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Time-Dependent Mechanism Reliability Analysis with Envelope Functions and First Order Approximation

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Abstract

This work develops an envelope approach to time-dependent mechanism reliability defined in a period of time where a certain motion output is required. Since the envelope function of the motion error is not explicitly related to time, the time-dependent problem can be converted into a time-independent problem. The envelope function is approximated by piecewise hyper-planes. To find the expansion points for the hyper-planes, the approach linearizes the motion error at the means of random dimension variables, and this approximation is accurate because the tolerances of the dimension variables are small. The expansion points are found with the maximum probability density at the failure threshold. The time-dependent mechanism reliability is then estimated by a multivariable normal distribution at the expansion points. As an example, analytical equations are derived for a four-bar function generating mechanism. The numerical example shows the significant accuracy improvement.

1. Introduction

One of the functions of a mechanism is to realize desired motion. For example, a functional relationship $\psi = \psi(\mathbf{X}, \theta)$ is required to map the motion input θ into the motion output ψ. The required motion can be achieved by determining the mechanism dimension variables $\mathbf{X}$. The required motion output, however, may not be realized exactly for two reasons. The limitations of a mechanism, such as its type, degree of freedom, and the number of synthesis points, can only allow for an approximation to the desired motion output. In addition, the randomness in the mechanism dimensions makes the actual motion output fluctuate around the nominal motion output. As a result, the motion error, which is the difference between the actual motion output and the required motion output [1-5], is inevitable.

The motion errors due to the aforementioned two reasons are structural error and random error, respectively [6]. The total motion error is the sum of the two. For brevity, we refer to the total motion error as the motion error.

There are many probabilistic methodologies for handling the random motion output [7-16], especially the reliability-based methodologies. The mechanism reliability is the probability of the output member's position or orientation falling within a specified range from the desired position and/or orientation [17]. Kinematic reliability has applied to a wide range of mechanisms [18-28].

For mechanism reliability analysis, the dominating methodologies are Monte Carlo simulation (MCS) and the First Order Second Moment (FOSM) method. MCS is accurate, but is computationally expensive. FOSM is less accurate but is much more efficient. For a linear function with normal random variables, the solution from FOSM is exact. The dimension variables of a mechanism are commonly assumed following normal distributions. In this work, we also use normal distributions for dimensional variables. The tolerances of the dimension variables are small, so are the standard deviations of the dimension variables. The motion error is therefore near linear with respect to the dimension variables in the vicinity of the means of the dimension variables. As a result, the accuracy of FOSM is satisfactory. It is the reason why FOSM is widely used for mechanism reliability analysis and synthesis [9, 18, 19, 21].

The above methods, however, are only for the point reliability, which provides us with the likelihood of realizing the desired function only at a specific time instant θ regardless whether the function has been realized or not prior to that instant. In many applications, it is more important to know the probability of realizing the desired function over a range (time period) of the input motion $[\theta_0, \theta_e]$ where the desired function is defined. Such a probability is called the time-dependent reliability in the literature of structural reliability. More precisely, the probability may be terms as the interval reliability because it is defined on a time interval.

The most popular time-dependent reliability method, the Rice's formula, has been recently introduced into mechanism reliability analysis [29]. The method is based on the concept of upcrossing, which is an event that the motion error exceeds the failure threshold at a time instant. The method assumes that all the upcrossings during the period of time under consideration are independent. The method is efficient and is also accurate when the upcrossings are close to independent or the upcrossings are rare. When there are multiple dependent upcrossings during the period of time, the error is large and usually much larger than the true value.

In this work, we develop a new method to improve the accuracy of the time-dependent mechanism reliability analysis. Basics of the mechanism error are given in Section 2, and the Rice's formula is also reviewed. In Section 3, the new method is introduced, followed by the full development of the reliability analysis for four-bar function generating mechanisms in Section 4. A numerical example is provided in Section 5. Conclusions are made in Section 6.

2. Background

In this section, we review the definitions of the motion error and mechanism reliability.

2.1 Motion error

The motion error is the difference between the actual motion output ψ and the desired motion output ψ_d. It is given by

$$g(\mathbf{X},\theta) = \psi(\mathbf{X},\theta) - \psi_d(\theta) \tag{1}$$

where $\mathbf{X} = (X_i)_{i=1,n}$ is an *n*-dimensional random vector, which consists of the dimension variables of the mechanism. $\psi(\mathbf{X},\theta)$ and $\psi_d(\theta)$ are the actual motion output and desired motion output, respectively.

Eq. (1) contains both the structural and random errors. To ensure the mechanism work properly, the motion error must be less than the allowable error ε. Due to the uncertainty in the dimension variables, the requirement may not be satisfied completely. The motion reliability is used to quantify the probability of the satisfaction. The reliability is measured by the probability that the desired function is realized within the specified error ε over the range of the input motion $[\theta_0, \theta_e]$. It is evaluated by

$$R(\theta_0, \theta_e) = \Pr\left\{\left|g(\mathbf{X}, \theta)\right| \leqslant \varepsilon,\ \forall \theta \in [\theta_0, \theta_e]\right\} \tag{2}$$

The reliability is defined on a time interval and is time dependent. The time-dependent probability of failure is

$$p_f(\theta_0, \theta_e) = \Pr\left\{\left|g(\mathbf{X}, \theta)\right| > \varepsilon,\ \exists \theta \in [\theta_0, \theta_e]\right\} \tag{3}$$

Many time-dependent reliability methodologies are available in the area of structural reliability. Time-dependent structural reliability problems involve time-dependent factors, such as decaying material properties and randomly varying load in time [30]. There are two basic types of methods for time-dependent reliability: extreme value methods [31-35] and first-passage methods [30, 33, 36-42].

An extreme value method uses the global extreme values of the performance function under consideration. A failure occurs when the extreme value on the time interval is greater than or less than the threshold. If the distribution of the extreme value is available, a time-independent reliability method can be used to solve for the time-dependent reliability [31, 32].

A first-passage method is based on the first time when the performance exceeds or falls below a threshold. The method usually calculates the rate (upcrossing rate or downcrossing rate) of the likelihood that the performance exceeds or falls below the threshold. The most commonly used method is the Rice's formula [43] as previously mentioned. It is difficult to obtain the crossing rate for general stochastic processes [44]. Many methods focus on the asymptotic solutions for the crossing rate [36-39]. For special stochastic processes, such as a stationary Gaussian processes, an analytical outcrossing rate is available [33]. A new analytical derivation of the crossing rate has also been reported [30] for general stochastic processes. This method is based on the First Order Reliability Method (FORM).

Mechanism reliability problems are different from structural reliability problems. The error function $g(\mathbf{X},\theta)$ does not directly involve any stochastic processes in its input, and the random dimension variables $\mathbf{X}$ are time independent. However, as $g(\mathbf{X},\theta)$ is a function of the time factor θ, it is still a stochastic process. Since the actual motion output $\psi(\mathbf{X},\theta)$ and the desired motion output $\psi_d(\theta)$ are generally nonlinear functions of θ, the moment functions of $g(\mathbf{X},\theta)=\psi(\mathbf{X},\theta)-\psi_d(\theta)$ are also time dependent. As a result, $g(\mathbf{X},\theta)$ is a non-stationary stochastic process. The autocorrelation of $g(\mathbf{X},\theta)$ at two different time instants is also usually high and may not decay with respect to time. The nonstationality and high autocorrelation of the error function make the reliability analysis complicated. The common structural reliability methods for stationary processes or special processes are not applicable for mechanism reliability problems.

The Rice's formula has been recently introduced into mechanism reliability analysis [29]. As mentioned previously, the method is efficient, but not accurate if the independent crossing assumption does not hold. The appendix reviews the application of the Rice formula in the mechanism reliability analysis.

3 The Envelope Methodology

To improve the accuracy of mechanism reliability analysis, we propose an envelope method. The method uses the envelope function of the motion error $g(\mathbf{X}, \theta)$ on a time interval $[\theta_0, \theta_e]$. Once a specific time interval $[\theta_0, \theta_e]$ is given, the envelope function is time independent. We then denote the envelope function by $G(\mathbf{X})$. If we use $G(\mathbf{X})$ for the reliability analysis, the problem will be time independent. We can then convert time-dependent reliability analysis into time-independent one.

3.1 The envelope function

We now generate the envelope functions $G^{+}(\mathbf{X}) = 0$ and $G^{-}(\mathbf{X}) = 0$ for failure boundaries $g(\mathbf{X}, \theta) = \varepsilon$ and $g(\mathbf{X}, \theta) = -\varepsilon$, respectively. We start from $G^{+}(\mathbf{X})$. $g(\mathbf{X}, \theta) = \varepsilon$ can be considered as a parametric function where θ is the parameter. The function represents a family of hyper-surfaces when θ varies on $[\theta_0, \theta_e]$. The envelope $G^{+}(\mathbf{X}) = 0$ corresponds to $g(\mathbf{X}, \theta) = \varepsilon$ and always keeps in touch with or is tangent to each member of $g(\mathbf{X}, \theta) = \varepsilon$. A point on $G^{+}(\mathbf{X}) = 0$ can be considered as the intersection of two adjacent (close enough) hyper-surface of $g(\mathbf{X}, \theta) = \varepsilon$.

Let two hyper-surfaces be $g(\mathbf{X}, \theta) = \varepsilon$ and $g(\mathbf{X}, \theta + \delta\theta) = \varepsilon$, where $\delta\theta > 0$, we have

$$\frac{g(\mathbf{X}, \theta + \delta\theta) - g(\mathbf{X}, \theta)}{\delta\theta} = 0 \tag{4}$$

Letting $\delta\theta \to 0$, we obtain

$$\frac{\partial g(\mathbf{X}, \theta)}{\partial\theta} = 0 \text{ or } \dot{g}(\mathbf{X}, \theta) = 0 \tag{5}$$

where the dot means the time derivative $\frac{\partial}{\partial\theta}(\cdot)$.

$G^{+}(\mathbf{X})$ is then determined by

$$\begin{cases} g(\mathbf{X}, \theta) = \varepsilon \\ \dot{g}(\mathbf{X}, \theta) = 0 \end{cases} \tag{6}$$

A point on the envelope corresponds to a time instant where the motion error is ε and the velocity error is zero because $\dot{g}(\mathbf{X},\theta) = \dot{\psi}(\mathbf{X},\theta) - \dot{\psi}_d(\theta) = 0$ where $\dot{\psi}(\mathbf{X},\theta)$ and $\dot{\psi}_d(\theta)$ are the actual output velocity and desired output velocity, respectively. θ can be solved from either equation, and it is then plugged into the other equation. For example, θ is obtained from the second equation as

$$\theta = \dot{g}^{-1}(\mathbf{X}) \tag{7}$$

where $\dot{g}^{-1}(\cdot)$ is the inverse function of $\dot{g}(\cdot)$ with respect to θ . Then from the first equation, we have

$$g(\mathbf{X}, \dot{g}^{-1}(\mathbf{X})) = \varepsilon \tag{8}$$

$G^{+}(\mathbf{X})$ is then given by

$$G^{+}(\mathbf{X}) = g(\mathbf{X}, \dot{g}^{-1}(\mathbf{X})) - \varepsilon = 0 \tag{9}$$

It is time independent now.

Similarly, the other envelop $G^{-}(\mathbf{X}) = 0$ is determined by

$$\begin{cases} g(\mathbf{X}, \theta) = -\varepsilon \\ \dot{g}(\mathbf{X}, \theta) = 0 \end{cases} \tag{10}$$

After the envelope functions are found, the time-dependent reliability is given by

$$R(\theta_0, \theta_e) = \Pr\{G^{+}(\mathbf{X}) < 0 \cap G^{-}(\mathbf{X}) > 0\} \tag{11}$$

In the above equation, event $G^{+}(\mathbf{X}) < 0 \cap G^{-}(\mathbf{X}) > 0$ is the intersection of two events, which are the event of the maximum motion error being less than ε and the event of the minimum motion error being greater than $-\varepsilon$. The analysis is now converted to a time-independent problem. Directly obtaining the envelope function in Eq. (9) is difficult because it is hard to get the inverse function $\dot{g}^{-1}(\cdot)$. Next, we derive equations for $G^{+}(\mathbf{X})$ and $G^{-}(\mathbf{X})$ with approximations.

3.2 Approximate the envelope functions

A time-independent reliability analysis method can be used to solve for the reliability in Eq. (11). The popular mechanism reliability methods are the first order methods, such as the first order second moment (FOSM) method and the first order reliability method (FORM). Both methods linearize the performance function at a single point. Since an envelope function may be highly nonlinear, a single expansion point may not be good enough. We then approximate the envelope function with piece-wise linear hyper-planes at multiple expansion points.

As described previously, we assume that the dimension variables are independently and normally distributed. Since the standard deviations of the dimension variables are small, we approximate the motion error by the first order Taylor expansion series at the means $\boldsymbol{\mu}$ of $\mathbf{X}$ as

$$g(\mathbf{X}, \theta) \approx a_0(\theta) + \mathbf{a}(\theta) \cdot (\boldsymbol{\mu} - \mathbf{X}) \tag{12}$$

where

$$a_0(\theta) = \psi(\boldsymbol{\mu}, \theta) - \psi_d(\theta) \tag{13}$$

$$\mathbf{a}(\theta) = \left(\left. \frac{\partial \psi}{\partial X_i} \right|_{\boldsymbol{\mu}} \right)_{i=1,n} \tag{14}$$

The first term on the right-hand side of Eq. (12) is the motion error when uncertainty is not considered and is evaluated at the means of random variables. It gives the structural motion error. The second terms is the random motion error caused by the deviation of the random variables from their means.

To make the derivation easy, we transform $\mathbf{X}$ into a vector $\mathbf{U}$ consisting of standard normal random variables by

$$U_i = \frac{X_i - \mu_i}{\sigma_i} \tag{15}$$

where σ_i is the standard deviation of X_i. Then

$$g(\mathbf{X}, \theta) \approx L(\mathbf{U}, \theta) = b_0(\theta) + \mathbf{b}(\theta) \cdot \mathbf{U} \tag{16}$$

where

$$b_0(\theta) = a_0(\theta) \tag{17}$$

$$\mathbf{b}(\theta) = \left(b_i(\theta) \right)_{i=1,n} = \left(a_i(\theta)\sigma_i \right)_{i=1,n} \tag{18}$$

The approximation is accurate because the variations around $\boldsymbol{\mu}$ are small. The reason is that the tolerance of a dimension variable is small.

The task now becomes to find the envelope functions $G^{+}(\mathbf{U})$ and $G^{-}(\mathbf{U})$ for $L(\mathbf{U},\theta)=\varepsilon$ and $L(\mathbf{U},\theta)=-\varepsilon$, respectively.

According to Eq. (6), $G^{+}(\mathbf{U}) = 0$ is given by

$$\begin{cases} L = b_0(\theta) + \mathbf{b}(\theta)\cdot\mathbf{U} = \varepsilon \\ \dot{L} = \dot{b}_0(\theta) + \dot{\mathbf{b}}(\theta)\cdot\mathbf{U} = 0 \end{cases} \tag{19}$$

$G^{+}(\mathbf{U}) = 0$ becomes an envelope of a family of linear functions $L(\mathbf{U},\theta)=\varepsilon$. It is the envelope that encloses all the failure regions at all the time instants on $[\theta_0, \theta_e]$. The expansion points of $G^{+}(\mathbf{U}) = 0$ should be close to the origin $\mathbf{U} = 0$ while keep in touch with $L(\mathbf{U},\theta)=\varepsilon$. They should therefore come from the closest points of $L(\mathbf{U},\theta)=\varepsilon$ to $\mathbf{U} = 0$ at some time instants. Suppose a closest point is $\mathbf{U}$ at instant θ. For the linear function $L(\mathbf{U},\theta)=\varepsilon$, $\mathbf{U}$ is perpendicular to $L(\mathbf{U},\theta)=\varepsilon$ since the distance between $\mathbf{U}$ and the original is the shortest; in other words, $\mathbf{U}$ and the gradient of L at $\mathbf{U}$ are collinear. Therefore

$$\mathbf{U} = c\frac{\mathbf{b}(\theta)}{\sqrt{\mathbf{b}(\theta)\cdot\mathbf{b}(\theta)}} \tag{20}$$

where c is a constant, and the second term on the right-hand side is a unit vector in the direction of the gradient. $G^{+}(\mathbf{U}) = 0$ should satisfy both Eq. (20) and $G^{+}(\mathbf{U}) = 0$. From the first line of Eq. (19),

$$b_0(\theta) + c\frac{\mathbf{b}(\theta)\cdot\mathbf{b}(\theta)}{\sqrt{\mathbf{b}(\theta)\cdot\mathbf{b}(\theta)}} = \varepsilon \tag{21}$$

This gives

$$c = \frac{\varepsilon - b_0(\theta)}{\sqrt{\mathbf{b}(\theta)\cdot\mathbf{b}(\theta)}} \tag{22}$$

Then

$$\mathbf{U} = \frac{\left[\varepsilon - b_0(\theta)\right]\mathbf{b}(\theta)}{\mathbf{b}(\theta)\cdot\mathbf{b}(\theta)} \tag{23}$$

Plugging it into the second line of Eq. (19) yields

$$\dot{b}_0(\theta) + \left(\varepsilon - b_0\right)\frac{\dot{\mathbf{b}}(\theta)\cdot\mathbf{b}(\theta)}{\mathbf{b}(\theta)\cdot\mathbf{b}(\theta)} = 0 \tag{24}$$

Eq. (24) is an equation with a single variable θ. There may be multiple solutions for θ. As indicated in Eq. (6), the motion error at θ should be positive. (If no positive solutions could be found, then $G^{+}(\mathbf{U}) = 0$ does not exist.) We then calculate the motion error at θ and eliminate those solutions where the motion errors are negative. Let the remaining solutions be θ_i^{+}, where $i = 1, 2, \ldots, m^{+}$. The expansion points are then

$$\mathbf{U}(\theta_i^{+}) = \frac{\left[\varepsilon - b_0(\theta_i^{+})\right]\mathbf{b}(\theta_i^{+})}{\mathbf{b}(\theta_i^{+})\cdot\mathbf{b}(\theta_i^{+})} \tag{25}$$

where $i = 1, 2, \ldots, m^{+}$.

The other envelope $G^{-}(\mathbf{U})$ is given by

$$\begin{cases} L = b_0(\theta) + \mathbf{b}(\theta)\cdot\mathbf{U} = -\varepsilon \\ \dot{L} = \dot{b}_0(\theta) + \dot{\mathbf{b}}(\theta)\cdot\mathbf{U} = 0 \end{cases} \tag{26}$$

With the same principle, we have the following equation for the expansion points:

$$\dot{b}_0(\theta) - \left(\varepsilon + b_0\right)\frac{\dot{\mathbf{b}}(\theta)\cdot\mathbf{b}(\theta)}{\mathbf{b}(\theta)\cdot\mathbf{b}(\theta)} = 0 \tag{27}$$

After obtaining the solutions to the above equation, we eliminate those solutions where the motion errors are positive. Denote the remaining solutions by θ_i^-, where $i = 1, 2, \ldots, m^-$. The expansion points are then

$$\mathbf{U}(\theta_i^-) = -\frac{\left[\varepsilon + b_0(\theta_i^-)\right]\mathbf{b}(\theta_i^-)}{\mathbf{b}(\theta_i^-)\cdot\mathbf{b}(\theta_i^-)} \tag{28}$$

where $i = 1, 2, \ldots, m^-$.

Since b_0, $\mathbf{b}$, $\dot{b}_0$ and $\dot{\mathbf{b}}$ are the outputs of a mechanism analysis and may also be analytically available, numerically finding θ_i^+ and θ_i^- are not difficult. The curves of Eqs. (24) and (27) can also be displayed, and it is therefore easy to find all the solutions for θ_i^+ and θ_i^-.

The envelope function $G^+ = 0$ can now be approximated by hyper-planes $L(\mathbf{U}(\theta_i^+)) = \varepsilon$ where $i = 1, 2, \cdots, m^+$. Likewise, $G^- = 0$ can be approximated by hyperplanes $L(\mathbf{U}(\theta_i^-)) = -\varepsilon$ where $i = 1, 2, \cdots, m^-$. Hence the time-dependent reliability is calculated by

$$R = \Pr\left\{\left[\bigcap_{i=1}^{m^+} L(\mathbf{U}, \theta_i^+) < \varepsilon\right] \cap \left[\bigcap_{i=1}^{m^-} L(\mathbf{U}, \theta_j^-) > -\varepsilon\right]\right\} \tag{29}$$

Also considering the two end points θ_0 and θ_e of the time interval, we define a sign function

$$s(\theta_i) = \begin{cases} +1 & \text{if } \theta = \theta_i^+ \text{ or } L(\mathbf{U}, \theta) \geq 0 \text{ when } \theta = \theta_0 \text{ or } \theta_e \\ -1 & \text{if } \theta = \theta_i^- \text{ or } L(\mathbf{U}, \theta) < 0 \text{ when } \theta = \theta_0 \text{ or } \theta_e \end{cases} \tag{30}$$

Then

$$R = \Pr\left\{ \bigcap_{i=1}^{m} s(\theta_i) L(\mathbf{U}, \theta_i) < \varepsilon_i \right\} \tag{31}$$

where θ_i includes θ_0, θ_i^+, θ_i^-, and θ_e, $m = m^+ + m^- + 2$, and $\varepsilon_i = s_i \varepsilon$

3.3 Estimate the reliability

As shown in Eq. (16), the approximated motion error $L(\mathbf{U}, \theta)$ is normally distributed, and so is $s(\theta_i) L(\mathbf{U}, \theta_i)$ in Eq. (31). The reliability can therefore be estimated by a multivariate normal distribution function with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$, or $\Phi_m(\boldsymbol{\varepsilon}, \boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = (s(\theta_i)\mu_L(\theta_i))_{i=1,m} = (s(\theta_i) b_0(\theta_i))_{i=1,m} \tag{32}$$

$$\boldsymbol{\Sigma} = (\sigma_{ij})_{i,j=1,m} \tag{33}$$

where σ_{ij} is the covariance between $s(\theta_i) L(\mathbf{U}, \theta_i)$ and $s(\theta_j) L(\mathbf{U}, \theta_j)$.

According to Eq. (19)

$$\sigma_{ij} = s(\theta_i) s(\theta_j) b(\theta_i) b(\theta_j) \tag{34}$$

To be a valid covariance matrix, $\boldsymbol{\Sigma}$ should be a positive definite matrix. In other words, the rank of $\boldsymbol{\Sigma}$ should be equal to m. This means that $L(\mathbf{U}, \theta_i)$ $(i = 1, 2, \ldots, m)$ should be independent. If the requirement is not satisfied, not all the time instants are needed. Fig. 1 shows such a situation where three expansion points $\mathbf{U}(\theta_1^+)$, $\mathbf{U}(\theta_2^+)$, and

$\mathbf{U}(\theta_3^+)$ are found for envelope function $L(\mathbf{U})=\varepsilon$. The three lines of $L(\mathbf{U})=\varepsilon$ expanded at the three expansion points are plotted. The joint failure event $L(\mathbf{U}(\theta_1^+))<\varepsilon \cap L(\mathbf{U}(\theta_2^+))<\varepsilon$ is equivalent to the safety event $L(\mathbf{U}(\theta_1^+))<\varepsilon \cap L(\mathbf{U}(\theta_2^+))<\varepsilon \cap L(\mathbf{U}(\theta_3^+))<\varepsilon$. It is also seen that the probability of failure $\Pr\left\{L(\mathbf{U}(\theta_3^+))>\varepsilon\right\}$ is minimum because the line expanded at $\mathbf{U}(\theta_3^+)$ or the failure boundary is the farthest from the original. The expansion point $\mathbf{U}(\theta_3^+)$ is redundant in calculating the reliability because $L(\mathbf{U}(\theta_1^+))<\varepsilon \cap L(\mathbf{U}(\theta_2^+))$ leads to $L(\mathbf{U}(\theta_3^+))<\varepsilon$. In other words, $L(\mathbf{U}(\theta_1^+)$, $L(\mathbf{U}(\theta_2^+)$, and $L(\mathbf{U}(\theta_3^+)$ are dependent. If $\mathbf{U}(\theta_3^+)$ is kept, the covariance matric $\mathbf{\Sigma}$ will not be positive definite. $\mathbf{U}(\theta_3^+)$ should be eliminated, and the envelope function can then be approximated by only the two lines passing through $\mathbf{U}(\theta_1^+)$ and $\mathbf{U}(\theta_2^+)$, respectively. Since the maximum number of the time instants should be equal to the rank of $\mathbf{\Sigma}$, letting the rank be r, we need to eliminate $m-r$ time instants.

Note that although keeping $\mathbf{U}(\theta_3^+)$ and eliminating another expansion point can also make the covariance matrix positive definite, the approximated envelope will not be accurate and may result in a large error in the reliability analysis.

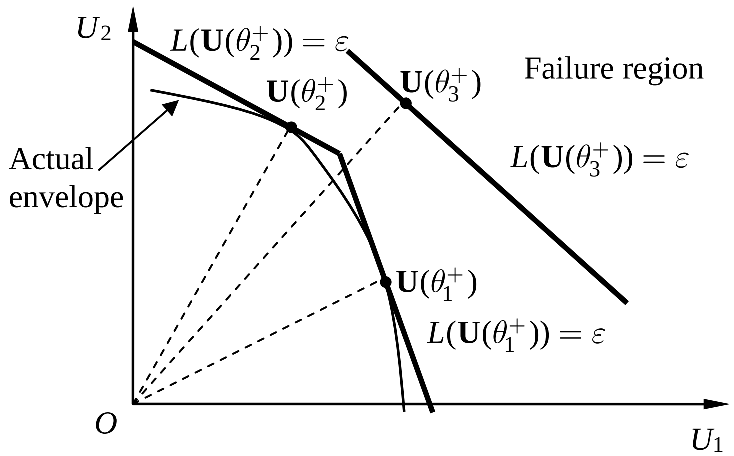

Fig. 1 Expansion points

For the reason explained above, we eliminate those time instants that have smallest point probabilities of failure. The point probability of failure is calculated by

$$\begin{aligned} p_f(\theta_i) &= \Pr\{s(\theta_i)L(\mathbf{U},\theta_i) > \varepsilon\} \\ &= \Pr\{s(\theta_i)[b_0(\theta_i) + \mathbf{b}(\theta)\mathbf{U}] > \varepsilon\} \\ &= 1 - \Phi\left\{\frac{s(\theta_i)[\varepsilon - b_0(\theta_i)]}{\sqrt{\mathbf{b}(\theta_i)\cdot\mathbf{b}(\theta_i)}}\right\} \end{aligned} \tag{35}$$

After $p_f(\theta_i)$, where $i = 1, 2, \cdots, m$, are calculated, we sort $p_f(\theta_i)$ with a decreasing order. This produces $p_f(\theta_i^{'})$, where $i = 1, 2, \cdots, m$. Then we keep the first r instants $\theta_i^{'}$, where $i = 1, 2, \cdots, r$. The new mean vector is

$$\boldsymbol{\mu}^{'} = (s(\theta_i^{'})\, b_0(\theta_i^{'})) \tag{36}$$

The new covariance is

$$\boldsymbol{\Sigma}^{'} = (\sigma_{ij}^{'}) \tag{37}$$

where $\sigma_{ij}^{'} = s(\theta_i^{'})s(\theta_j^{'})b(\theta_i^{'})b(\theta_j^{'})$.

The multivariate normal cumulative distribution function (CDF) $\Phi_r(\boldsymbol{\varepsilon}^{'}, \boldsymbol{\mu}^{'}, \boldsymbol{\Sigma}^{'})$ for the reliability can be calculated by the following integral with a numerical algorithm:

$$R(\theta_0, \theta_e) = \int_0^{\boldsymbol{\varepsilon}^{'}} \frac{1}{(2\pi)^{r/2}\left|\boldsymbol{\Sigma}^{'}\right|} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}^{'})\left(\boldsymbol{\Sigma}^{'}\right)^{-1}(\mathbf{x}-\boldsymbol{\mu}^{'})^{T}\right\} d\mathbf{x} \tag{38}$$

3.4 Numerical Procedure

We at first solve for the time instants $\theta_i^{'}$ for $G^{+}(\mathbf{U}) = 0$ with Eq. (24). This requires to call the mechanism analysis to obtain $b_0(\theta)$, $\dot{b}_0(\theta)$, $\mathbf{b}(\theta)$, and $\dot{\mathbf{b}}(\theta)$. All these variables may be analytically available. Likewise we can also obtain $\theta_i^{'}$ for $G^{-}(\mathbf{U}) = 0$ with Eq. (27).

Then in the second step, we calculate the mean and covariance matrix $\boldsymbol{\Sigma}$ with a size of $m \times m$ using Eqs. (32) and (34). If the rank r of $\boldsymbol{\Sigma}$ is less than m, we eliminate $m - r$ time instants where the point probabilities of failure are the least. A new covariance matrix $\boldsymbol{\Sigma}^{'}$ with a reduced size can then be found. We also calculate the associated mean vector $\boldsymbol{\mu}^{'}$.

In the last step, we evaluate the multivariate normal CDF. Since the dimensions of the CDF are not high for mechanisms, the CDF can be easily obtained with a numerical method.

4 Four-Bar Function Generating Mechanisms

We now use the proposed method for the reliability analysis of a four-bar function generating mechanism as shown in Fig. 2.

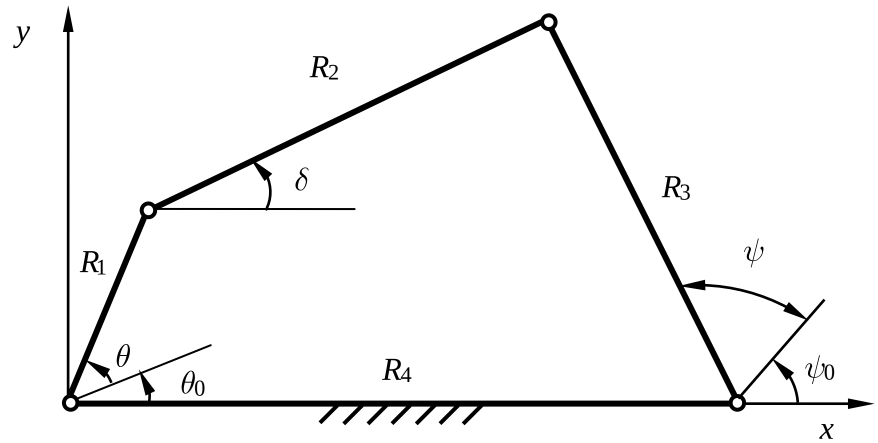

Fig. 2 Four-bar function generator mechanism

The dimension variables are $\mathbf{X}=(R_1, R_2, R_3, R_4)$. The motion output is derived using the following loop equations:

$$\begin{cases} R_1\cos\theta + R_2\cos\delta - R_3\cos\psi - R_4 = 0 \\ R_1\sin\theta + R_2\sin\delta - R_3\cos\psi = 0 \end{cases} \tag{39}$$

Solving for the two unknowns ψ and δ, we obtain

$$\psi = 2\arctan\frac{A \pm \sqrt{A^2+B^2-C^2}}{B+C} \tag{40}$$

where

$$A = -2R_1R_3\sin\theta \tag{41}$$

$$B = 2R_3(R_4 - R_1\cos\theta) \tag{42}$$

$$C = R_2^2 - R_1^2 - R_3^2 - R_4^2 + 2R_1R_4\cos\theta \tag{43}$$

$$\delta = \arctan \frac{R_3 \sin\psi - R_1 \sin\theta}{R_4 + R_3 \cos\psi - R_1 \cos\theta} \tag{44}$$

Vector $\mathbf{b} = (b_1, b_2, b_3, b_4)$ is obtained by taking derivatives of ψ.

$$b_i = a_i \sigma_i \tag{45}$$

where

$$a_1 = \frac{\partial\psi}{\partial R_1} = \frac{\cos(\delta - \theta)}{R_3 \sin(\delta - \psi)} \tag{46}$$

$$a_2 = \frac{\partial\psi}{\partial R_2} = \frac{1}{R_3 \sin(\delta - \psi)} \tag{47}$$

$$a_3 = \frac{\partial\psi}{\partial R_3} = -\frac{\cos(\psi - \delta)}{R_3 \sin(\delta - \psi)} \tag{48}$$

$$a_4 = \frac{\partial\psi}{\partial R_4} = \frac{\cos\delta}{R_3 \sin(\psi - \delta)} \tag{49}$$

The derivative vector $\dot{\mathbf{b}} = (\dot{b}_1, \dot{b}_2, \dot{b}_3, \dot{b}_4)$ is then given by

$$\dot{b}_i = \dot{a}_i \sigma_i \tag{50}$$

where

$$\dot{a}_1 = \frac{\sin(\delta - \theta)[R_2 \sin(\delta - \psi) - R_1 \sin(\psi - \theta)]}{R_2 R_3 \sin(\delta - \psi)^2} \tag{51}$$

$$\dot{a}_2 = \frac{R_1[R_3 \sin(\psi - \theta) - R_2 \sin(\delta - \theta)]}{R_2 R_3^2 \sin(\delta - \psi)} \tag{52}$$

$$\dot{a}_3 = \frac{R_1[R_3 \sin(\psi - \theta) - R_2 \sin(\delta - \theta)]}{R_2 R_3^2 \sin^3(\delta - \psi)} \tag{53}$$

$$\dot{a}_4 = \frac{R_1}{R_2 R_3^3 \sin^2(\delta - \psi)}[R_3 \sin(\psi - \theta)\cos\delta\cos(\delta - \psi) - R_2 \sin(\delta - \theta)\cos\delta\cos(\delta - \psi) + R_3 \sin(\delta - \psi)\sin(\psi - \theta)\sin\psi] \tag{54}$$

All the equations for solving for the time instants for the expansion points are now analytically available. Then the procedure in Sec. 3.4 can be followed to calculate the time-dependent reliability.

5 A Numerical Example

A four-bar linkage mechanism shown in Fig. 2 is required to achieve the following function:

$$\psi_d(\theta) = 76^\circ + 60^\circ \sin\left[\frac{3}{4}(\theta - 95.5^\circ)\right]$$

on $[\theta_0, \theta_e]$, where $\theta_0 = 95.5^\circ$ and $\theta_e = 215.5^\circ$. The distributions of the dimension variables are given in Table 1.

Table 1 Distributions of dimensions variables

Variables	Mean (mm)	Standard deviation (mm)	Distribution
R_1	$\mu_1 = 53.0$	$\sigma_1 = 0.1$	Normal
R_2	$\mu_2 = 122.0$	$\sigma_2 = 0.1$	Normal
R_3	$\mu_3 = 66.5$	$\sigma_3 = 0.1$	Normal
R_4	$\mu_4 = 100.0$	$\sigma_4 = 0.1$	Normal

Both the envelope method and the Rice's formula were used to calculate the probability of failure on $[\theta_0, \theta_e]$ for various failure thresholds ε. To compare the accuracy, we also performed Monte Carlo simulation (MCS) with a sample size of 10^7. Given such a large sample size, the MCS solution is regarded as accurate.

Fig. 3 shows the motion error at the means of dimension variables. The motion error takes both positive and negative values.

We now show the result for the threshold $\varepsilon = 0.4°$. The time instants for the expansion points for $G^{+} = 0$ were found to be $122.982°$ and $186.9673°$. Since at the former instant the motion error is positive while at the latter instant the motion error is negative, we keep the former; and therefore $\theta_1^{+} = 122.982°$. The time instants for the expansion points for $G^{-} = 0$ were found to be $122.1041°$ and $186.8522°$. Since at the former instant the motion error is positive while at the latter instant the motion error is negative, we keep the latter, and therefore $\theta_1^{-} = 186.8522°$.

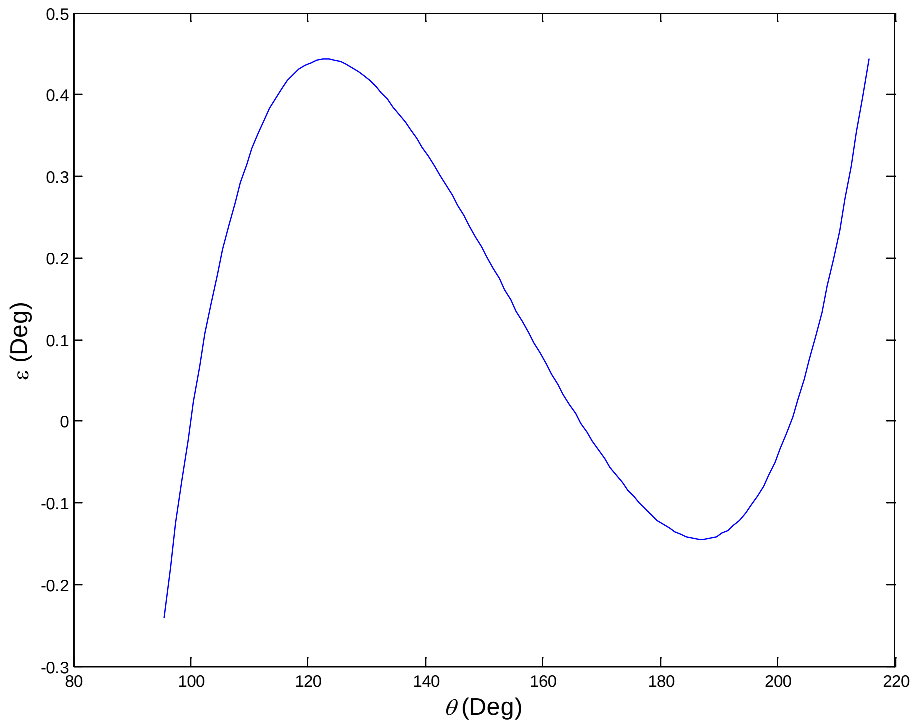

Fig. 3 Motion error at the means of dimension variables

Considering the two endpoints of the time interval $[95.5^{\circ}, 215.5^{\circ}]$, we obtained four time instants, $\theta_0 = 95.5^{\circ}$, $\theta_1^+ = 122.982^{\circ}$, $\theta_1^- = 186.8522^{\circ}$, and $\theta_e = 215.5^{\circ}$. By calculating the motion errors at the two endpoints, we obtained all the sign functions $s(\theta_0) = -1$, $s(\theta_1^+) = 1$, $s(\theta_1^-) = -1$, and $s(\theta_e) = 1$. The mean vector of the motion errors at the above time instants is $\boldsymbol{\mu} = (-0.2399^{\circ}, 0.4427^{\circ}, -0.1444^{\circ}, 0.4444^{\circ})$. The covariance matrix at the four time instants is

$$\boldsymbol{\Sigma} = \begin{pmatrix} 0.5373 & -0.4787 & 0.4466 & -0.4193 \\ -0.4787 & 0.4823 & -0.5235 & 0.4933 \\ 0.4466 & -0.5235 & 0.6576 & -0.6262 \\ -0.4193 & 0.4933 & -0.6262 & 0.6028 \end{pmatrix} \times 10^{-5}$$

The rank of the matrix is $r = 3$, less than the dimension of the matrix, which is four. Hence we needed to eliminate one time instant. The point probabilities of failures were calculated. They are $p_f(\theta_0) = 0.1139$, $p_f(\theta_1^+) = 0.6342$, $p_f(\theta_1^-) = 0.0411$, and $p_f(\theta_e) = 0.6237$. Eliminating the least probability of failure or the instant θ_1^-, we obtained $(\theta_1', \theta_2', \theta_3') = (95.5^{\circ}, 122.982^{\circ}, 215.5^{\circ})$ and $(s(\theta_1'), s(\theta_2'), s(\theta_3')) = (-1, 1, 1)$. The mean vector with the reduced size is $\boldsymbol{\mu}' = (0.4444^{\circ}, 0.4427^{\circ}, -0.2399^{\circ})$, and the covariate matrix with the reduced size is

$$\boldsymbol{\Sigma}' = \begin{pmatrix} 0.6028 & 0.4933 & -0.4193 \\ 0.4933 & 0.4823 & -0.4787 \\ -0.4193 & -0.4787 & 0.5373 \end{pmatrix} \times 10^{-5}$$

Then the time-dependent reliability was calculated with a three-dimensional integral or a three-dimensional normal CDF given in Eq. (38) where $\varepsilon_1 = -0.4^\circ$, $\varepsilon_2 = 0.4^\circ$, and $\varepsilon_3 = 0.4^\circ$. The integral is given by

$$R(\theta_0,\theta_e) = \int_0^{-0.4^\circ}\int_0^{0.4^\circ}\int_0^{0.4^\circ} \frac{1}{(2\pi)^{3/2}\left|\mathbf{\Sigma}'\right|} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}')\left(\mathbf{\Sigma}'\right)^{-1}(\mathbf{x}-\boldsymbol{\mu}')^T\right\} dx_1 dx_2 dx_3$$

where $\mathbf{x} = (x_1, x_2, x_3)$.

The results for all the error thresholds are shown in Table 2.

Table 2 Probability of failure $p_f(95.5^\circ, 215.5^\circ)$

ε	Rice	Envelope	MCS
0.4°	9.9043×10^{-1}	7.9536×10^{-1}	8.0842×10^{-1}
0.5°	8.9227×10^{-1}	4.2287×10^{-1}	4.2760×10^{-1}
0.6°	5.0934×10^{-1}	1.5858×10^{-1}	1.5975×10^{-1}
0.7°	1.2874×10^{-1}	3.9466×10^{-2}	3.9769×10^{-2}
0.8°	1.5408×10^{-2}	6.2935×10^{-3}	6.3632×10^{-3}
0.9°	1.0173×10^{-3}	6.3494×10^{-4}	6.4310×10^{-4}

The probabilities are also plotted in Fig. 4. The results show that the Rice's formula has large errors when the failure threshold is low or the reliability is low. The envelope method produced very accurate results for all the failure threshold values.

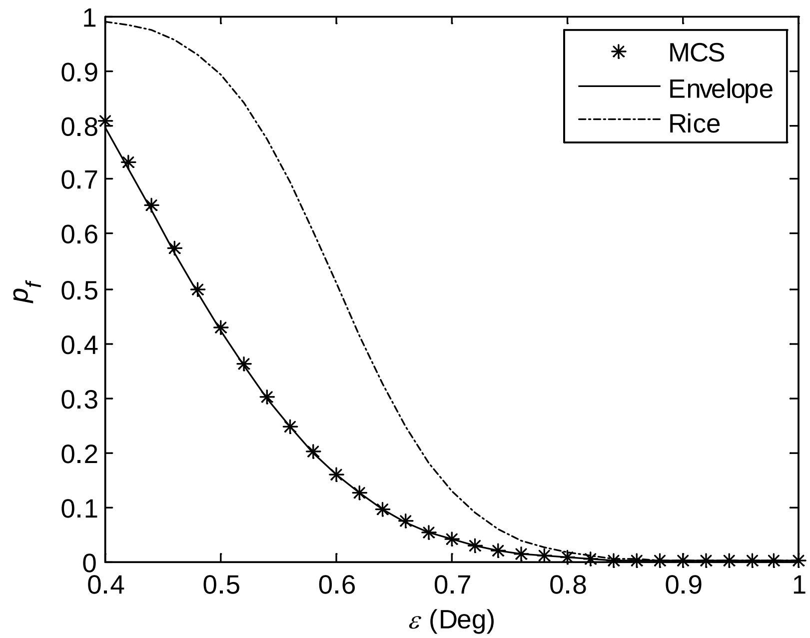

Fig. 4 Probability if failure on $[95.5^{\circ}, 215.5^{\circ}]$

As indicated in both Table 2 and Fig. 4, the probabilities of failure from the envelope method are slightly less than those from MCS. The reason is explained as follows. The probability of failure is the probability of the union of all the events (at all the time instants) where the motion error is larger than the allowable error. The number of the events or time instants is infinite. The envelope method uses a limited number of time instants, but most significant ones, to estimate the probability of failure. As a result, the estimated probability of failure is less than or equal to the true value. But as demonstrated in this example, the envelope method is still accurate.

We also tested the enveloped method using larger standard deviations of 0.3 mm for all the links. A standard deviation of 0.3 mm corresponds to a large tolerance of 0.1 mm according to the 3-sigma rule (tolerance = 3 sigma). Accurate results were also obtained.

The numbers of function evaluations are also provided in Table 3. For a given set of input angles, finding a set of output angles by calling the mechanism analysis is defined as one function evaluation. The envelope method is slightly more efficient than the Rice's formula.

Table 3 Number of Function Evaluations

ε	Rice	Envelope	MCS
0.4°	229	150	10^7
0.5°	229	148	10^7
0.6°	229	164	10^7
0.7°	229	150	10^7
0.8°	229	144	10^7
0.9°	169	146	10^7

6 Conclusions

This work develops an envelope approach to improve the accuracy of time-dependent mechanism reliability analysis. The envelope of the motion error function is derived and then approximated with hyper-planes at several expansion points. The time instants that correspond to the expansion points are found by the linearization of the motion error function. The accuracy of the proposed method is high for mechanisms with both low and high reliability. The proposed method needs both the motion displacement and velocity. Since analytical equations for the displacement and velocity are usually available, the method is also efficient. With the high accuracy and efficiency, the method can also be used for reliability-based mechanism synthesis.

As mentioned above, the envelope method is applicable for time-dependent probabilistic mechanism analysis and synthesis where analytical equations of displacements and velocities exist. Although Monte Carlo simulation (MCS) could be used with analytical equations, its efficiency may still not be adequate, especially for mechanism synthesis with a high reliability requirement. Using the envelope method to replace MCS can alleviate the computational burden significantly. When analytical equations are not available, however, the efficiency of the envelope method will be much lower.

For special cases where the tolerances of the dimension variables are large, the envelope method can be modified to maintain high accuracy. One possible way is to expand the motion error function at the Most Probable Point (MPP) that is used in the First Order Reliability Method (FORM), instead of at the means of random input variables. Using the MPP will be our future work.

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Appendix Rice's Formula for Mechanism Reliability

An upcrossing at a time instant is the event when the motion error exceeds its failure threshold ε at that instant, and a downcrossing at a time instant is the event when the motion error falls below its failure threshold $-\varepsilon$ at that instant. With the assumption that all the upcrossings and downcrossings on $[\theta_0, \theta_e]$ are independent, the reliability $R(\theta_0, \theta_f)$ is calculated by

$$R(\theta_0, \theta_e) = R(\theta_0)\exp\left\{-\int_{\theta_0}^{\theta_e}\left[v^+(\theta) + v^-(\theta)\right]d\theta\right\}$$

where $v^+(\theta)$ and $v^-(\theta)$ are the upcrossing and downcrossing rates, respectively; $R(\theta_0)$ is the initial point reliability at μ_0 and is computed by

$$R(\theta_0) = 1 - p_f(\theta_0)$$

where $p_f(\theta_0)$ is calculated with Eq. (35).

With the linearization of the motion error, the upcrossing rate is given by [29]

$$v^+(\theta) = || \dot{\mathbf{b}}(\theta) || \, \phi[\beta_+(\theta)]\Psi\left\{\frac{\dot{\beta}_+(\theta)}{|| \dot{\mathbf{b}}(\theta) ||}\right\}$$

where

$$\Psi(x) = \phi(x) - x\Phi(-x)$$

$$\beta_+(\theta) = \frac{\varepsilon - \mu_g(\theta)}{\sigma_g(\theta)}$$

The downcrossing rate is given by [29]

$$v^-(\theta) = || \dot{\mathbf{b}}(\theta) || \, \phi[\beta_-(\theta)]\Psi\left\{\frac{\dot{\beta}_-(\theta)}{|| \dot{\mathbf{b}}(\theta) ||}\right\}$$

where

$$\beta_-(\theta) = \frac{\varepsilon + \mu_g(\theta)}{\sigma_g(\theta)}$$

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