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Robust Design for Multivariate Quality Characteristics Using Extreme Value Distribution

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Abstract

Quality characteristics are important product performance variables that determine customer satisfaction. Their expected values are optimized and their standard deviations are minimized during robust design. Most of robust design methodologies consider only a single quality characteristic, but a product is often judged by multiple quality characteristics. It is a challenging task to handle dependent and oftentimes conflicting quality characteristics. This work proposes a new robustness modeling measure that uses the maximum quality loss among multiple quality characteristics for problems where the quality loss is the same no matter which quality characteristics or how many quality characteristics are defective. This treatment makes it easy to model robust design with multivariate quality characteristics as a single objective optimization problem and also account for the dependence between quality characteristics. The new method is then applied to problems where bivariate quality characteristics are involved. A numerical method for robust design with bivariate quality characteristics is developed based on the First Order Second Moment method. The method is applied to the mechanism synthesis of a four-bar linkage and a piston engine design problem.

1. INTRODUCTION

In engineering applications, there are many uncertainties. For example, the loading acting on mechanical systems is stochastic because of random operation conditions. The actual dimensions of a mechanical part fluctuate randomly around their designed nominal values due to the random manufacturing imprecision. The uncertainties may have significant impact on the quality, safety, and performance of a product. There are several design methodologies that mitigate the effects of uncertainty, and robust design (RD) [1] is one of them.

As an optimization technique, RD optimizes the performance (objective) of a product subject to design requirements (constraints). In addition, RD also minimizes the variation of the objective function caused by the aforementioned uncertainties. As a result, the performance of the product will be insensitive to uncertainty and remains within a small range around its designed value in the presence of uncertainty.

The robustness is commonly measured by the Taguchi quality loss function (QLF) [2]. Let a performance function be

$$Y = f(\mathbf{d}, \mathbf{X}, \mathbf{P}) \tag{1}$$

where Y is a quality characteristic (QC), which is a response variable that determines the performance of a product and customer satisfaction. $\mathbf{d}$ is a vector of deterministic design variables, and $\mathbf{X}$ is a vector of random design variables, whose means $\boldsymbol{\mu}_\mathbf{x}$ are to be determined during optimization. $\mathbf{P}$ is a vector of random parameters or noise factors. Due to the uncertainties in random variables $\mathbf{Z} = (\mathbf{X}, \mathbf{P})$, Y is also a random variable.

For a nominal-the-best (NTB) type QC, the Taguchi's QLF is given by

$$L = k(Y - m)^2 \tag{2}$$

where L is the quality loss, m is a target for Y, and k is a quality loss constant. The smaller-the-better QCs can also be treated as NTB QCs by setting their targets to zero. The QLF of the larger-the-better (LTB) type can be found in [2]. Next we only discuss the NTB and STB QCs.

RD minimizes the expected quality loss, which is computed by

$$E(L) = k\left[\left(\mu_Y - m\right)^2 + \sigma_Y^2\right] \tag{3}$$

where μ_Y and σ_Y are the mean and standard deviation of Y, respectively. Minimizing $E(L)$ can bring μ_Y to the target m and simultaneously minimize the variation in Y.

Most of the methodologies of RD are only for problems with a single QC. In real applications, the common task that engineers encounter is to select the optimal design variables by considering multiple QCs simultaneously. It is a challenging task because of the following complexities:

- The QCs may be in different orders of magnitude with different measuring units.
- The importance of the QCs may be different.
- The directions of the improvement of the QCs may be different.
- The QCs may conflict with one another. The improvement on one QC may lead to degradation of other QCs and vice versa.
- The QCs are statistically dependent if they are determined by common design variables and are affected by common random variables.

The most straightforward way of handling QCs is to use the simple summation of their quality losses. The summation is given by

$$E(L) = \sum_{i=1}^{n} E(L_i) = \sum_{i=1}^{n} k_i \left[\left(\mu_{Y_i} - m_i \right)^2 + \sigma_{Y_i}^2 \right] \tag{4}$$

where n is the number of QCs.

Since there is only one objective in the above objective function, it is easy to solve the RD problem. But there are several disadvantages. The expected quality loss obtained from summing up the individual quality losses may be much higher than the actual quality loss. The larger is the number of QCs under consideration, the higher is the expected quality loss than the true quality loss. The simple summation cannot take the dependence between the QCs into consideration as only the means and standard deviations of QCs appear in Eq. (4).

Other RD approaches [3-10] have also been proposed for multiple QCs. For example, a multivariate quality loss function is used to account for the joint losses of all pairs of quality characteristics [9]. The dependence of any pairs of two QCs is considered by accounting for the quality loss if both QCs are defective. But it might be difficult to determine such a joint quality loss, and the dependence between more than two QCs cannot be included in the overall quality loss.

The expected overall quality loss function in [9] is just an approximation. The analytical expected overall quality loss is derived in [10] for bivariate quality loss functions. The method is based on the First Order Second Moment (FOSM) and is efficient because of the analytical formulas.

In addition to the use of quality loss function, other robustness measures have also been proposed. A multivariate process capability index measure is defined in [11] where the ratio of the volume of the specification region over that of the process spread region is employed. Another multivariate process capability index is also developed in [12]. A more general RD framework is set up in [13] where all the three types of QCs are included; conflicting and dependent QCs can be handled.

Another way to deal with multiple QCs is to use the joint probability density function (PDF) of the QCs to minimize the associated variation. With the joint PDF available, the joint probability that the QCs are bounded within their specification limits can be estimated. Using this strategy, the work in [14] minimizes the determinant of the covariance matrix of the QCs in order to ensure the joint robustness. This method involves only one objective function and does not use a quality loss function; thus, no quality loss is provided.

This work is concerned with the situation where the quality loss is the same no matter which QCs or how many QCs are defective. This problem is commonly encountered in practices. We propose to use the maximum quality loss among the quality losses of all the QCs.

We explain why the maximum quality loss is needed and also give the robustness analysis model in Section 2. We then derive analytical equations for the robust analysis when bivariate QCs are involved in Section 3. The robust design model is given in Section 4 followed by examples in Section 5. Conclusions and future work are discussed in Section 6.

2. Robust Design with the Maximum Quality Loss

In this section, we discuss the use of the maximum quality loss to model a RD problem with multiple QCs.

2.1 Maximum quality loss

This work is concerned with a design problem where the quality loss is a constant if at least one QC reaches its specification limit regardless which QC it is. This problem is common and is encountered in the following situations:

- If at least one QC is defective, the component or product should be discarded and replaced by a new one. No matter what QCs are defective, the cost is the same.
- If at least one QC is defective, the component or product must be reworked. The rework requires the same process and materials, and the cost is the same regardless which QCs or how many QCs are defective.

Next we use a problem with three NTB QCs Y_1, Y_2 and Y_3 as an example to discuss how to quantify the quality loss for the above situations.

Let the quality loss of Y_i $(i = 1, 2, 3)$ be

$$L_i = k_i (Y_i - m_i)^2 \tag{5}$$

Let the specification limits of Y_i be $m_i \pm \Delta_i$ and the quality loss be A if Y_i is at its specification limit or when $Y_i = m_i \pm \Delta_i$. Since $L_i = A$,

$$A = k_i \Delta_i^2 \tag{6}$$

Then the quality constant is given by $k_i = A/\Delta_i^2$.

When at least one QC reaches its specification limit, the quality loss is A. The situation is expressed by

$$L = \begin{cases} A \text{ if } (Y_1 - m_1)^2 = \Delta_1^2, (Y_2 - m_2)^2 < \Delta_2^2, (Y_3 - m_3)^2 < \Delta_3^2 \\ A \text{ if } (Y_2 - m_2)^2 = \Delta_2^2, (Y_1 - m_1)^2 < \Delta_1^2, (Y_3 - m_3)^2 < \Delta_3^2 \\ A \text{ if } (Y_3 - m_3)^2 = \Delta_3^2, (Y_1 - m_1)^2 < \Delta_1^2, (Y_2 - m_2)^2 < \Delta_2^2 \\ A \text{ if } (Y_1 - m_1)^2 = \Delta_1^2, (Y_2 - m_2)^2 = \Delta_2^2, (Y_3 - m_3)^2 < \Delta_3^2 \\ A \text{ if } (Y_1 - m_1)^2 = \Delta_1^2, (Y_3 - m_3)^2 = \Delta_3^2, (Y_2 - m_2)^2 < \Delta_2^2 \\ A \text{ if } (Y_2 - m_2)^2 = \Delta_2^2, (Y_3 - m_3)^2 = \Delta_3^2, (Y_1 - m_1)^2 < \Delta_1^2 \\ A \text{ if } (Y_1 - m_1)^2 = \Delta_1^2, (Y_2 - m_2)^2 = \Delta_2^2, (Y_3 - m_3)^2 = \Delta_3^2 \end{cases} \tag{7}$$

$(Y_i - m_i)^2 = \Delta_i^2$ means $L_i = k_i (Y_i - m_i)^2 = (A/\Delta_i^2)\Delta_i^2 = A$, and $(Y_i - m_i)^2 < \Delta_i^2$ means $L_i < A$. Thus Eq. (7) is rewritten as

$$L = \begin{cases} A \text{ if } L_1 = A, L_2 < A, L_3 < A \\ A \text{ if } L_2 = A, L_1 < A, L_3 < A \\ A \text{ if } L_3 = A, L_1 < A, L_2 < A \\ A \text{ if } L_1 = L_2 = A, L_3 < A \\ A \text{ if } L_1 = L_3 = A, L_2 < A \\ A \text{ if } L_2 = L_3 = A, L_1 < A \\ A \text{ if } L_1 = L_2 = L_3 = A \end{cases} \tag{8}$$

The equation is equivalent to

$$L = A \text{ if } \max\{L_1, L_2, L_3\} = A \tag{9}$$

Eq. (9) suggests that the maximum quality loss should be used for the multiple QCs. Suppose there are n QCs denoted by $\{Y_i\}_{i=1,n}$ and their quality losses are $\{L_i\}_{i=1,n}$. Denote the maximum quality loss by W, which is given by

$$W = \max\{L_i\}_{i=1,n} \tag{10}$$

Since L_i is random, so is W. Then during RD we should minimize the expected maximum quality loss $E(W)$.

The cumulative distribution function (CDF) $F_W(\cdot)$ of W is given by

$$\begin{aligned} &F_W(w) = \Pr\{W < w\} \\ &= \Pr\{L_1 < w, L_2 < w, \cdots, L_n < w\} = \Pr\left\{\bigcap_{i=1}^{n} L_i < w\right\} \\ &= \int_{\Omega} f_{\mathbf{Z}}(z)dz \end{aligned} \tag{11}$$

where $\Omega = \bigcap_{i=1}^{n} L_i < w$, and $f_{\mathbf{Z}}(\cdot)$ is the joint PDF of $\mathbf{Z} = (\mathbf{X}, \mathbf{P})$.

The PDF of w is

$$f_W(w) = \frac{dF_W(w)}{dw} \tag{12}$$

Then the expected maximum quality loss is

$$E(W) = \int_0^{\infty} w f_W(w) dw \tag{13}$$

As indicated in the above equation, the key to obtaining $E(W)$ is to know the PDF $f_W(w)$. We will discuss how to get it in Section 3.

2.2 Advantages of using the maximum quality loss

Using the maximum quality loss has the following advantages:

(1) The RD is a single objective optimization problem.

There is no need to assign weights to QCs. This avoids the subjectivity and difficulty of identifying different weights.

(2) Using the maximum quality loss is more realistic.

As discussed above for the situation that this work addresses, if a QC is beyond its specification limit, the product or component needs to be reworked, repaired, or discarded, and the cost is the same no matter what QC is defective or how many QCs are defective. Simply summing up the quality losses of multiple QCs will give a much higher quality loss, sometimes far away from the true quality loss, as mentioned previously.

(3) The maximum quality loss can automatically account for the dependence between QCs. The reason is explained below.

According to Eq. (1), QC Y_i $(i=1,n)$ is given by $Y_i = f_i(\mathbf{d}, \mathbf{X}, \mathbf{P})$. All the QCs share the common variables $\mathbf{d}$, $\mathbf{X}$, and $\mathbf{P}$, and they are therefore dependent. The degree of dependence between two QCs affects the likelihood of the following two events: both of the QCs are defective and either QC is defective. The dependence therefore also determines the expected quality loss with multiple QCs. Simply adding the quality losses of individual QCs cannot account for the dependence because the expectation of the sum of quality losses does not require knowing the covariance of the QCs.

The expected maximum quality loss accounts for the dependence of QCs. Let us look at a simple example where two quality losses L_1 and L_2 follow normal distributions $N(\mu_{L_1}, \sigma_{L_1}^2)$ and $N(\mu_{L_2}, \sigma_{L_2}^2)$, respectively, where μ and σ stand for a mean and a standard deviation, respectively. L_1 and L_2 are dependent with the coefficient of correlation $\rho_{L_1 L_2}$. Let the maximum quality loss be W.

$$L(W) = \max(L_1, L_2) \tag{14}$$

The expected maximum quality loss is given by [15]

$$E(W) = \mu_{L_1} \Phi\left(\frac{\mu_{L_1} - \mu_{L_2}}{\theta}\right) + \mu_{L_2} \Phi\left(\frac{\mu_{L_2} - \mu_{L_1}}{\theta}\right) + \theta \phi\left(\frac{\mu_{L_1} - \mu_{L_2}}{\theta}\right) \tag{15}$$

where $\theta = \sqrt{\sigma_{L_1}^2 + \sigma_{L_2}^2 - 2\rho_{L_1 L_2} \sigma_{L_1} \sigma_{L_2}}$

As indicated above, $E(W)$ is determined by the dependence of L_1 and L_2, or the coefficient of correlation $\rho_{L_1 L_2}$. It is therefore also determined by the dependence of QCs Y_1 and Y_2. Although it may not likely that a quality loss follows a normal distribution, this simple example demonstrates that the maximum expected QC is affected by the dependence of the QCs.

Eq. (13) indicates that the joint PDF of the QCs is needed to determine the distribution of the maximum quality loss, and thereby the expected maximum quality loss.

3. Robustness Assessment for Bivariate NTB and STB QCs

The task of robustness assessment is to calculate the expected maximum quality loss. In this work, we apply the principle of the maximum quality loss to QCs which may belong to a NTB or STB type. We then derive analytical equations for the expected maximum quality loss based on the First Order Second Moment (FOSM) method [10, 16], which is widely used in RD. (Since a LTB QC is the reciprocal of the square of the QC, the analytical expression of the expected maximum quality loss may not exist when the LTB QC is involved. It is possible in the future work to derive equations that approximate the expected maximum quality loss.) In this section we focus on analytical derivations for bivariate QCs and also discuss how to extend the result to higher dimensions.

FOSM approximates a QC Y_i $(i=1,2)$ at the means of $\mathbf{Z}=(\mathbf{X},\mathbf{P})$, or $\boldsymbol{\mu}_{\mathbf{Z}}=(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{P}})$, where $\boldsymbol{\mu}_{\mathbf{X}}=\{\mu_{X_j}\}_{j=1,n_x}$ and $\boldsymbol{\mu}_{\mathbf{P}}=\{\mu_{P_j}\}_{j=1,n_p}$.

$$Y_i \approx f_i(\mathbf{d},\boldsymbol{\mu}_{\mathbf{Z}}) + \sum_{j=1}^{n_z} \frac{\partial f_i}{\partial Z_j}\Big|_{\boldsymbol{\mu}_{\mathbf{Z}}} (Z_j - \mu_{Z_j}) \tag{16}$$

where $n_z = n_x + n_p$.

Assume $Z_j \sim N(\mu_{Z_j}, \sigma_{Z_j}^2)$, $j = 1, 2, \cdots, n_z$. Also assume that all the random variables are independent. If Z_j is not normally distributed, it can be transformed into a normal variable. One can also transform dependent random variables into independent ones by Rosenblatt or Nataf transformation [17] before using Eq. (16).

Then Y_i is a linear combination of normal random variables and is also a normal random variable. Its mean and standard deviation are

$$\mu_{Y_i} = f_i(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{Z}}) \tag{17}$$

and

$$\sigma_{Y_i} = \left[\sum_{j=1}^{n_z} \left(\frac{\partial f_i}{\partial Z_j}\Big|_{\boldsymbol{\mu}_{\mathbf{Z}}} \right)^2 \sigma_{z_j}^2 \right]^{0.5} \tag{18}$$

respectively.

The coefficient of correlation of Y_1 and Y_2 is

$$\rho_{Y_1 Y_2} = \sum_{j=1}^{n_z} \left(\frac{\partial f_1}{\partial Z_j}\Big|_{\boldsymbol{\mu}_{\mathbf{Z}}} \right) \left(\frac{\partial f_2}{\partial Z_j}\Big|_{\boldsymbol{\mu}_{\mathbf{Z}}} \right) \sigma_{Z_j}^2 \Big/ \sigma_{Y_1} \sigma_{Y_2} \tag{19}$$

For the quality losses $L_i = k_i (Y_i - m_i)^2 \ \ (i = 1, 2)$, let

$$Q_i = \sqrt{L_i} = \sqrt{k_i} (Y_i - m_i) \ \ (i = 1, 2) \tag{20}$$

which follows a normal distribution $N(\mu_i, \sigma_i^2)$, with its mean and standard deviation given by

$$\mu_i = \sqrt{k_i}(\mu_{Y_i} - m_i) \tag{21}$$

and

$$\sigma_i = \sqrt{k_i}\sigma_{Y_i} \tag{22}$$

respectively.

The coefficient of correlation of Q_1 and Q_2 is

$$\rho = \rho_{Y_1Y_2} \tag{23}$$

Since the two quality losses are Q_1^2 and Q_2^2 now, the maximum quality loss is then

$$W = \max(Q_1^2, Q_2^2) \tag{24}$$

Our task now is to find the expected value of W or $E(W)$. According to Eq. (11), the CDF of W is

$$\begin{aligned} F_W(w) &= \Pr\{W < w\} = \Pr\{Q_1^2 < w, Q_2^2 < w\} \\ &= \Pr\{|Q_1| < \sqrt{w}, |Q_2| < \sqrt{w}\} \\ &= \int_{-\sqrt{w}}^{\sqrt{w}} \int_{-\sqrt{w}}^{\sqrt{w}} f_{12}(q_1, q_2) dq_1 dq_2 \end{aligned} \tag{25}$$

where $f_{12}(\cdot,\cdot)$ is the joint PDF of Q_1 and Q_2. It is a bivariate normal PDF with mean (μ_1, μ_2) and covariance matrix $\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$.

Define $H(w,q_2) = \int_{-\sqrt{w}}^{\sqrt{w}} f_{12}(q_1,q_2)dq_1$. Then the PDF of W is

$$
\begin{aligned}
f_W(w) &= \frac{dF_W(w)}{dw} = \frac{d}{dw}\int_{-\sqrt{w}}^{\sqrt{w}} H(w,q_2)dq_2 \\
&= H(w,\sqrt{w})\frac{1}{2\sqrt{w}} - H(w,-\sqrt{w})(-\frac{1}{2\sqrt{w}}) + \int_{-\sqrt{w}}^{\sqrt{w}} \frac{\partial H(w,q_2)}{\partial w}dq_2 \\
&= \frac{1}{2\sqrt{w}}\left[H(w,\sqrt{w}) + H(w,-\sqrt{w}) \right] \\
&+ \int_{-\sqrt{w}}^{\sqrt{w}} \left[f_{12}(\sqrt{w},q_2)\frac{1}{2\sqrt{w}} - f_{12}(-\sqrt{w},q_2)(-\frac{1}{2\sqrt{w}}) \right] dq_2 \\
&= \frac{1}{2\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}} \left[f_{12}(q_1,\sqrt{w}) + f_{12}(q_1,-\sqrt{w}) \right] dq_1 \\
&+ \frac{1}{2\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}} \left[f_{12}(\sqrt{w},q_2) + f_{12}(-\sqrt{w},q_2) \right] dq_2
\end{aligned}
\tag{26}
$$

Let $v = \sqrt{w}$, we have

$$
\begin{aligned}
f_W(w) &= \frac{1}{2v}\int_{-v}^{v} \left[f_{12}(q_1,v) + f_{12}(q_1,-v) \right] dq_1 \\
&+ \frac{1}{2v}\int_{-v}^{v} \left[f_{12}(v,q_2) + f_{12}(-v,q_2) \right] dq_2
\end{aligned}
\tag{27}
$$

and

$$
E(W) = \int_0^{\infty} w f_W(w)dw = 2\int_0^{\infty} v^3 f_W(w)dv \tag{28}
$$

Plugging Eq. (27) into Eq. (28) yields

$$E(W)=\int_0^{\infty} v^2 \int_{-v}^{v} \left[f_{12}(q_1,v)+f_{12}(q_1,-v) \right] dq_1 dv$$
$$+\int_0^{\infty} v^2 \int_{-v}^{v} \left[f_{12}(v,q_2)+f_{12}(-v,q_2) \right] dq_2 dv \quad (29)$$

Next let us evaluate the inner integrals. We start from $\int_{-v}^{v} f_{12}(q_1,v)dq_1$, which can be written as

$$f_{12}(q_1,v)=f_{Q_2}(v)\,f_{Q_1|Q_2=v}(q_1) \quad (30)$$

$f_{Q_2}(v)$ is the PDF of a normal random variable with μ_2 and σ_2 at v. $f_{Q_1|Q_2=v}$ is the PDF of Q_1 conditional on $Q_2=v$, and its mean and standard deviation are given by

$$\mu_1^{+}=\mu_{Q_1|Q_2=v}=\mu_1+\rho\sigma_1\left(\frac{v-\mu_2}{\sigma_2}\right) \quad (31)$$

and

$$\sigma_1^{+}=\sigma_{Q_1|Q_2=v}=\sqrt{1-\rho^2}\,\sigma_1 \quad (32)$$

respectively.

Similarly

$$f_{12}(q_1,-v)=f_{Q_2}(-v)\,f_{Q_1|Q_2=-v}(q_1) \quad (33)$$

where $f_{Q_1|Q_2=-v}(q_1)$ is the PDF of the normal random variable whose mean and standard deviation are

$$\mu_1^- = \mu_{Q_1|Q_2=-v} = \mu_1 + \rho\sigma_1\left(\frac{-v-\mu_2}{\sigma_2}\right) \tag{34}$$

and

$$\sigma_1^- = \sigma_{Q_1|Q_2=-v} = \sqrt{1-\rho^2}\,\sigma_1 \tag{35}$$

respectively.

$$f_{12}(v,q_2) = f_{Q_1}(v)\,f_{Q_2|Q_1=v}(q_2) \tag{36}$$

where $f_{Q_1}(v)$ is the PDF of Q_1, and $f_{Q_2|Q_1=v}(q_2)$ is the PDF of the normal random variable whose mean and standard deviation are

$$\mu_2^+ = \mu_{Q_2|Q_1=v} = \mu_2 + \rho\sigma_2\left(\frac{v-\mu_1}{\sigma_1}\right) \tag{37}$$

and

$$\sigma_2^+ = \sigma_{Q_2|Q_1=v} = \sqrt{1-\rho^2}\,\sigma_2 \tag{38}$$

respectively.

$$f_{12}(-v,q_2) = f_{Q_1}(-v)\,f_{Q_2|Q_1=-v}(q_2) \tag{39}$$

where $f_{Q_2|Q_1=-v}(q_2)$ is the PDF of the normal random variable whose mean and standard deviation are

$$\mu_2^- = \mu_{Q_2|Q_1=-v} = \mu_2 + \rho\sigma_2\left(\frac{-v-\mu_1}{\sigma_1}\right) \tag{40}$$

and

$$\sigma_2^- = \sigma_{Q_2|Q_1=-v} = \sqrt{1-\rho^2}\,\sigma_2 \tag{41}$$

respectively.

Then the PDF of w is given by

$$\begin{aligned} f_W(w) = \frac{1}{2v}\Big[& f_{Q_2}(v)\int_{-v}^{v} f_{Q_1|Q_2=v}(q_1)dq_1 \\ & + f_{Q_2}(-v)\int_{-v}^{v} f_{Q_1|Q_2=-v}(q_1)dq_1 \\ & + f_{Q_1}(v)\int_{-v}^{v} f_{Q_2|Q_1=v}(q_2)dq_2 \\ & + f_{Q_1}(-v)\int_{-v}^{v} f_{Q_2|Q_1=-v}(q_2)dq_2 \Big] \end{aligned} \tag{42}$$

For a normal PDF $f(x)$ with μ and σ, $\int_a^b f(x)dx = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$ and $f(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)$, we have

$$f_W(w) = \frac{1}{2\sigma_2 v}\phi\left(\frac{v-\mu_2}{\sigma_2}\right)\left[\Phi\left(\frac{v-\mu_1^+}{\sigma_1^+}\right) - \Phi\left(\frac{-v-\mu_1^+}{\sigma_1^+}\right)\right]$$
$$+\frac{1}{2\sigma_2 v}\phi\left(\frac{-v-\mu_2}{\sigma_2}\right)\left[\Phi\left(\frac{v-\mu_1^-}{\sigma_1^-}\right) - \Phi\left(\frac{-v-\mu_1^-}{\sigma_1^-}\right)\right]$$
$$+\frac{1}{2\sigma_1 v}\phi\left(\frac{v-\mu_1}{\sigma_1}\right)\left[\Phi\left(\frac{v-\mu_2^+}{\sigma_2^+}\right) - \Phi\left(\frac{-v-\mu_2^+}{\sigma_2^+}\right)\right]$$
$$+\frac{1}{2\sigma_1 v}\phi\left(\frac{-v-\mu_1}{\sigma_1}\right)\left[\Phi\left(\frac{v-\mu_2^-}{\sigma_2^-}\right) - \Phi\left(\frac{-v-\mu_2^-}{\sigma_2^-}\right)\right] \quad (43)$$

Then the PDF $f_W(w)$ is analytically available, and $E(W)$ can be easily calculated using Eq. (28) and given by

$$E(W) = \int_0^{\infty} \left\{ \frac{v^2}{\sigma_2}\phi\left(\frac{v-\mu_2}{\sigma_2}\right)\left[\Phi\left(\frac{v-\mu_1^+}{\sigma_1^+}\right) - \Phi\left(\frac{-v-\mu_1^+}{\sigma_1^+}\right)\right]\right.$$
$$+\frac{v^2}{\sigma_2}\phi\left(\frac{-v-\mu_2}{\sigma_2}\right)\left[\Phi\left(\frac{v-\mu_1^-}{\sigma_1^-}\right) - \Phi\left(\frac{-v-\mu_1^-}{\sigma_1^-}\right)\right]$$
$$+\frac{v^2}{\sigma_1}\phi\left(\frac{v-\mu_1}{\sigma_1}\right)\left[\Phi\left(\frac{v-\mu_2^+}{\sigma_2^+}\right) - \Phi\left(\frac{-v-\mu_2^+}{\sigma_2^+}\right)\right]$$
$$\left.+\frac{v^2}{\sigma_1}\phi\left(\frac{-v-\mu_1}{\sigma_1}\right)\left[\Phi\left(\frac{v-\mu_2^-}{\sigma_2^-}\right) - \Phi\left(\frac{-v-\mu_2^-}{\sigma_2^-}\right)\right]\right\} dv \quad (44)$$

The above derivations can be extended to QCs with higher dimensions. We next briefly discuss how to deal with three QCs Q_1, Q_2, and Q_3. Extending Eq. (25), we have the following CDF of the maximum QC:

$$F_W(w) = \int_{-\sqrt{w}}^{\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}} f_{123}(q_1, q_2, q_3)dq_1 dq_2 dq_3 \quad (45)$$

Thus, the associated PDF is given by

$$f_W(w) = \frac{dF_W(w)}{dw} = \frac{d}{dw}\int_{-\sqrt{w}}^{\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}} f_{123}(q_1, q_2, q_3)dq_1dq_2dq_3 \quad (46)$$

With the same idea for the two-dimensional problem above, we obtain

$$\begin{aligned} f_W(w) = &\frac{1}{2\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}}\left[f_{123}(q_1, q_2, \sqrt{w}) + f_{123}(q_1, q_2, -\sqrt{w})\right]dq_1dq_2 \\ &+\int_{-\sqrt{w}}^{\sqrt{w}}\left[\frac{\partial}{\partial w}\int_{-\sqrt{w}}^{\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}} f_{123}(q_1, q_2, q_3)dq_1dq_2\right]dq_3 \end{aligned} \quad (47)$$

We now use the results we have obtained from Eqs. (25) and (26), which result in the following equation:

$$\begin{aligned} \frac{d}{dw}\int_{-\sqrt{w}}^{\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}} f_{12}(q_1, q_2)dq_1dq_2 = &\frac{1}{2\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}}\left[f_{12}(q_1, \sqrt{w}) + f_{12}(q_1, -\sqrt{w})\right]dq_1 \\ &+\frac{1}{2\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}}\left[f_{12}(\sqrt{w}, q_2) + f_{12}(-\sqrt{w}, q_2)\right]dq_2 \end{aligned} \quad (48)$$

If we replace $f_{12}(\cdot)$ with $f_{123}(\cdot)$ in the above equation and plug it into the second term on the right-hand side of Eq. (47), we obtain

$$\begin{aligned} f_W(w) = &\frac{1}{2\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}}\left[f_{123}(q_1, q_2, \sqrt{w}) + f_{123}(q_1, q_2, -\sqrt{w})\right]dq_1dq_2 \\ &+\frac{1}{2\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}}\left[f_{123}(\sqrt{w}, q_2, q_3) + f_{123}(-\sqrt{w}, q_2, q_3)\right]dq_2dq_3 \\ &+\frac{1}{2\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}}\int_{-\sqrt{w}}^{\sqrt{w}}\left[f_{123}(q_1, \sqrt{w}, q_3) + f_{123}(q_1, -\sqrt{w}, q_3)\right]dq_1dq_3 \end{aligned} \quad (49)$$

Then we can use Eq. (28) to obtain the expected maximum quality loss $E(W)$ in a form of a triple integral, which can be reduced to a double integral by using the conditional joint PDFs such as $f_{Q_1,Q_2|Q_3=\sqrt{w}}(\cdot)$, $f_{Q_2,Q_3|Q_1=\sqrt{w}}(\cdot)$, $f_{Q_1,Q_3|Q_2=\sqrt{w}}(\cdot)$, etc. It is also possible to derive analytical equations for more than three QCs.

4. Robust Design with the Maximum QC

With the maximum quality loss, the RD optimization model can be formulated as

$$\begin{cases} \min\limits_{(\mathbf{d},\boldsymbol{\mu}_\mathbf{X})} \; E(W) = E\left\{\max\left[L_i\left(\mathbf{d},\mathbf{X},\mathbf{P}\right)\right]_{i=1,n}\right\} \\ \text{s. t.} \quad \mu_{g_j}\left(\mathbf{d},\mathbf{X},\mathbf{P}\right) + \beta_{g_j}\sigma_{g_j}\left(\mathbf{d},\mathbf{X},\mathbf{P}\right) \le 0, \;\; j=1,2,\cdots,n_g \end{cases} \tag{50}$$

where g_j is a constraint function, whose satisfaction is guaranteed at a probability level $\Phi(\beta)$ approximately [10]. μ_{g_j} and σ_{g_j} are the mean and standard deviation of $g_j(\cdot)$, which are given by

$$\mu_{g_j} = g_j\left(\mathbf{d},\boldsymbol{\mu}_\mathbf{X},\boldsymbol{\mu}_\mathbf{P}\right),\; j=1,2,\cdots,n_g \tag{51}$$

and

$$\sigma_{g_j} = \left[\sum_{k=1}^{n_z}\left(\frac{\partial g_j}{\partial Z_k}\Big|_{\boldsymbol{\mu}_\mathbf{Z}}\right)^2 \sigma_{z_k}^2\right]^{0.5},\; j=1,2,\cdots,n_g \tag{52}$$

respectively.

5. EXAMPLES

In this section, we apply the RD method with the maximum quality loss to two design problems. The first is the robust mechanism synthesis, and the second is the robust piston engine design.

5.1 Robust mechanism synthesis

A four-bar linkage shown in Fig. 1 is to be designed to realize the following motion [10, 18, 19]:

Place Fig. 1 here

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(1) When the crank angle is $\theta = 0^{\circ}$, the rocker output angle is $\psi = 35^{\circ}$.

(2) When the crank angle is $\theta = 100^{\circ}$, the rocker output angle is $\psi = 95^{\circ}$.

(3) The minimal transmission angle is $\lambda = 40^{\circ}$.

The QCs are the motion output angles ψ at $\theta = 0^{\circ}$ and $\theta = 100^{\circ}$, which are given by

$$Q_i = f_i(\mathbf{d}, \mathbf{X}, \mathbf{P}) = \psi(\mathbf{d}, \mathbf{X}, \mathbf{P}; \theta)$$

where $i = 1, 2$; and $\theta = 0^{\circ}$ for $i = 1$, and $\theta = 100^{\circ}$ for $i = 2$. The output angle is given by [18]

$$\psi(\mathbf{X}, \theta) = 2\arctan\left(\frac{-E \pm \sqrt{E^2 + D^2 - F^2}}{F - D}\right)$$

where $D = 2R_4(R_1 - R_2\cos\theta)$, $E = -2R_2R_4\sin\theta$, and $F = R_1^2 + R_2^2 + R_4^2 - R_3^2 - 2R_1R_2\cos\theta$.

The specification limits of the QCs are $m_1 \pm 0.1^\circ$ and $m_2 \pm 0.2^\circ$, where $m_1 = 35^\circ$ and $m_2 = 95^\circ$. If either QC or both QCs are at their specification limits, the quality loss is \$100. The two quality loss constants are therefore $k_1 = 100/0.1^2 = 10000$ \$/deg^2 and $k_2 = 100/0.2^2 = 2500$ \$/deg^2.

The deterministic design variables are $\mathbf{d} = (\theta_0, \psi_0)$, which are the initial crank angle θ_0, and the initial output angle ψ_0, respectively. The random design variables are $\mathbf{X} = (R_2, R_3, R_4)$, which are the link lengths. The random parameter R_1 is the distance between revolute joints *A* and *D*, and $\mathbf{P} = (R_1)$. The variables are given in Table 1.

Place Table 1 here

The constraints are for the existence of a crank and the permitted transmission angle, and they should be satisfied at a probability level of $\beta = 4$, which is equivalent to a probability of failure 3.17×10^{-5}. The constraint functions are given by [20]

$$g_1 = R_2 + R_3 - (R_1 + R_4) \le 0$$

$$g_2 = R_2 + R_4 - (R_1 + R_3) \le 0$$

$$g_3 = R_2 + R_1 - (R_3 + R_4) \le 0$$

$$g_4 = R_3^2 + R_4^2 - (R_1 - R_2)^2 - 2R_3R_4 \cos\lambda \le 0$$

$$g_5 = -[R_3^2 + R_4^2 - (R_1 + R_2)^2] - 2R_3R_4\cos\lambda \le 0$$

The equations for the QCs are given in [20]. The derivatives of the QCs and the constraint functions are analytically available, and so are the means and standard deviations of the constraint functions.

To compare different methods, we provide the following design models. Model 1 is for the deterministic optimization and is given by

$$\begin{cases} \min\limits_{(\mathbf{d},\mathbf{X})} \ f = (\psi_1 - m_1)^2 + (\psi_2 - m_2)^2 \\ \text{subject to} \\ \qquad g_j \le 0 \ (j = 1, 2, \cdots, 5) \end{cases}$$

where the mean values of random variables are used, and the sum of the motion errors is minimized.

The robust design model with the summation of individual quality losses is formulated by Model 2 as

$$\begin{cases} \min\limits_{(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}})} \ \sum E(L) = E(L_1) + E(L_2) \\ \text{subject to} \\ \qquad \mu_{g_j} + \beta_j \sigma_{g_j} \le 0 \ (j = 1, 2, \cdots, 5) \end{cases}$$

where $E(L_i) = k_i \left\{ \left[\mu_{\psi i}(\mathbf{d}, \mathbf{X}, \mathbf{P}; \theta_i) - m_i \right]^2 + \sigma_{\psi i}^2(\mathbf{d}, \mathbf{X}, \mathbf{P}; \theta_i) \right\}$, $\theta_1 = 0°$, and $\theta_2 = 100°$.

The robust design model with the maximum quality loss is provided by Model 3 as

$$\begin{cases} \min\limits_{(\mathbf{d},\mu_{\mathbf{X}})} E(L_{\max}) = E\left\{\max\left[L_1, L_2\right]\right\} \\ \text{subject to} \\ \qquad \mu_{g_j} + \beta_j \sigma_{g_j} \le 0 \ (j = 1, 2, \ldots, 5) \end{cases}$$

where $L_i = k_i\left[\psi(\mathbf{d},\mathbf{X},\mathbf{P};\theta_i) - m_i\right]^2$.

The optimal design variables from the three models are given in Table 2, and the robustness of the three designs is summarized in Table 3.

Place Table 2 here

Place Table 3 here

The three methods produced the desired average motion because the means of the output angles are almost on their targets. The deterministic optimization (Model 1) has the largest expected maximum quality loss $862.1. The robust design with the sum of quality losses (Model 2) improves the design significantly with an expected maximum quality loss $461.3. With the incorporation of the maximum quality loss, the proposed method (Model 3) further improves the robustness. It produces the lowest expected maximum quality loss $459.2.

The coefficients of correlation between the two QCs are also given in Table 3. The different coefficients of correlation are also a contributing factor for different maximum quality losses. The sums of individual expected quality loss $\sum E(L)$ from the three methods are also provided in Table 3.

The expected maximum quality losses $E\{L_{\max}\}$ in Table 3 were calculated by the proposed methodology. They were confirmed by Monte Carlo simulation (MCS) with a sample size of 10^5 as shown as $E\{L_{\max}\}(\mathrm{MCS})$ in the last row of Table 3. The MCS solutions indicate that the proposed bivariate robust analysis using maximum quality loss is accurate.

5.2 Robust Piston Design

The second example is the robust piston engine design. Piston slap noise is the engine noise that results from the secondary motion of the piston [21, 22]. It is considered as a major QC because it is one of the key factors for customer dissatisfaction. One of the objectives is to find a design with the minimal piston slap noise that is invariant to noise factors. A designer should also consider the friction between the piston skirt and the cylinder liner, between the rings and the piston, and between the rings and the liner. The friction largely affects the noise and is also treated as another QC. It is desirable to have smaller friction. Both of the QCs therefore belong to the STB type. As indicated from the simulation study in [21, 22], reducing piston noise will increase piston friction. The two QCs hence conflict with each other.

In this application, we used the models from [21, 22] to perform the robust design. The model was developed based on computationally intensive multi-body dynamics simulations. Details of the simulation model are given in [21]. Table 4 shows the four deterministic design variables and two random parameters. There are no random design variables in this design. The deterministic design variables are $\mathbf{d} = [d_1, d_2, d_3, d_4] = [SL, SP, SO, PO]$, which are the skirt length (*SL*), skirt profile (*SP*), skirt ovality (*SO*), and pin offset (*PO*), respectively. The random parameters are $\mathbf{P} = [P_1, P_2] = [CL, LP]$, which are the piston-to-bore clearance (*CL*) and the location of combustion peak pressure (*LP*). As shown in Table 4, both *CL* and *LP* are normally distributed.

Place Table 4 here

The two QCs are

$$\mathbf{Y} = Y_i(\mathbf{d}, \mathbf{P})$$

where $i = 1, 2$; and $i = 1$ is for the slap noise while $i = 2$ is for the piston friction.

The specification limits of the noise and friction QCs are 65 dB and 8.0 N, respectively. If either QC or both QCs are at their specification limits, the quality loss is \$2000. The two quality loss constants are therefore $k_1 = 1000/65^2 = 0.2367$ \$/dB^2 $k_2 = 1000/8^2 = 15.625$ \$/N^2.

As in Example 1, to compare different methods, we give the following design models. The deterministic optimization model is given by Model 1 as

$$\begin{cases} \min\limits_{(\mathbf{d})} \ f = Y_1^2 + Y_2^2 \\ \text{subject to} \\ \quad 21 \le d_1 \le 25 \\ \quad 1 \le d_2 \le 3 \\ \quad 1 \le d_3 \le 3 \\ \quad 0.5 \le d_4 \le 1.3 \end{cases}$$

where the mean values of random parameters are used.

The robust design model with the summation of individual quality losses is formulated by Model 2 as

$$\begin{cases} \min\limits_{(\mathbf{d})} \ \sum E(L) = E(L_1) + E(L_2) \\ \text{subject to} \\ \quad 21 \le d_1 \le 25 \\ \quad 1 \le d_2 \le 3 \\ \quad 1 \le d_3 \le 3 \\ \quad 0.5 \le d_4 \le 1.3 \end{cases}$$

where $E(L_i) = k_i(\mu_{Y_i}{}^2 + \sigma_{Y_i}^2)$, and $i = 1, 2$.

The robust design model with the maximum quality loss is provided by Model 3 as

$$\begin{cases} \min\limits_{(\mathbf{d})} \ E(W) = E\left\{\max\left[L_1, L_2\right]\right\} \\ \text{subject to} \\ \quad 21 \le d_1 \le 25 \\ \quad 1 \le d_2 \le 3 \\ \quad 1 \le d_3 \le 3 \\ \quad 0.5 \le d_4 \le 1.3 \end{cases}$$

where $L_i = k_i Y_i^2$, and $i = 1, 2$.

Place Table 5 here

The deterministic optimization (Model 1) produced the largest expected maximum quality loss \$718.94, and RD with the sum of quality (Model 2) losses reduced the expected maximum quality loss to \$711.38. The proposed method (Model 3) generated the minimum expected maximum quality loss with the minimum value of \$654.53. The expected maximum quality losses from the three methods were calculated by the equations derived in Section 3. Their accuracy was checked by MCS with a sample size of 10^5, and the MCS solutions are in the last row of Table 5. The solutions from the proposed method are close to those from MCS.

6. CONCLUSIONS

This work investigates robust design that involves multiple quality characteristics for problems where the same quality loss occurs if at least one quality characteristic is defective. It is found that the maximum quality loss should be used to model the robustness. Then the probability distribution is derived for the maximum quality loss.

The methodology is applied to robust design with two quality characteristics based on the First Order Second Moment method. An analytical equation for the probability density function of the maximum quality loss is derived, and the expected maximum quality loss can then be evaluated numerically.

The efficiency of the robust design in this study depends on both the optimization algorithm used and the proposed robustness analysis. The proposed robustness analysis method is efficient because of the linear approximation of the quality characteristic function with respect to random variables. If the efficiency is measured by the number of function calls of the quality characteristics, then the efficiency is directly promotional to the number of random variables. More specifically, if the derivatives of the quality characteristic function are analytically available, the number of function call is only one for a robustness analysis; if the derivatives are evaluated numerically by the finite difference method, the number of function calls is the number of random variables plus one.

The accuracy of the proposed robustness analysis depends on the nonlinearity of the quality characteristic function and the degree of uncertainty in the random variables. The accuracy is higher for functions that are less nonlinear or for random variables with smaller standard deviations. For the two examples presented, the accuracy is good.

The analytical equations are derived for the smaller-the-better and the nominal-the-best quality characteristics and are for problems with two quality characteristics. Extending the method to the larger-the-better quality characteristics should be one direction of the future work. The other future work is the derivation of the probability density function of the maximum quality loss among more than two quality characteristics.

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List of Table Captions

List of Figure Caption

Table 1 Design variables and parameters

Variable	Distribution	Mean (mm)	STD (mm)
R_1	Normal	1000.0	2.0
R_2	Normal	μ_{R_2}	1.0
R_3	Normal	μ_{R_3}	1.0
R_4	Normal	μ_{R_4}	1.0

Table 2 Optimal design variables

Design variables	Model 1	Model 2	Model 3 (proposed)
μ_{R_2}	416.20 mm	440.96 mm	439.73 mm
μ_{R_3}	898.78 mm	853.52 mm	855.94 mm
μ_{R_4}	604.50 mm	688.28 mm	684.46 mm
θ_0	37.01°	66.49°	68.68°
ψ_0	38.83°	56.59°	57.45°

Table 3 Optimal points

	Model 1	Model 2	Model 3 (proposed)
μ_{ψ_1}	35.0°	35.0024°	35.0006°
μ_{ψ_2}	95.0°	94.9904°	94.9908°
σ_{ψ_1}	0.2877°	0.1978°	0.1965°
σ_{ψ_2}	0.270°	0.2956°	0.3006°
ρ	0.6604	0.6889	0.6992
$\sum E(L)$	\$1009.7	\$609.9	\$611.2
$E\{L_{\max}\}$	\$862.1	\$461.3	\$459.2
$E\{L_{\max}\}$ (MCS)	\$858.6	\$459.6	\$457.6

Table 4 Design variables and parameters

Variable	Distribution	Mean	STD
$d_1(SL)$	Deterministic	(mm)	-
$d_2(SP)$	Deterministic	/	-
$d_3(SO)$	Deterministic	/	-
$d_4(PO)$	Deterministic	(mm)	-
$P_1(CL)$	Normal	50 mm	3 mm
$P_2(LP)$	Normal	14.5°	1°

Table 5 Optimal results

	Model 1	Model 2	Model 3 (proposed)
μ_{f_1}	55.1133 dB	54.8231 dB	52.5863 dB
μ_{f_2}	2.9269 N	2.9812 N	4.81 N
σ_{f_1}	0.1628 dB	0.1304 dB	0.2320 dB
σ_{f_2}	0.2232 N	0.1879 N	0.0634 N
ρ	-0.6974	-0.7496	0.6051
$\sum E(L)$	\$853.57	\$850.81	\$1023.6
$E\{L_{\max}\}$	\$718.94	\$711.38	\$654.53
$E\{L_{\max}\}$ (MCS)	\$719.53	\$712.0	\$655.23

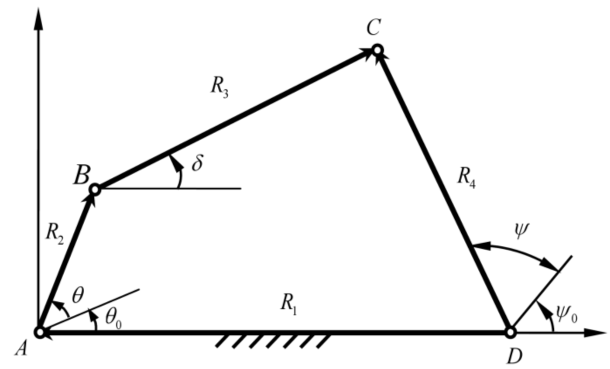

Figure 1 Four-bar linkage