] \cap \left[\max_\tau H(\mathbf{X}(\tau),\tau) < -w\right]\right\} \tag{22}$$

For a general nonlinear function $H(\cdot)$ with general stochastic processes $\mathbf{X}(t)$, one available numerical method for computing the above probability is the Poisson upcrossing method [22-23]. The method has been widely used for time-dependent structural reliability analysis, but its accuracy may be poor. New numerical methods for evaluating Eq. (22) need to be developed. Our primary purpose herein is the robustness modeling for time-dependent design problems, and developing new algorithms is beyond the scope of this work. Nevertheless, the above derivations will serve as a guideline for developing such new numerical methods. To easily validate the proposed time-dependent robustness metrics, we employ the Monte Carlo simulation (MCS). Next we develop a MCS procedure for the time-dependent robustness analysis and then use it for the methodology validation.

5. Monte Carlo Simulation for Time-Dependent Robustness Metrics

The primary reason we use MCS is to easily evaluate the new robustness metrics. Other reasons are as follows:

(1) Once new approximation methods for time-dependent robustness analysis are developed, MCS results can serve as a benchmark for their accuracy evaluation.

(2) Unlike time-dependent reliability analysis associated with rare events, the robustness analysis here requires significantly less number of samples because we are interested in only the expectation estimation.

(3) Using MCS is feasible in some applications if the evaluation of QCs is quick.

(4) For many applications with general non-stationary stochastic processes, MCS is the only approach we may use.

Fig. 7 shows the MCS procedure where the key is to efficiently draw samples for stochastic processes $\mathbf{X}(t)$. This task is much more complicated than generating random variables. For example, sampling on a Gaussian process may involve solving a large-scale eigenvalue problem because a large number of discretization time points are needed to ensure the stochastic process being accurately represented [24-25]. How to efficiently draw samples from a general stochastic process needs further research. In this work, we consider only Gaussian processes.

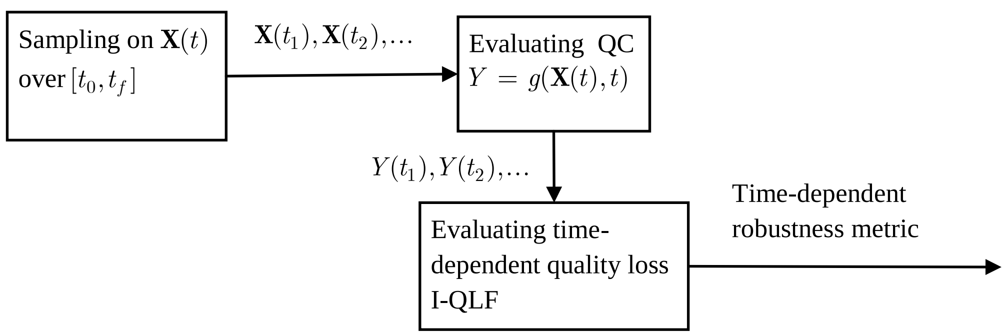

Fig. 7 MCS approach

At first, we discretize the time interval $[t_0, t_f]$ into equal sub-intervals $t_0 = t_1 < t_2 < \ldots < t_{p-1} < t_p = t_f$. Then we follow the three steps described below.

(1) Obtain samples of $\mathbf{X}$. The vector $\mathbf{X}$ may include both random variables and stochastic processes. Let $\mathbf{X} = (\mathbf{X}_S(t), \mathbf{X}_R)$ where $\mathbf{X}_S(t)$ are stochastic processes and $\mathbf{X}_R$ are time-invariant random variables. The methodologies in [24-25] can be used to generate N samples for $\mathbf{X}_S(t)$ over $[t_0, t_f]$. Suppose the generated samples are $\mathbf{X}_{Si}(t_1), \mathbf{X}_{Si}(t_2), \ldots, \mathbf{X}_{Si}(t_f)$ $(i = 1, 2, \ldots, N)$. We also generate samples for $\mathbf{X}_R$. The samples are denoted by X_{Ri} $(i = 1, 2, \ldots, N)$.

(2) Evaluate the QC at $t_0 = t_1 < t_2 < \ldots < t_{p-1} < t_p = f_f$. The samples of the QC are $Y_i(t_1) = Y(\mathbf{X}_{Si}(t_1), \mathbf{X}_{Ri})$, $Y_i(t_2) = Y(\mathbf{X}_{Si}(t_2), \mathbf{X}_{Ri})$, …, $Y_i(t_p) = Y(\mathbf{X}_{Si}(t_p), \mathbf{X}_{Ri})$ ($i = 1, 2, \ldots, N$). Then calculate the I-QLFs with the following equation:

$$L_i(t_0, t_f) = \max_{j=0}^{p}\left\{A(t_j)[Y(t_j) - m(t_j)]^2\right\} \tag{23}$$

where $i = 1, 2, \ldots, N$.

(3) Evaluate the time-dependent robustness metric or the expected I-QLF using the following equation.

$$E_L(t_0, t_f) \approx \frac{1}{N}\sum_{i=1}^{N} L_i(t_0, t_f) \tag{24}$$

6. Numerical Example

We now validate the proposed robustness metrics through the robust design of a four-bar linkage. The linkage serves as a function generator. As shown in Fig. 8. the crank *AB* is the input member with the input angle $\theta + \theta_0$, where θ_0 is the initial input angle, and the rocker *CD* is the output member with the output angle $\psi + \psi_0$, where ψ_0 is the initial output angle. R_1, R_2, R_3, and R_4 are the random lengths of the mechanism. The random vector is therefore $\mathbf{X} = (R_1, R_2, R_3, R_4)^T$. The distributions of the dimension variables are given in Table 1.

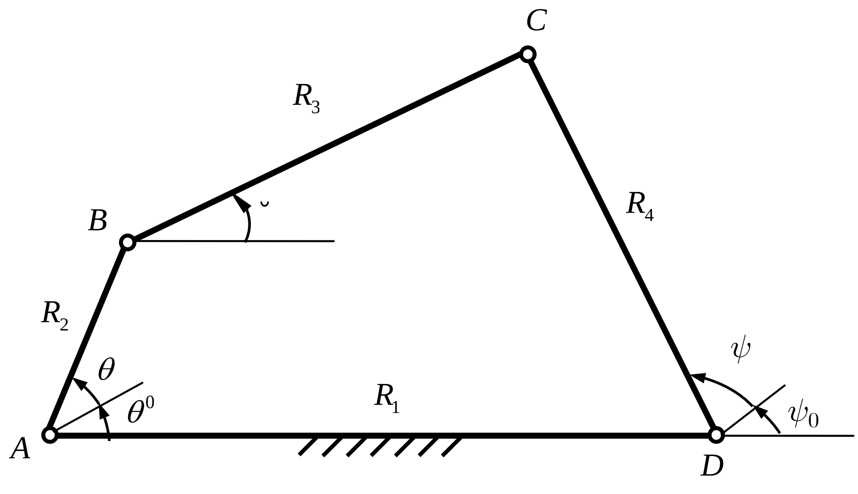

Fig. 8 Four-bar function generator mechanism

Table 1 Random dimensions for the sine function generator

Variable	Mean (mm)	Standard deviation (mm)	Distribution
R_1	$\mu_1 = 100$	$\sigma_1 = 0.2$	Normal
R_2	μ_2	$\sigma_2 = 0.2$	Normal
R_3	μ_3	$\sigma_3 = 0.2$	Normal
R_4	μ_4	$\sigma_4 = 0.2$	Normal

The function generator mechanism is supposed to realize a function defined by $y = \sin x$ with $x \in [x_0, x_f] = [0, 90^\circ]$. The range of the input angle θ is $[\theta_0, \theta_f] = [\theta_0, \theta_0 + 120^\circ]$, and the range of the output angle $\psi + \psi_0$ is $[\psi_0, \psi_f] = [\psi_0, \psi_0 + 50^\circ]$. According to [26], the required functional relationship between the output ψ_d and input θ is $\psi_d(\theta) = \psi_0 + 50^\circ \sin\left[\frac{3}{4}(\theta - \theta_0)\right]$ for function $y = \sin x$. The design variables of the mechanism synthesis are μ_1, μ_2, μ_3, θ_0, and ψ_0.

The motion output Ã varies over the interval of the motion input µ 2 [0±; 120±], over which the mechanism is supposed to perform its intended function. In this problem, the time factor is t = µ. The quality characteristic (QC) Y is the motion error of the mechanism. The motion error is measured by the deviation of the actual motion output Ã from its desired motion output. Because of the randomness in the lengths of the links, Y varies with respect to θ. Y is therefore a stochastic process.

To perform robust synthesis, we are interested in knowing the robustness of the mechanism during the time interval $[0^\circ, 120^\circ]$ over which the desired function is defined. We first derive the QC and then define its quality loss function.

The loop-closure equations of the four-bar mechanism are given by

$$\begin{bmatrix} R_2\cos(\theta + \theta_0) + R_3\cos\gamma - R_1 - R_4\cos(\psi + \psi_0) \\ R_2\sin(\theta + \theta_0) + R_3\sin\gamma - R_4\sin(\psi + \psi_0) \end{bmatrix} = \mathbf{0} \tag{25}$$

Solving the above equations yields the actual motion output below.

$$\psi(\mathbf{X}, \theta) + \psi_0 = 2\arctan\left(\frac{-E \pm \sqrt{E^2 + D^2 - F^2}}{F - D}\right) \tag{26}$$

where $D = 2R_4\left[R_1 - R_2\cos(\theta + \theta_0)\right]$, $E = -2R_2R_4\sin(\theta + \theta_0)$ and $F = R_1^2 + R_2^2 + R_4^2 - R_3^2 - 2R_1R_2\cos(\theta + \theta_0)$.

The QC Y is then given by

$$Y(\mathbf{X},\theta) = \psi(\mathbf{X},\theta) + \psi_0 - \psi_d(\theta)$$
$$= 2\arctan\left(\frac{-E \pm \sqrt{E^2 + D^2 - F^2}}{F - D}\right) - 50^\circ \sin\left[\tfrac{3}{4}(\theta - \theta_0)\right] \tag{27}$$

A close-to-zero motion error is desired, and the target of the QC is then $m(\theta) = 0$. Therefore, the deviation of the QC from its target is

$$g(\mathbf{X},\theta) = Y(\mathbf{X},\theta) - m(\theta)$$
$$= 2\arctan\left(\frac{-E \pm \sqrt{E^2 + D^2 - F^2}}{F - D}\right) - 50^\circ \sin\left[\tfrac{3}{4}(\theta - \theta_0)\right] \tag{28}$$

The P-QLF is given by

$$L(\theta) = A[g(\mathbf{X},\theta)]^2,\ 0^\circ \le \theta \le 120^\circ \tag{29}$$

and the I-QLF is given by

$$L(0,120^\circ) = A\max_\theta[g(\mathbf{X},\theta)]^2,\ 0^\circ \le \theta \le 120^\circ \tag{30}$$

where A is assumed to be \$1000/degree^2.

The I-QLF based robust design is modeled as

$$\begin{cases} \min\limits_{(\mu_1,\mu_2,\mu_3,\theta_0,\psi_0)} L(0,120^\circ) = E\left\{A\max\limits_\theta[g(\mathbf{X},\theta)]^2\right\},\ 0^\circ \le \theta \le 120^\circ \\ \text{subject to} \\ \mu_2 + \mu_1 - \mu_3 - \mu_4 \le 0 \\ \mu_2 + \mu_3 - \mu_1 - \mu_4 \le 0 \\ \mu_2 + \mu_4 - \mu_1 - \mu_3 \le 0 \\ 30^\circ - \cos^{-1}\left\{\left[\mu_3^2 + \mu_4^2 - (\mu_1 - \mu_2)^2\right] / \ 2\mu_3\mu_4\right\} \le 0 \\ \cos^{-1}\left\{\left[\mu_3^2 + \mu_4^2 - (\mu_1 + \mu_2)^2\right] / \ 2\mu_3\mu_4\right\} - 170^\circ \le 0 \\ 20 \text{ mm} \le \mu_2 \le 450 \text{ mm}, 20 \text{ mm} \le \mu_3, \mu_4 \le 450 \text{ mm}, \\ 0^\circ \le \theta_0, \psi_0 \le 360^\circ \end{cases} \tag{31}$$

The I-QLF is in the objective function and is to be minimized. The constraints are for the existence of a crank and the requirement of the transmission angle, whose nominal value should be greater than or equal to 30°.

To compare the new approach with the traditional one, we will give the P-QLF based robust design model below.

$$\begin{cases} \min\limits_{(\mu_1,\mu_2,\mu_3,\theta_0,\psi_0)} \quad L(0,120^\circ) = \dfrac{1}{k}\sum\limits_{i=1}^{k} E\left\{[g(\mathbf{X},\theta_i)]^2\right\} \ (\theta_1 = 0^\circ, \theta_k = 120^\circ) \\ \text{subject to} \\ \mu_2 + \mu_1 - \mu_3 - \mu_4 \le 0 \\ \mu_2 + \mu_3 - \mu_1 - \mu_4 \le 0 \\ \mu_2 + \mu_4 - \mu_1 - \mu_3 \le 0 \\ 30^\circ - \cos^{-1}\left\{\left[\mu_3^2 + \mu_4^2 - (\mu_1 - \mu_2)^2\right] / \, 2\mu_3\mu_4\right\} \le 0 \\ \cos^{-1}\left\{\left[\mu_3^2 + \mu_4^2 - (\mu_1 + \mu_2)^2\right] / \, 2\mu_3\mu_4\right\} - 170^\circ \le 0 \\ 20 \text{ mm} \le \mu_2 \le 450 \text{ mm}, 20 \text{ mm} \le \mu_3, \mu_4 \le 450 \text{ mm}, \\ 0^\circ \le \theta_0, \psi_0 \le 360^\circ \end{cases} \tag{32}$$

The time interval $[0^\circ, 120^\circ]$ is discretized into $k = 181$ equal small intervals. The average P-QLF is to be minimized.

For this mechanism synthesis problem, the computational cost is not high. The MCS-based robustness analysis is performed with a sample size of 300. Because we evaluate only expectations, this sample size is good enough.

The starting point used for both P-QLF and I-QLF based robust design was $(\mu_2, \mu_3, \mu_4, \theta_0, \psi_0) = (10 \text{ mm}, 100 \text{ mm}, 160 \text{ mm}, 0^\circ, 0^\circ)$. The optimal results are given in Table 2.

Table 2 Optimal results

Method	P-QLF	I-QLF
μ_{R_2}(mm)	48.14	43.08
μ_{R_3}(mm)	100.48	103.57
μ_{R_4} (mm)	74.13	66.51
θ_0 (deg)	100.46	97.15
ψ_0 (deg)	99.01	93.89
Average expected P-QLF $\overline{L}_E(\theta)$ ($)	21.60	24.76
Maximal expected P-QLF $\max L_E(\theta)$ ($)	70.0	28.33
Expected I-QLF $L_E(0^\circ, 120^\circ)$ ($)	84.21	43.87

From the traditional approach (P-QLF in the table), the average expected quality loss is $21.60 while its counterpart from the new approach (I-QLF in the table) is $24.76. The new approach has a (24.76-21.60)/ 21.60 =14.63% increase in the average expected quality loss. However, the expected interval quality loss from the new approach is $43.87, which is much smaller than that from the traditional approach. The latter approach produced $84.21 of the expected interval quality loss. The reduction of the new approach is (84.21-43.87)/43.87=92.0%. The new approach also produces a maximal expected point quality loss of \$28.33, which is also much smaller than that (\$70.0) from the traditional approach.

The expected P-QLF $E_L(\theta)$ and expected I-QLF $E_L(0,\theta)$ are depicted in Figs. 9 and 10. The expected P-QLF fluctuates over time while the expected I-QLF increases with time. The latter is always greater than the former except at the initial time when both of them are equal.

The motion of the mechanism is periodic, and hence the motion error or the QC is cyclic. The above analysis is only for $\theta \in [\theta_0, \theta_f] = [0^\circ, 120^\circ]$ in the first motion cycle. We can easily extend the results to later cycles. From the second cycle when $\theta \in [360^\circ i, 120^\circ + 360^\circ i]$ $(i = 1, 2, 3, \cdots)$, the expected point quality loss function (P-QLF in Fig. 9) will repeat itself as over $\theta \in [0^\circ, 120^\circ]$. On the other hand, the expected interval quality loss function (I-QLF in Fig. 9) will maintain constant or will be \$43.87. Therefore, we can conclude that over the entire time period $\theta \in [360^\circ i, 120^\circ + 360^\circ i]$ $(i = 1, 2, 3, \cdots)$, the expected quality loss over the service time is \$42.86.

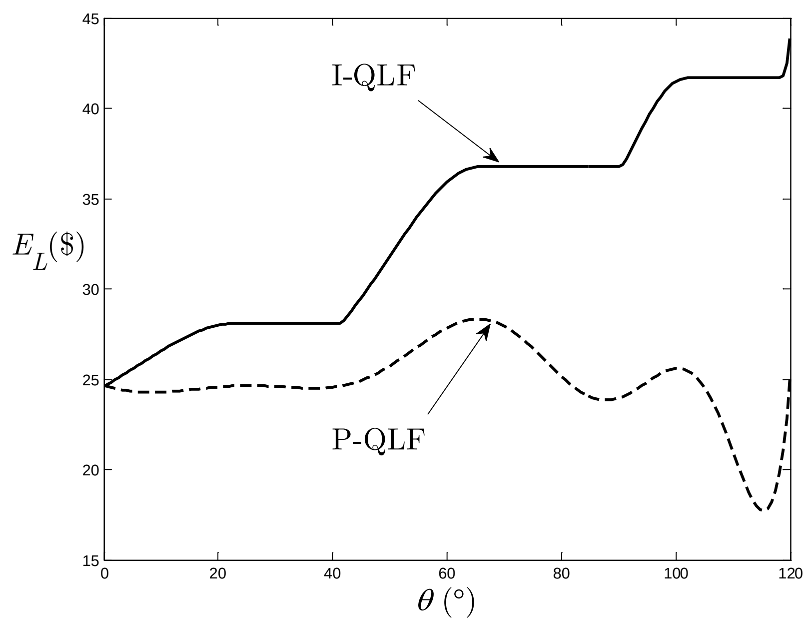

Fig. 9 Expected QLFs from I-QLF based robust design

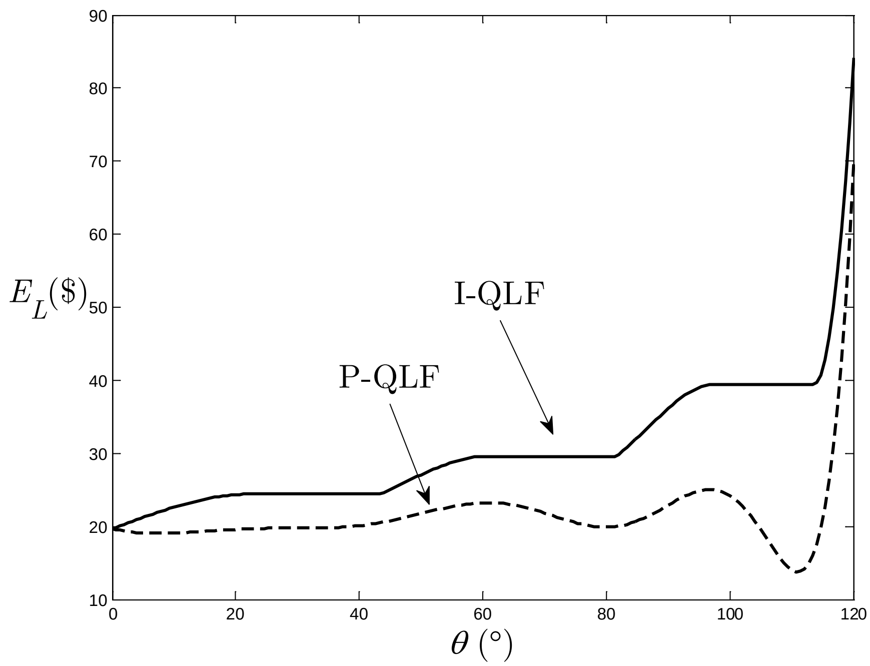

Fig. 10 Expected QLFs from P-QLF based robust design

For an easy comparison of the two approaches, we plot the quality losses from the two approaches in Fig. 11, where the results from the new approach are represented by bold curves. Both the expected I-QLFs are in solid curves while both the expected P-QLFs are in dashed curves.

As mentioned previously, the QC is a stochastic process with auto dependency. To show this, in Fig. 12, we pick a specific time instance, the initial time $\theta_0 = 0^\circ$, and we plot the coefficients of autocorrelation of the quality loss at this instance with other instances of time over $\theta \in [\theta_0, \theta_f] = [0^\circ, 120^\circ]$. The figure shows that the autocorrelations of both the designs are positive and that the two designs have different autocorrelation structures in their quality losses.

The I-QLF based robust design also helps to reduce the average motion error over $\theta \in [\theta_0, \theta_f] = [0^\circ, 120^\circ]$. Fig. 13 shows the average motion errors from both approaches. The average motion error from the new approach is smaller than that from the traditional approach in most of the time over $\theta \in [\theta_0, \theta_f] = [0^\circ, 120^\circ]$.

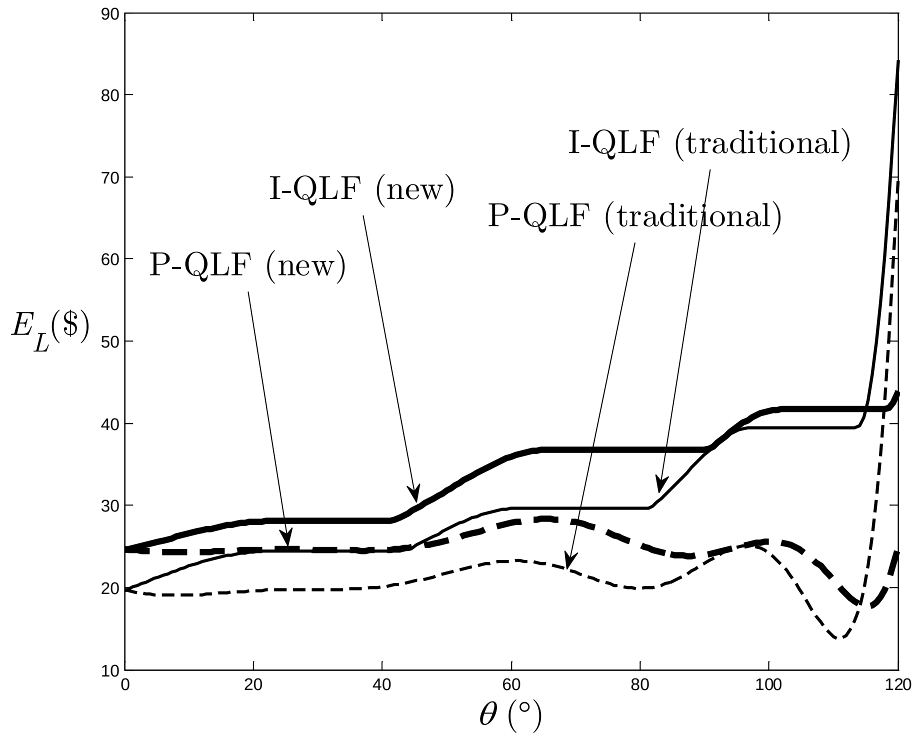

Fig. 11 Expected quality losses from both approaches

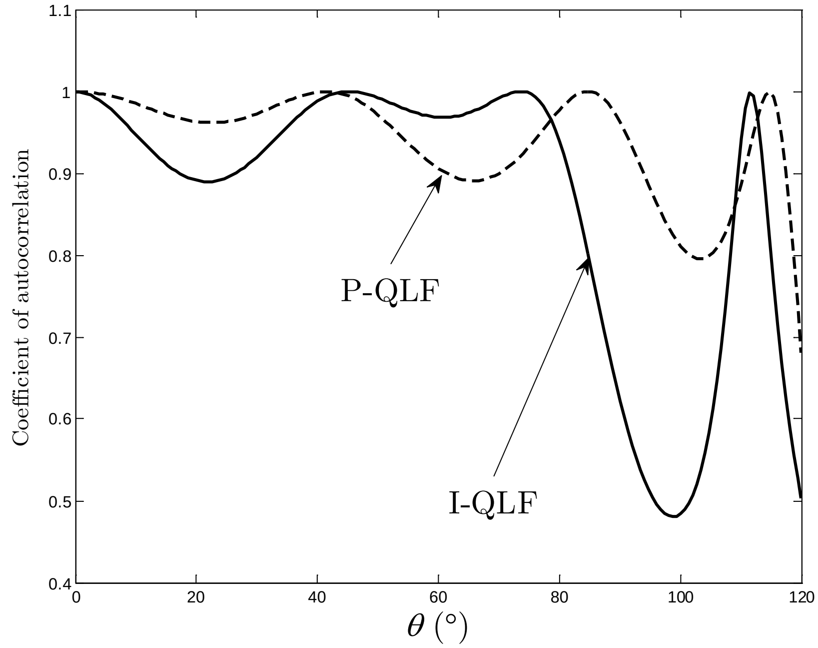

Fig. 12 Coefficients of autocorrelation of QLF

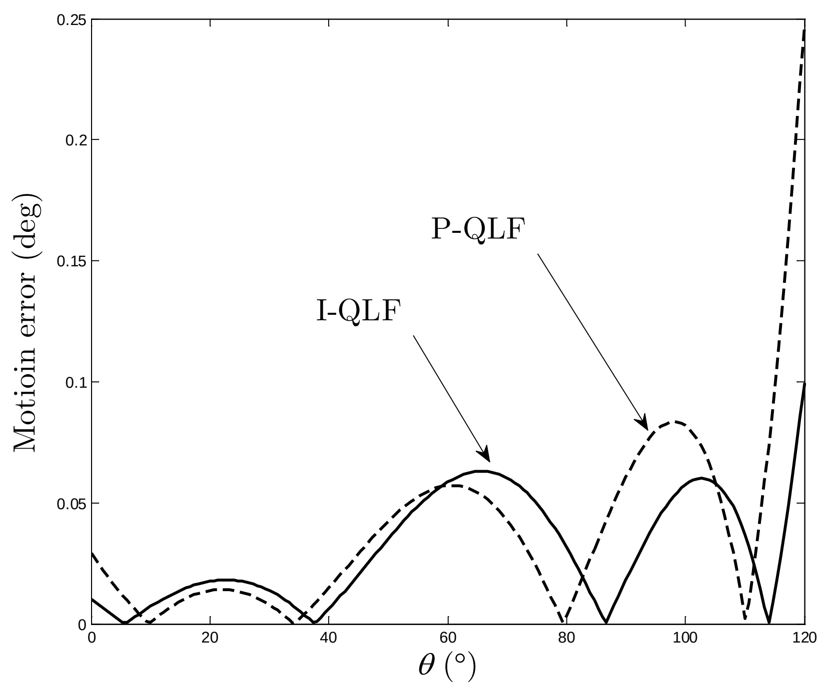

Fig. 13 Average motion errors

7. Conclusions

When quality characteristics vary within the product service time, the traditional instantaneous (point) robustness metrics may not work. The major reason is that they cannot fully reflect the quality loss over the service time interval. The quality losses expressed by the point robustness metrics may not increase with time. The point robustness metrics cannot reflect the auto-dependence properties of the time-dependency quality characteristics.

New robustness metrics for time-dependent quality characteristics are defined in this work to overcome the drawbacks of the point robustness metrics. The new robustness metrics in this work are only two possible metrics. Other metrics could also be developed. To provide a general guidance, we have proposed criteria for the new metrics. A metric must represent the maximal expected quality loss over the time interval of interest. The metric should capture the auto-dependency properties of quality losses over the time interval. The metric expressed in the form of a quality loss should be a non-decreasing function with respect to time.

The expected interval quality loss is the major robustness metric proposed in this work. It is the expected maximal point quality loss over product service time interval. It can better describe the quality loss over the time interval. It can also capture the auto correlation between the quality losses over the time interval. During robust design optimization, the expected interval quality loss will be minimized.

The two new robustness metrics require demanding computations because they involve the extreme values of the quality characteristics over the time interval. There are two possible ways to approach this challenge. The first way is to develop new robustness analysis algorithms that can efficiently estimate the new robustness metrics. The other way is to define other new but similar time-dependent robustness metrics that are much more efficient to be evaluated.

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