ASME Journal of Mechanical Design

doi:10.1115/1.4031906

System Reliability Analysis with Dependent Component Failures during Early Design Stage – A Feasibility Study

Yao Cheng and Xiaoping Du[1]

Department of Mechanical and Aerospace Engineering

Missouri University of Science and Technology

[1]400 West 13th Street, Toomey Hall 272,Rolla, MO 65409, U.S.A.; Tel: 1-573-341-7249; E-mail: dux@mst.edu

AUTHORS INFORMATION:

Yao Cheng, M.S.
Research Assistant
Department of Mechanical and Aerospace Engineering
Missouri University of Science and Technology
258A Toomey Hall
400 West 13th Street
Rolla, MO 65409-0500
E-mail: ycbm7@mst.edu

Xiaoping Du, Ph.D.
Professor
Department of Mechanical and Aerospace Engineering
Missouri University of Science and Technology
272 Toomey Hall
400 West 13th Street
Rolla, MO 65409-0500
573-341-7249 (voice)
573-341-4607 (fax)
E-mail: dux@mst.edu

Abstract

It is desirable to predict product reliability accurately in the early design stage, but the lack of information usually leads to the use of independent component failure assumption. This assumption makes the system reliability prediction much easier, but may produce large errors since component failures are usually dependent after the components are put into use within a mechanical system. The bounds of the system reliability can be estimated, but are usually wide. The wide reliability bounds make it difficult to make decisions in evaluating and selecting design concepts, during the early design stage. This work demonstrates the feasibility of considering dependent component failures during the early design stage with a new methodology that makes the system reliability bounds much narrower. The following situation is addressed: the reliability of each component and the distribution of its load are known, but the dependence between component failures is unknown. With a physics-based approach, an optimization model is established so that narrow bounds of the system reliability can be generated. Three examples demonstrate that it is possible to produce narrower system reliability bounds than the traditional reliability bounds, thereby better assisting decision making during the early design stage.

Keywords: Conceptual design, System reliability analysis, Dependent component, Optimization

1. Introduction

There are four design stages in a design process, including problem definition, conceptual design, embodiment design, and detail design [1]. The early design stage includes problem definition and conceptual design. During the problem definition stage, the problem and working criteria/goals are defined, information such as voice of customer is gathered, and functional modeling is performed [2]. During the conceptual design stage, design concepts are generated, analyzed, and selected [3]. In this work, we consider reliability in the conceptual design stage. Reliability is the ability of a product to perform its intended function without failure, and it is usually quantified by the probability of such ability [4]. In the past, reliability issues were usually addressed when field failure data and/or life testing data became available. This treatment is too late because losses have already occurred. It is therefore necessary to perform reliability analysis in the early design stage. Considering reliability upfront will not only ensure high reliability, robustness, safety, and availability, but also reduce risk and product lifecycle cost [5]. Specifically, predicting system reliability helps decision making in the early design stage [6]. For example, after several design concepts are generated, the best design concept(s) should be selected. In many cases, the product reliability is a major decision factor for keeping or eliminating design concepts. Reliable decision making relies on the accurate system reliability prediction.

Although methodologies exist for early reliability prediction [7-9], predicting reliability early is still a challenging task due to various reasons. Herein, we focus on one of the most important reasons – the lack of dependence information between component failures. Nowadays it is a common practice for a product (or system) to have its components designed and manufactured from different companies (suppliers). These components are individually and independently designed, tested, and manufactured. The reliability of each component may be known to the designers of a new product. When the components are assembled into a system for operation, they are dependent, and the dependent relationship needs to be considered for obtaining the system reliability. The dependence comes from the following reasons: components operate under the same environment, they are subjected to the same load, they deform dependently due to geometric constraints, and the output of one component is the input to other components, and vice versa.

Lacking dependent component states poses a challenge for the early product design because it is difficult to define the exact dependent relationship of components due to the limited information available to the designers of the new product. Even if the designers could acquire the reliability of each component from the supplier who designed and manufactured the component, they do not have access to all the details that are necessary for the system reliability prediction, such as the material properties, geometry, and critical parameters of the component. As a result, the joint probability density of the states of all the components is not available in general.

For the above reasons, approximations to the system reliability are usually used. The commonly used reliability engineering methods are based on the assumption of independent component failures [10-12] on the condition that component reliabilities are given. The independent component state assumption makes the system reliability analysis much easier, but may produce large errors and may therefore lead to erroneous decisions for design concept evaluation and selection. Besides, Park et al. [13] demonstrated that the error due to ignoring dependence can be negligible for a highly reliable system. The conclusion is verified by various conditions. But for design concepts that may not have high reliability, considering component dependence is still necessary for concept evaluation and selection with respect to reliability.

Efforts have been made to improve the accuracy of system reliability by considering component dependence. Humphreys and Jenkins [14] reviewed and summarized the development of techniques of dealing with dependent component failures before 1991. Zhang and Horigome [15] proposed a method to predict system reliability by considering both dependent component failures and time-varying failure rates under several assumptions about system states and time-varying failure and repair rates. This study is suitable for system and component failures due to a cumulative shock-damage process. Pozsgai and Neher [16] summarized approaches to the reliability of mechanical systems with the dependence consideration, such as common-mode failures, load-sharing, and functional dependence. Neil et al. [17] developed hybrid Bayesian Networks (BNs) to model dependable systems with a new iterative algorithm, which combines dynamic discretization with propagation algorithms to realize inference in hybrid BNs. This model uses several assumptions; for example, the repair time is negligible. Marriott and Bate [18] considered dependent failures of nuclear submarines. Their method is based on the unified partial model (UPM), which provides a way to assess the effects of dependent failures on a system in an auditable manner. The method, however, may not be applicable for early designs due to the limited information available for the input of the UPM model. Recently, Youn et al. [19], Nguyen et al. [20], and Wang et al. [21] presented system reliability analysis models for problems where all the component parameters are known. In summary, it may not be easy to apply these methodologies in the early design stage because of limited information about component dependence.

The alternative way is to estimate the bounds of the system reliability. For instance, for a series system, with the inclusion-exclusion principle [22], the system reliability analysis involves the joint probabilities associated with the components of the system. When the component states are dependent, it is difficult to calculate the probabilities of the intersections for a large number of components; thus system reliability bounds $\left[R_S^{\min}, R_S^{\max}\right]$ are of interest, where $R_S^{\min}$ and $R_S^{\max}$ are the minimum and maximum system reliabilities, respectively. The analysis may require the marginal component probabilities, $\Pr(C_i)$ for component C_i, and the joint probabilities of small sets of components, for example, bicomponent probabilities $\Pr(C_i C_j)$ for components *i* and *j*; tricomponent probabilities $\Pr(C_i C_j C_k)$ for components *i*, *j*, and *k*; and so on. Even the bicomponent joint probability $\Pr(C_i C_j)$, however, still needs knowing the joint probability of C_i and C_j. Without using joint probabilities, Boole [23] derived an inequality equation to calculate the system probability bounds for series systems with only the unicomponent probabilities $\Pr(C_i)$, namely, component reliabilities. The bounds produced, however, may be too wide for practical use, as will be discussed in the next section.

In the area of structural reliability which is based on computational models derived from physics principles, narrower system reliability bounds could be produced because joint probabilities are computationally available [24]. The first-order approximation method for system reliability analysis proposed by Hohenbichler and Rackwitz [25] produces narrow system reliability bounds. The method is efficient, but cannot be used in conceptual design because it requires all detailed information about components, such as component limit-state functions, which may not be available during conceptual design. Kounias [26], Hunter [27], and Ditlevsen [28] also developed methodologies for series systems with both unicomponent probabilities $\Pr(C_i)$ and bicomponent probabilities $\Pr(C_i C_j)$. Zhang [29] generalized the methodologies with high order joint probabilities, such as tricomponent and quadricomponent probabilities. These methods still have some drawbacks. The system reliability bounds have the order-dependency problem, meaning that different orders of components may result in different system reliability bounds. The computational demand is also intensive since all the possible ordering alternatives need to be considered. Song and Kiureghian [30] later used linear programming (LP) to address some of these drawbacks. The LP method has no restrictions on component ordering and can incorporate incomplete component probabilities and inequality constraints on component probabilities. Its efficiency deteriorates as the dimension of the problem increases because the size of the problem expands exponentially with respect to the number of components. Ramachandran [31] reviewed and summarized progresses made on structural reliability bounds before 2004. Recently, Domyancic and Millwater [32] summarized and compared different computational methods such as first order bounds, Ditlevsen bounds, KAT lower bound, and LP bounds and demonstrated the applications in series systems. However, as the computational models may not be available during the early design stage, these methods could hardly be applied for the system reliability analysis of a new product.

The purpose of this work is to explore possible ways to accurately and efficiently produce narrow system reliability bounds during the early design stage using a physics-based method with limited information. We demonstrate the feasibility for the following situation: component reliabilities are provided to the designers of a new product from individual suppliers, and the system designers know the load, to which the new product is subjected. We also assume that a component has only one major failure mode that is related to the strength of the component. With a physics-based approach, we establish an optimization model to produce narrower bounds of the product (system) reliability, which will better assist the decision making process in the early design stage.

We review the methodologies of system reliability modeling in Section 2. We then present the proposed system reliability analysis in Section 3, followed by three examples in Section 4. More discussions on the uncertainty in input variables are provided in Section 5. Conclusions and future work are given in Section 6.

2. Review of System Reliability Modeling

There are three typical types of systems, including series systems, parallel systems, and mixed systems. Herein we focus on series systems. The proposed methodology in this work can be extended to the other two types of systems.

A series system consists of components in series as shown in Fig. 1. The failure of one component can result in the failure of the entire system. This type of system is also referred to as a weakest link system.

Place Figure 1 here

We denote the components by $C_1,\ C_2,\ ...,\ C_n$. Correspondingly, their reliabilities are denoted by $R_1,\ R_2,\ ...,\ R_n$. If the states of the components are assumed to be independent, the system reliability is

$$R_S = \prod_{i=1}^{n} R_i \tag{1}$$

The direct use of the above method with the independent component assumption may not be applicable to many mechanical systems. For example, the speed reducer system shown in Fig. 2 consists of one motor, one belt, one drum, two couplings, three shafts, four gears, four keys, and eight bearings, with a total of 24 components. For a simple demonstration, assume the reliability of each component is $R = 0.9999$ or the probability of failure is $p_f = 10^{-4}$, then the system reliability is $R_S = 0.9999^{24} = 0.9976$ according to Eq. (1), or the probability of system failure is $p_{f,S} = 1 - R_S = 2.4 \times 10^{-3}$. The calculated probability of system failure is so high that the design would be rejected for any practical applications. In reality, however, given the high component reliability 0.9999, the actual system reliability of the speed reducer system should be much higher than the calculated value 0.9976. The reason is that the states of the components are dependent because all the components share the common load in this speed reducer system.

On the other hand, without considering the dependence, the design could be extremely conservative. For instance, if the required system reliability of the speed reducer in Fig. 2 is $R_S = 0.999$ and the reliability of each component is the same, then the required component reliability should be at least $\sqrt[24]{0.999} = 0.999958$, or the probability of failure of each component should be less than or equal to $p_f = 4.1687 \times 10^{-5}$. For the aforementioned reason of dependent components, the actual required maximum component reliability should be much lower than 0.999958, or the actual required minimum probability of component failure should be much larger than $p_f = 4.1687 \times 10^{-5}$.

Place Figure 2 here

Since it is difficult to obtain the system reliability without knowing the dependence between component failures, the bounds of the system reliability are usually used. The upper bound is given by [33]

$$R_S \le \min\{R_i\},\ i = 1, ..., n \tag{2}$$

The component dependence could be positive or negative. If a failure of one component leads to an increased tendency for other components to fail, the dependence is positive, and vice versa. For most mechanical systems, the dependence is positive [34], and we therefore consider only positive dependence. For positive dependence, the lower bound of the system reliability is given by [33]

$$\prod_{i=1}^{n} R_i \le R_S,\ i = 1, ..., n \tag{3}$$

Therefore

$$\prod_{i=1}^{n} R_i \le R_S \le \min\{R_i\},\ i = 1, ..., n \tag{4}$$

Or the bounds of the probability of system failure are

$$\max\left\{p_{f,i}\right\} \le p_{f,S} \le 1 - \prod_{i=1}^{n}(1 - p_{f,i}),\ i = 1, 2,\ ...,\ n \tag{5}$$

where $p_{f,S}$ is the probability of system failure, which is equal to $1 - R_S$; $p_{f,i}$ is the probability of component failure and $p_{f,i} = 1 - R_i$. In Eq. (4), estimating the reliability bounds requires only knowing component reliabilities, but the width or the distance between the lower and upper bounds is usually too large. Take the above speed reducer system as an example. If the component reliability is 0.9999, the system reliability bounds are then $0.9976 \le R_S \le 0.9999$, or the bounds of the probability of system failure are $10^{-4} \le p_{f,S} \le 2.4 \times 10^{-3}$.

The wide gap between the lower and upper bounds makes decision making extremely difficult. For example, during the early design stage, if the bounds of the system reliability of two design concepts are as shown in Fig. 3 (a), designers will not be able to differentiate one design from the other with respect to reliability because the two bounds are so wide and they overlap with each other. If the bounds of the system reliability of two design concepts were narrower as shown in Fig. 3 (b), designers would easily differentiate one design from the other and will conclude that design 2 is more reliable than design 1.

Place Figure 3 here

To address the above problem, we propose a physics-based approach that produces narrower bounds for the system reliability.

3. System Reliability Analysis with Dependent Components

The objective of this work is to explore a possible way to produce narrower bounds of system reliability in order to assist decision making in the early design stage. To show the feasibility, we focus on problems where the failure of a system can be predicted using the physics-based stress-strength interference model. The overview of the proposed method is discussed in the next subsection followed by details in the subsequent subsection.

3.1 Overview of the proposed method

As mentioned previously, we focus on series systems. The components of the system may be designed, manufactured, and tested independently by different companies or suppliers. The reliability analysis of the components is the responsibility of the suppliers. The reliability of each component of a new product is available to the system designers, who are responsible to predict the system reliability. The system designers may also have knowledge about the factors of safety that the suppliers may have used in their component designs. In addition to component reliabilities, the system designers may also have other information, such as the load to which the system is subjected. The system designers, however, do not have access to all the detailed information (usually proprietary) about the component designs, such as the analysis models and material properties, e.g., the distributions of the strengths of the components.

With the above information available, we develop a system reliability prediction methodology based on the stress-strength interference model. Instead of providing a single-valued system reliability, the proposed method produces system reliability bounds, which are much narrower than those from the traditional method discussed in Section 2. The task of the proposed method is then to search for the maximum and minimum system reliabilities, and this is accomplished by establishing an optimization model for the system reliability bounds. The objective of the optimization model is the system reliability, the design variables are unknown distribution parameters of components, and the constraints are those related to component reliabilities and factors of safety of the components.

The above assumptions, along with other assumptions we use in this work, are summarized as follows:

- The new product is a series system. The reason we select series systems is that they are commonly encountered in mechanical applications, such as the speed reducer in Fig. 2. The proposed method can be extended to parallel systems and mix systems.
- Each physical component has only one major failure mode related to the strength of the component. If a physical component has multiple failure modes, to use the proposed method, one can treat each failure mode as a single component. For example, if there are two physical components, each having two failure modes, then there are four components from the viewpoint of system analysis.
- The load and strength of each component are independent. This assumption holds for many problems where material strengths do not depend on the load applied to the component.
- The system designers of the new product know the load, to which the new product is subjected. The examples of the system load include the output torque of the speed reducer in Fig. 2, the wind velocity or water velocity of a wind turbine or hydrokinetic turbine, the force acting on the slider of a crank-slider mechanism. The system designers also know the distribution types of the strengths of the components, but the distribution parameters of the strengths are unknown.
- Component reliabilities are provided by component suppliers to the system designers of the new product.

3.2 System reliability model

We start from the models for the case with general distributions and then present the models for a special case with normal distributions.

3.2.1 General optimization model

In order to obtain the system reliability bound with dependent components, the designers of the new product need to ask component suppliers to provide component reliabilities. The limit-state function of the *i*-th component is defined by

$$Y_i = S_{Ste,i} - S_{Stn,i} \tag{6}$$

where $S_{Ste,i}$ is the stress in the component, $S_{Stn,i}$ is the strength of the component, and $-Y_i$ or $S_{Stn,i} - S_{Ste,i}$ is the design margin. $S_{Ste,i}$ is determined by the component load $w_i L$ or a function of $w_i L$. Substituting $S_{Ste,i}$ with $w_i L$ in Eq. (6), we could rewrite the limit-state function as

$$Y_i = w_i L - S_{R,i} \tag{7}$$

where $S_{R,i}$ is the general resistance of the component to the load. $S_{R,i}$ is in general a function of the component strength $S_{Stn,i}$ and other parameters, such as the dimension variables of the component. The information about some of the parameters may be proprietary to the component supplier. As will be discussed later, the proposed method does not require the designers of the new product to know such proprietary information.

For the system to which component i belongs, L is the total load to the system, and w_i indicates the fraction of the load that component i shares, and w_i is a constant. If the load acting on the component is equal to the load acting on the system, $w_i = 1$; if the load acting on the component is less than the load acting on the system, $w_i < 1$. w_i can be determined by the simplified free-body diagram of component i as shown in Fig. 4, where L_i is the load applied to the component. Note that Fig. 4 is only a schematic diagram, which shows how the system load is shared by components, and it is not a real free-body diagram. Also note that L_i is the resultant force acting on the component and could produce point forces, distributed forces, bending moments, and torques that exert on the component.

Place Figure 4 here

The reliability and probability of failure of component i are given by

$$R_i = \Pr\{Y_i < 0\} \tag{8}$$

and

$$p_{f,i} = \Pr\{Y_i > 0\} \tag{9}$$

We assume that the component resistance $S_{R,i}$ and the load to the system L are independent. Let the probability density functions (pdf) of the component load and resistance be $f_{L_i}(l)$ and $f_{S_{R,i}}(s)$, respectively, and let their joint pdf be $f_{L_i,S_{R,i}}(l,s)$. Then the component reliability is calculated by

$$R_i = \Pr\{Y_i < 0\} = \iint_{w_i L < S_{R,i}} f_{L_i,S_{R,i}}(l,s)dlds \tag{10}$$

Given all the component limit-state functions, the safe condition of the system is determined by the intersection $\{Y_1 < 0 \cap Y_2 < 0 \cap ... \cap Y_n < 0\}$, or $\{Y_1 < 0, Y_2 < 0, ..., Y_n < 0\}$. Then the system reliability is given by

$$R_S = \Pr(Y_1 < 0,\ Y_2 < 0, ...,\ Y_n < 0) = \Pr(\mathbf{Y} < 0) \tag{11}$$

where $\mathbf{Y} = (Y_1,\ Y_2,\ ...,\ Y_n)$. Using the joint pdf $f_{\mathbf{Y}}(\mathbf{y})$ of $\mathbf{Y} = (Y_1,\ Y_2,\ ...,\ Y_n)$, we have

$$R_S = \Pr(\mathbf{Y} < 0) = \int f_{\mathbf{Y}}(\mathbf{y})\,\mathrm{d}\,\mathbf{y} \qquad (12)$$

If the distributions of the loads and resistances of all the components are available, $f_{\mathbf{Y}}(\mathbf{y})$ will also be available, and the system reliability can then be obtainable by Eq. (12). As discussed previously, for the system designers of the new product, however, the distribution parameters of component resistances are unknown. We denote $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n)$ for the distribution parameters of component resistances, where $\mathbf{d}_i$ contains the distribution parameters of the resistance of component i. For example, if the resistance of component i is normally distributed, then $\mathbf{d}_i = (\mu_{S_{R,i}}, \sigma_{S_{R,i}})$, where μ and σ denote the mean and standard deviation, respectively. Some of the parameters in $\mathbf{d}$ are proprietary to the component suppliers. Without knowing the distributions of the component resistances, the designers of the new product will not be able to obtain an exact system reliability prediction. As mentioned previously, the proposed method uses all the information available to the designers of the new product to produce narrow bounds of the system reliability with the assumption that the distribution types of the component resistances are known while the distribution parameters are unknown.

The system reliability bounds are found by solving for the minimum and maximum system reliabilities through using optimization models. We now discuss such optimization models, including their design variables, objective functions, and constraint functions.

The design variables are those of unknown distribution parameters of the component resistances, denoted by $\mathbf{d}$. For example, if the component resistances follow normal distributions, the design variables will be means and standard deviations $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, ..., \mathbf{d}_n) = (\mu_{S_{R,1}}, \sigma_{S_{R,1}}, \mu_{S_{R,2}}, \sigma_{S_{R,2}}, ..., \mu_{S_{R,n}}, \sigma_{S_{R,n}})$.

The objective function is the system reliability given in Eq. (12). It is a function of known distribution parameters of the system load $\mathbf{p}_L$, and unknown design variables $\mathbf{d}$. The objective function is denoted by $R_S(\mathbf{d};\mathbf{p}_L)$. Maximizing $R_S(\mathbf{d};\mathbf{p}_L)$ produces the maximum system reliability $R_S^{\max}$ while minimizing $R_S(\mathbf{d};\mathbf{p}_L)$ produces the minimum system reliability $R_S^{\min}$.

There are multiple constraint functions. The reliability of a component gives an equality constraint according to Eq. (10), and there are therefore n equality constraints, as shown below.

$$h_i(\mathbf{d};\mathbf{p}_L) = \iint_{w_i L < S_{R,i}} f_{L_i,S_{R,i}}(l,s)\,dlds = R_i,\ i = 1,2,...,n \tag{13}$$

Although the designers of the new product may not know the actual factors of safety used by component designers from the suppliers, they have good knowledge about the range of the factors of safety of the components. Denote the lower and upper bounds of the factors of safety by $n_{s,i}^{\min}$ and $n_{s,i}^{\max}$, respectively, we have the following inequality constraints.

$$n_{s,i}^{\min} \le n_{s,i}(\mathbf{d};\mathbf{p}_L) \le n_{s,i}^{\max} \tag{14}$$

There are therefore totally $2n$ inequality constraints given by

$$g_i(\mathbf{d};\mathbf{p}_L) = n_{s,i}^{\min} - n_{s,i}(\mathbf{d};\mathbf{p}_L) \le 0,\ i = 1,2,\ldots,n \tag{15}$$

and

$$g_{i+n}(\mathbf{d};\mathbf{p}_L) = n_{s,i}(\mathbf{d};\mathbf{p}_L) - n_{s,i}^{\max} \le 0,\ i = 1,2,\ldots,n \tag{16}$$

In addition, the designers may also have good knowledge about the coefficients of variation, which are the ratios of standard deviations to means of component resistances. Denote a coefficient of variation by c, and its lower and upper bounds by $c_i^{\min}$ and $c_i^{\max}$, respectively. From $c_i^{\min} \le c_i(\mathbf{d};\mathbf{p}_L) \le c_i^{\max}$, we have other $2n$ inequality constraints.

$$g_{i+2n}(\mathbf{d};\mathbf{p}_L) = c_i^{\min} - c_i(\mathbf{d};\mathbf{p}_L) \le 0,\ i = 1, 2, \ldots, n \tag{17}$$

and

$$g_{i+3n}(\mathbf{d};\mathbf{p}_L) = c_i(\mathbf{d};\mathbf{p}_L) - c_i^{\max} \le 0,\ i = 1, 2, \ldots, n \tag{18}$$

The optimization model for the minimum system reliability is then given by

$$\begin{cases} \min\limits_{\mathbf{d}}\ R_S(\mathbf{d};\mathbf{p}_L) \\ \text{subject to} \\ h_i(\mathbf{d};\mathbf{p}_L) = \iint\limits_{w_i L < S_{R,i}} f_{L_i, S_{R,i}}(l,s)\,dl\,ds = R_i,\ i = 1, 2, ..., n \\ g_i(\mathbf{d};\mathbf{p}_L) = n_{s,i}^{\min} - n_{s,i}(\mathbf{d};\mathbf{p}_L) \le 0, \\ g_{i+n}(\mathbf{d};\mathbf{p}_L) = n_{s,i}(\mathbf{d};\mathbf{p}_L) - n_{s,i}^{\max} \le 0, \\ g_{i+2n}(\mathbf{d};\mathbf{p}_L) = c_i^{\min} - c_i(\mathbf{d};\mathbf{p}_L) \le 0, \\ g_{i+3n}(\mathbf{d};\mathbf{p}_L) = c_i(\mathbf{d};\mathbf{p}_L) - c_i^{\max} \le 0, \end{cases} \tag{19}$$

For the maximum system reliability, we just change the first line of the optimization model in Eq. (19) from $\min\limits_{\mathbf{d}}\ R_S(\mathbf{d};\mathbf{p}_L)$ to $\max\limits_{\mathbf{d}}\ R_S(\mathbf{d};\mathbf{p}_L)$. The two optimization models will produce the minimum and maximum system reliabilities, thereby the system reliability bounds.

3.2.2 Optimization model for normal distributions

After having presented the general case, we now discuss a special case where all random variables are normally distributed. Suppose $S_{R,i}$ and L follow normal distributions $S_{R,i} \sim N(\mu_{S_{R,i}}, \sigma_{S_{R,i}}^2)$ and $L \sim N(\mu_L, \sigma_L^2)$, respectively. From Eq. (7), the mean and standard deviation of Y_i are

$$\mu_i = w_i \mu_L - \mu_{S_{R,i}} \tag{20}$$

$$\sigma_i = \sqrt{(w_i \sigma_L)^2 + \sigma_{S_{R,i}}^2} \tag{21}$$

The reliability of component i is then calculated by

$$R_i = \Pr(Y_i < 0) = \Phi\left(-\frac{\mu_{Y_i}}{\sigma_{Y_i}}\right) = \Phi\left(-\frac{w_i \mu_L - \mu_{S_{R,i}}}{\sqrt{(w_i \sigma_L)^2 + \sigma_{S_{R,i}}^2}}\right),\ i = 1, 2, ..., n \tag{22}$$

where Φ is the cumulative distribution function of a standard normal variable. It can be shown that every linear combination of $(Y_1,\ Y_2,\ ...,\ Y_n)$ is normally distributed if the resistances $S_{R,i}$ ($i = 1, 2, \ldots, n$) and load L are independently and normally distributed. As a result, vector $\mathbf{Y} = (Y_1,\ Y_2,\ ...,\ Y_n)$ follows a multivariate normal distribution denoted by $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where the mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ are given by

$$\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_n) \tag{23}$$

and

$$\boldsymbol{\Sigma} = \begin{pmatrix} \mathrm{cov}_{11} & \ldots & \mathrm{cov}_{1n} \\ \vdots & \ddots & \vdots \\ \mathrm{cov}_{n1} & \cdots & \mathrm{cov}_{nn} \end{pmatrix} \tag{24}$$

where

$$\mathrm{cov}_{ij} = \begin{cases} \sigma_i^2 & i = j \\ \mathrm{cov}(Y_i, Y_j) & i \neq j \end{cases} \tag{25}$$

From Eq. (7), we can derive the covariance between Y_i and Y_j, and it is given by

$$\mathrm{cov}_{ij} = \mathrm{cov}(Y_i, Y_j) = \mathrm{cov}\left[w_i L - S_{R,i}, w_j L - S_{R,j}\right] = w_i w_j \sigma_L^2 \tag{26}$$

Thus, the covariance matrix $\boldsymbol{\Sigma}$ in Eq. (24) is rewritten as

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{Y_1}^2 & \cdots & w_1 w_n \sigma_L^2 \\ \vdots & \ddots & \vdots \\ w_n w_1 \sigma_L^2 & \cdots & \sigma_{Y_n}^2 \end{pmatrix} \tag{27}$$

After $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are obtained, the pdf of $\mathbf{Y}$ is fully defined by

$$f(y_1, y_2, ..., y_n) = \frac{1}{(2\pi)^{\frac{n}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu}) \right\} \tag{28}$$

The system reliability is then obtained by integrating the probability density function using Eq. (12).

For the system designers of the new product, however, the distribution parameters of component resistances, for example, the means $\mu_{S_{R,i}}$ and standard deviations $\sigma_{S_{R,i}}$ $(i = 1, 2, \ldots, n)$ of normal distribution are unknown. As a result, the complete information that defines the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$ in Eq. (28) are not available to the designers. Thus, the exact system reliability cannot be obtained.

Narrow system reliability bounds can be found with the proposed optimization model. For this case, the design variables become $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, ..., \mathbf{d}_n) = (\mu_{S_{R,1}}, \sigma_{S_{R,1}}, \mu_{S_{R,2}}, \sigma_{S_{R,2}}, ..., \mu_{S_{R,n}}, \sigma_{S_{R,n}})$ as discussed previously, and the distribution parameters of the system load become $\mathbf{p}_L = (\mu_L, \sigma_L)$. The constraint functions associated with component reliabilities, according to Eq. (22), are given by

$$h_i(\mathbf{d}; \mu_L, \sigma_L) = \Phi\left(-\frac{w_i \mu_L - \mu_{S_{R,i}}}{\sqrt{\sigma_{S_{R,i}}^2 + (w_i \sigma_L)^2}} \right) = R_i, \ i = 1, 2, ..., n \tag{29}$$

And we have totally $2n$ inequality constraints according to the range of factors of safety $n_{s,i}$.

$$g_i(\mathbf{d};\mu_L,\sigma_L)=n_{s,i}^{\min}-\frac{\mu_{S_{R,i}}}{w_i\mu_L}\le 0,\ i=1,2,\ldots,n \tag{30}$$

and

$$g_{i+n}(\mathbf{d};\mu_L,\sigma_L)=\frac{\mu_{S_{R,i}}}{w_i\mu_L}-n_{s,i}^{\max}\le 0,\ i=1,2,\ldots,n \tag{31}$$

In addition, we have other $2n$ inequality constraints according to the ranges of the coefficients of variation c_i of the unknown distributions.

$$g_{i+2n}(\mathbf{d};\mu_L,\sigma_L)=c_i^{\min}-\frac{\sigma_{S_{R,i}}}{\mu_{S_{R,i}}}\le 0,\ i=1,2,\ldots,n \tag{32}$$

and

$$g_{i+3n}(\mathbf{d};\mu_L,\sigma_L)=\frac{\sigma_{S_{R,i}}}{\mu_{S_{R,i}}}-c_i^{\max}\le 0,\ i=1,2,\ldots,n \tag{33}$$

The optimization model for the minimum system reliability is then given by

$$\begin{cases} \min_{\mathbf{d}} \ R_S(\mathbf{d};\mu_L,\sigma_L) \\ \text{subject to} \\ h_i(\mathbf{d};\mu_L,\sigma_L) = \Phi\left(-\dfrac{w_i\mu_L - \mu_{S_{R,i}}}{\sqrt{\sigma_{S_{R,i}}^2 + (w_i\sigma_L)^2}}\right) = R_i,\ i=1,2,...,n \\ g_i(\mathbf{d};\mu_L,\sigma_L) = n_{s,i}^{\min} - \dfrac{\mu_{S_{R,i}}}{w_i\mu_L} \le 0, \\ g_{i+n}(\mathbf{d};\mu_L,\sigma_L) = \dfrac{\mu_{S_{R,i}}}{w_i\mu_L} - n_{s,i}^{\max} \le 0, \\ g_{i+2n}(\mathbf{d};\mu_L,\sigma_L) = c_i^{\min} - \dfrac{\sigma_{S_{R,i}}}{\mu_{S_{R,i}}} \le 0, \\ g_{i+3n}(\mathbf{d};\mu_L,\sigma_L) = \dfrac{\sigma_{S_{R,i}}}{\mu_{S_{R,i}}} - c_i^{\max} \le 0, \end{cases} \tag{34}$$

For the maximum system reliability, we just change the first line of the optimization model in Eq. (34) from $\min_{\mathbf{d}} \ R_S(\mathbf{d};\mathbf{p}_L)$ to $\max_{\mathbf{d}} \ R_S(\mathbf{d};\mathbf{p}_L)$.

There are n equality constraint functions, which may cause numerical difficulties in solving the optimization models. We could improve the optimization models by eliminating some of the design variables using the equality constraints. This will not only reduce the scale of the optimization but also improve the robustness of the solution process [35]. An equality constraint imposes a functional relationship on design variables, and design variables $\mu_{S_{R,i}}$ can then be substituted with remaining design variables. From Eq. (22), we obtain

$$\mu_{S_{R,i}} = w_i\mu_L + \Phi^{-1}(R_i)\sqrt{\sigma_{S_{R,i}}^2 + (w_i\sigma_L)^2} \tag{35}$$

Thus, design variables $\mu_{S_{R,i}}$ and all the equality constraints are eliminated. Plugging Eq. (35) into Eq. (34) yields

$$\begin{cases}\min\limits_{\mathbf{d}}\ R_S(\mathbf{d};\mu_L,\sigma_L)\\ \text{subject to}\\ g_i(\mathbf{d};\mu_L,\sigma_L)=n_{s,i}^{\min}-\dfrac{w_i\mu_L+\Phi^{-1}(R_i)\sqrt{\sigma_{S_{R,i}}^2+(w_i\sigma_L)^2}}{w_i\mu_L}\le 0,\ i=1,2,\ \dots,\ n\\ g_{i+n}(\mathbf{d};\mu_L,\sigma_L)=\dfrac{w_i\mu_L+\Phi^{-1}(R_i)\sqrt{\sigma_{S_{R,i}}^2+(w_i\sigma_L)^2}}{w_i\mu_L}{}_i-n_{s,i}^{\max}\le 0,\\ g_{i+2n}(\mathbf{d};\mu_L,\sigma_L)=c_i^{\min}-\dfrac{\sigma_{S_{R,i}}}{w_i\mu_L+\Phi^{-1}(R_i)\sqrt{\sigma_{S_{R,i}}^2+(w_i\sigma_L)^2}}\le 0,\\ g_{i+3n}(\mathbf{d};\mu_L,\sigma_L)=\dfrac{\sigma_{S_{R,i}}}{w_i\mu_L+\Phi^{-1}(R_i)\sqrt{\sigma_{S_{R,i}}^2+(w_i\sigma_L)^2}}-c_i^{\max}\le 0,\end{cases}\tag{36}$$

The new vector of the design variables in Eq. (36) is $\mathbf{d}=\boldsymbol{\sigma}_{S_R}=(\sigma_{S_{R,1}},\sigma_{S_{R,2}},\dots,\sigma_{S_{R,n}})$. The bounds of $\sigma_{S_{R,i}}$ can be determined by plugging Eq. (30) and Eq. (31) into Eq. (29), respectively.

$$\sigma_{S_{R,i}}^{\min}=\sqrt{\left(\frac{\left(n_{s,i}^{\min}-1\right)w_i\mu_L}{\Phi^{-1}(R_i)}\right)^2-\left(w_i\sigma_L\right)^2}\tag{37}$$

$$\sigma_{S_{R,i}}^{\max}=\sqrt{\left(\frac{\left(n_{s,i}^{\max}-1\right)w_i\mu_L}{\Phi^{-1}(R_i)}\right)^2-\left(w_i\sigma_L\right)^2}\tag{38}$$

The predicted system reliability bounds cover the true value if the true design point, which produces the true system reliability, falls into the feasible region defined by the constraint functions. It is therefore important to carefully select the parameters for the constraint functions. The designers of the new product could select these parameters based on their experiences, their knowledge about component design, and design standards in their specific areas.

4. Numerical Examples

In this section we provide three examples for three cases: (1) a system consists of different components with the same load, (2) a system consists of identical components with the same load, and (3) a system consists of different components with different loads. In the third example, we also demonstrate the superiority of the proposed method in early design decision making over that of the traditional method. Since the reliability is high, to easily show the accuracy of the results, we use the probability of failure.

4.1 Example 1: Three different components sharing the same load

A new design consists of three different components, supplied by three different companies, as shown in Fig. 5. They are subjected to the same load L. The resistances of the three components are known to the component designers, and their distributions are $S_1 \sim N(3500, 350^2)$ kN, $S_2 \sim N(3200, 260^2)$ kN, and $S_3 \sim N(4000, 400^2)$ kN. The three random variables are independent. The load L is known to both component designers and system designers of the new product. The distribution of the load is $L \sim N(2000, 200^2)$ kN. The probabilities of failure of the components obtained from the component designs are therefore $p_{f,1} = 9.920\times10^{-5}$, $p_{f,2} = 1.2696\times10^{-4}$, and $p_{f,3} = 3.87\times10^{-6}$ according to Eq. (22). The information about the component reliability is provided to the system designers of the new product. In addition, the system designers of the new product are confident that the factors of safety of the three components are between 1.5 and 2.5 and that the coefficients of variation of component resistances are between 0.08 and 0.2. The information available to the system designers of the new design is summarized in Table 1.

Place Figure 5 here

Place Table 1 here

For the system designers of the new product, the task is to estimate the system reliability of the new product using the information in Table 1. The simplified free-body diagrams of the three components are the same. Fig. 6 shows the simplified free-body diagram of component 1.

Place Figure 6 here

The three components are subjected to the same load L, and their limit-state functions are therefore given by

$$\begin{cases} Y_1 = L - S_1 \\ Y_2 = L - S_2 \\ Y_3 = L - S_3 \end{cases} \tag{39}$$

Thus, the system reliability of the new product is

$$R_S = \Pr(Y_1 < 0,\ Y_2 < 0,\ Y_3 < 0) = \int_{-\infty}^{0} f(\mathbf{y})\mathrm{d}\mathbf{y} \tag{40}$$

where $\mathbf{Y} = (Y_1,\ Y_2,\ Y_3) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. From Eq. (35), the means of component resistance μ_{S_i}, $i = 1, 2, 3$, are given by

$$\mu_{S_i} = \mu_L + \Phi^{-1}(R_i)\sqrt{\sigma_{S_i}^2 + (\sigma_L)^2} \tag{41}$$

The covariance between any two limit-state functions is $\mathrm{cov}(Y_i, Y_j) = \sigma_L^2$ according to Eq. (26), and the covariance matrix $\boldsymbol{\Sigma}$ is then given by

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{Y_1}^2 & \sigma_L^2 & \sigma_L^2 \\ \sigma_L^2 & \sigma_{Y_2}^2 & \sigma_L^2 \\ \sigma_L^2 & \sigma_L^2 & \sigma_{Y_3}^2 \end{pmatrix} \quad (42)$$

The design variables are $\mathbf{d} = (\sigma_{S_1}, \sigma_{S_2}, \sigma_{S_3})$. Thus, the optimization model is created using Eq. (36).

$$\begin{cases} \min_{\mathbf{d}} \ R_S(\mathbf{d}; \mu_L, \sigma_L) \\ \text{subject to} \\ g_i(\mathbf{d}; \mu_L, \sigma_L) = 1.5 - \dfrac{\mu_L + \Phi^{-1}(R_i)\sqrt{\sigma_{S_i}^2 + (\sigma_L)^2}}{2000} \le 0, \qquad i = 1, 2, 3 \\ g_{i+3}(\mathbf{d}; \mu_L, \sigma_L) = \dfrac{\mu_L + \Phi^{-1}(R_i)\sqrt{\sigma_{S_i}^2 + (\sigma_L)^2}}{2000} - 2.5 \le 0, \\ g_{i+6}(\mathbf{d}; \mu_L, \sigma_L) = 0.08 - \dfrac{\sigma_{S_i}}{\mu_L + \Phi^{-1}(R_i)\sqrt{\sigma_{S_i}^2 + (\sigma_L)^2}} \le 0, \\ g_{i+9}(\mathbf{d}; \mu_L, \sigma_L) = \dfrac{\sigma_{S_i}}{\mu_L + \Phi^{-1}(R_i)\sqrt{\sigma_{S_i}^2 + (\sigma_L)^2}} - 0.20 \le 0, \end{cases} \quad (43)$$

For the maximum system reliability, we just change the first line of the optimization model in Eq. (43) from $\min_{\mathbf{d}} \ R_S(\mathbf{d}; \mu_L, \sigma_L)$ to $\max_{\mathbf{d}} \ R_S(\mathbf{d}; \mu_L, \sigma_L)$. Table 2 shows the bounds of the probabilities of system failure obtained from the traditional method and the proposed method. The results indicate that the proposed method produces much narrower bounds than those from the traditional method. The two bounds are also plotted in Fig. 7.

Place Table 2 here

Place Figure 7 here

The true value of the probability of system failure is also provided in both Table 2 and Fig. 7, and it is calculated as if all the distributions of S_1, S_2, S_3, and L were known. Note that in reality, both component designers and system designers only know some of the distributions. Even though the exact value may never be known, we use it to verify the accuracy of the proposed method. As indicated by the results, the probability bounds from the proposed method do contain the exact value. To easily show the accuracy, we also use the percentage errors of the lower and upper bounds of the probabilities of system failure relative to the true value. The errors of the traditional and proposed methods are $[-44.68\%, 0.23\%]$ and $[-0.26\%, 0.22\%]$, respectively. They are also shown in Fig. 7.

4.2 Example 2: Three identical components sharing the same load

The system configuration is the same as that in Example 1. The three components are also subjected to the same load L. But the three components are identical here. The component resistance is known to the component designers, and its distribution is $S \sim N(4000, 130^2)$ kN. The load L is known to both component designers and system designers, and its distribution is $L \sim N(2400, 450^2)$ kN. The probability of failure of the component obtained from the component supplier is $p_f = 3.1789 \times 10^{-4}$ and is provided to the system designers. In addition, the system designers estimate that the factors of safety of the component are between 1.5 and 2.2 and that the coefficient of variation of component resistance is between 0.03 and 0.15. The information available to the system designers of the new design is summarized in Table 3.

Place Table 3 here

For the system designers of the new product, the task is to estimate the system reliability of the new product using the information in Table 3. The simplified free-body diagrams of the three components are the same as that in Example 1, as shown in Fig. 6.

The component limit-state functions are $Y_1 = Y_2 = Y_3 = L - S$ according to Eq. (39). Plugging their limit-state functions into Eqs. (40) through Eq. (43), we obtain the optimization model as follows.

$$\begin{cases} \min_{\mathbf{d}} \ R_S(\mathbf{d}; \mu_L, \sigma_L) \\ \text{subject to} \\ g_i(\mathbf{d}; \mu_L, \sigma_L) = 1.5 - \dfrac{\mu_L + \Phi^{-1}(R_i)\sqrt{\sigma_{S_i}^2 + (\sigma_L)^2}}{2400} \le 0, \qquad i = 1, 2, 3 \\ g_{i+3}(\mathbf{d}; \mu_L, \sigma_L) = \dfrac{\mu_L + \Phi^{-1}(R_i)\sqrt{\sigma_{S_i}^2 + (\sigma_L)^2}}{2400} - 2.2 \le 0, \\ g_{i+6}(\mathbf{d}; \mu_L, \sigma_L) = 0.03 - \dfrac{\sigma_{S_i}}{\mu_L + \Phi^{-1}(R_i)\sqrt{\sigma_{S_i}^2 + (\sigma_L)^2}} \le 0, \\ g_{i+9}(\mathbf{d}; \mu_L, \sigma_L) = \dfrac{\sigma_{S_i}}{\mu_L + \Phi^{-1}(R_i)\sqrt{\sigma_{S_i}^2 + (\sigma_L)^2}} - 0.15 \le 0, \end{cases} \tag{44}$$

For the maximum system reliability, we just change the first line of the optimization model in Eq. (44) from $\min_{\mathbf{d}} \ R_S(\mathbf{d}; \mu_L, \sigma_L)$ to $\max_{\mathbf{d}} \ R_S(\mathbf{d}; \mu_L, \sigma_L)$. Table 4 shows the bounds of the probabilities of system failure obtained from the traditional method and the proposed method. The results also indicate that the proposed method produces narrower bounds than those from the traditional method. The two bounds are also plotted in Fig. 8.

Place Table 4 here

Place Figure 8 here

The exact (true) value of the probability of system failure is also provided in both Table 4 and Fig. 8. The exact value is calculated as if the distributions of S and L were known. As indicated by the results, the bounds of the probability of system failure from the proposed method do contain the exact value of the probability of system failure. The relative errors of the two methods are $[-48.10\%, 55.65\%]$ and $[-3.52\%, 54.64\%]$ as shown in Fig. 8.

4.3 Example 3: Two different components sharing different loads

Two design concepts for a hoisting device with a load L are generated. They are shown in Fig. 9. Cables 1 and 2 are used in design concept 1 while cables 3 and 4 are used in design concept 2. All the cables are supplied by different companies. Both reliability and cost are two major factors for choosing one design concept between the two. The cost of design concept 2 is estimated 20% cheaper than that of design concept 1 because the components in design concept 2 are cheaper. The distribution of the weight of the block $L \sim N(1500, 160^2)$ kN is known to the system designers of the new hoisting device. The resistances of the two cables used in design concept 1 are only known to the component designers, and they are independently distributed with $S_1 \sim N(1200, 100^2)$ kN and $S_2 \sim N(2500, 250^2)$ kN. Using the distributions, the component designers estimate the probabilities of failure of the two cables are $p_{f,1} = 2.2078 \times 10^{-4}$ and $p_{f,2} = 3.7709 \times 10^{-4}$, and the results are provided to the system designers of the new product.

For design concept 2, the slope is $\theta = 30^{\circ}$, and the coefficient of kinetic friction between the block and surface is $\mu_R = 0.2$; they are known to system designers. The resistances of the two cables are only known to the component designers, and their distributions are $S_3 \sim N(600, 65^2)$ kN and $S_4 \sim N(1220, 140^2)$ kN. The two random variables are independent. The probabilities of failure of the two cables obtained from the component design are $p_{f,3} = 1.9475 \times 10^{-4}$ and $p_{f,4} = 2.5523 \times 10^{-4}$, and they are also provided to the system designers of the new product. In addition, for both concepts of the new product, the system designers estimate that the factors of safety of all the cables are between 1.5 and 2.5 and that the coefficients of variation of component resistances are between 0.08 and 0.2. The information available to the system designers of the two design concepts is summarized in Tables 5 and 6, respectively.

Place Figure 9 here

Place Table 5 here

Place Table 6 here

The simplified free-body diagram of design concept 1 is shown in Fig. 10.

Place Figure 10 here

We have

$$\begin{cases} L_1 = 0.5L \\ L_2 = L \end{cases} \tag{45}$$

The limit-state functions of the two cables in concept 1 are given by

$$\begin{cases} Y_1 = 0.5L - S_1 \\ Y_2 = L - S_2 \end{cases} \tag{46}$$

The simplified free-body diagram of design concept 2 is shown in Fig. 11.

Place Figure 11 here

Based on force equilibrium, we obtain

$$\begin{cases} L_3 = \dfrac{L(\sin\theta + \mu_R \cos\theta)}{3} \\ L_4 = \dfrac{2L(\sin\theta + \mu_R \cos\theta)}{3} \end{cases} \tag{47}$$

The limit-state functions of the two cables are then given by

$$\begin{cases} Y_3 = \dfrac{L(\sin\theta + \mu_R \cos\theta)}{3} - S_3 \\ Y_4 = \dfrac{2L(\sin\theta + \mu_R \cos\theta)}{3} - S_4 \end{cases} \tag{48}$$

The general limit-state function of the four cables for both design concepts is therefore

$$Y_i = w_i L - S_i \tag{49}$$

where $i = 1, 2, 3, 4$.

The system reliability of design concept 1 is then given by

$$R_{S_1} = \Pr(Y_1 < 0,\ Y_2 < 0) = \int_{-\infty}^{0} f(\mathbf{y}_1)\mathrm{d}\mathbf{y}_1 \tag{50}$$

where $\mathbf{Y}_1 = (Y_1,\ Y_2) \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$. The mean function of component resistance μ_{S_i} is given by

$$\begin{cases} \mu_{S_1} = w_1\mu_L + \Phi^{-1}(R_1)\sqrt{\sigma_{S_1}^2 + (w_1\sigma_L)^2} \\ \mu_{S_2} = w_2\mu_L + \Phi^{-1}(R_2)\sqrt{\sigma_{S_2}^2 + (w_2\sigma_L)^2} \end{cases} \tag{51}$$

The covariance between the two limit-state functions is $\mathrm{cov}(Y_1, Y_2) = w_1 w_2 \sigma_L^2$ according to Eq. (26), and the covariance matrix $\boldsymbol{\Sigma}_1$ is then given by

$$\boldsymbol{\Sigma}_1 = \begin{pmatrix} \sigma_{Y_1}^2 & w_1 w_2 \sigma_L^2 \\ w_1 w_2 \sigma_L^2 & \sigma_{Y_2}^2 \end{pmatrix} \tag{52}$$

The design variables are $\mathbf{d}_1 = (\sigma_{S_1}, \sigma_{S_2})$. Thus, the optimization model of concept 1 is created using Eq. (36).

$$
\begin{cases}
\min\limits_{\mathbf{d}_1} \; R_{S_1}(\mathbf{d}_1;\mu_L,\sigma_L) \\
\text{subject to} \\
g_1(\mathbf{d}_1;\mu_L,\sigma_L) = 1.5 - \dfrac{w_1\mu_L + \Phi^{-1}(R_1)\sqrt{\sigma_{S_1}^2 + (w_1\sigma_L)^2}}{1500 w_1} \le 0 \\
g_2(\mathbf{d}_1;\mu_L,\sigma_L) = 1.5 - \dfrac{w_2\mu_L + \Phi^{-1}(R_2)\sqrt{\sigma_{S_2}^2 + (w_2\sigma_L)^2}}{1500 w_2} \le 0 \\
g_3(\mathbf{d}_1;\mu_L,\sigma_L) = \dfrac{w_1\mu_L + \Phi^{-1}(R_1)\sqrt{\sigma_{S_1}^2 + (w_1\sigma_L)^2}}{1500 w_1} - 2.5 \le 0 \\
g_4(\mathbf{d}_1;\mu_L,\sigma_L) = \dfrac{w_2\mu_L + \Phi^{-1}(R_2)\sqrt{\sigma_{S_2}^2 + (w_2\sigma_L)^2}}{1500 w_2} - 2.5 \le 0 \\
g_5(\mathbf{d}_1;\mu_L,\sigma_L) = 0.08 - \dfrac{\sigma_{S_1}}{w_1\mu_L + \Phi^{-1}(R_1)\sqrt{\sigma_{S_1}^2 + (w_1\sigma_L)^2}} \le 0 \\
g_6(\mathbf{d}_1;\mu_L,\sigma_L) = 0.08 - \dfrac{\sigma_{S_2}}{w_2\mu_L + \Phi^{-1}(R_2)\sqrt{\sigma_{S_2}^2 + (w_2\sigma_L)^2}} \le 0 \\
g_7(\mathbf{d}_1;\mu_L,\sigma_L) = \dfrac{\sigma_{S_1}}{w_1\mu_L + \Phi^{-1}(R_1)\sqrt{\sigma_{S_1}^2 + (w_1\sigma_L)^2}} - 0.20 \le 0 \\
g_8(\mathbf{d}_1;\mu_L,\sigma_L) = \dfrac{\sigma_{S_2}}{w_2\mu_L + \Phi^{-1}(R_2)\sqrt{\sigma_{S_2}^2 + (w_2\sigma_L)^2}} - 0.20 \le 0
\end{cases}
\tag{53}
$$

where $w_1 = 0.5$ and $w_2 = 1$ from Eq. (45). For the maximum system reliability, we just change the first line of the optimization model in Eq. (53) from $\min\limits_{\mathbf{d}_1} R_{S_1}(\mathbf{d}_1;\mu_L,\sigma_L)$ to $\max\limits_{\mathbf{d}_1} R_{S_1}(\mathbf{d}_1;\mu_L,\sigma_L)$.

For design concept 2, the optimization model is similar to that in Eq. (53) with the following modifications: (1) change design variables from $\mathbf{d}_1 = (\sigma_{S_1},\sigma_{S_2})$ to $\mathbf{d}_2 = (\sigma_{S_3},\sigma_{S_4})$, (2) change component reliabilities from R_1 and R_2 to R_3 and R_4, and (3) change w_1 and w_2 to w_3 and w_4, where $w_3 = \left(\sin\theta + \mu'\cos\theta\right)/3$ and $w_4 = 2\left(\sin\theta + \mu'\cos\theta\right)/3$ according to Eq. (47).

Table 7 shows the bounds of the probabilities of system failure obtained from the traditional method and the proposed method for the two design concepts. The results not only indicate that the proposed method produces much narrower bounds for the probabilities of system failure than those from the traditional method, but also demonstrate the feasibility of the proposed method to assist the system designers to select a better concept with respect to reliability. The bounds of the two concepts are plotted in Fig. 12. It shows that design concept 2 is more reliable than design concept 1. This is because the probability of system failure of design concept 2 is lower than that of design concept 1 using proposed method. It is hard, however, to make decisions using the traditional method as the bounds for the probabilities of system failure of the two design concepts are wide and overlap as shown in Fig. 12. Thus, with the new system reliability analysis, the system designers may select design concept 2 because it has higher reliability and lower cost.

Place Table 7 here

Place Figure 12 here

The exact (true) value of the probability of system failure of each concept is also provided in Table 7. The exact value of design concept 1 is calculated as if all the distributions of S_1, S_2, and L were known; the exact value of design concept 2 is calculated as if all the distributions, S_3, S_4, and L, were known. As indicated by the results, the bounds of the probabilities of system failure using proposed method do contain the exact values of the probabilities of system failure.

4.4 Summary of the examples

The proposed method has produced narrow system reliability bounds where the true system reliability resides. The examples also demonstrate the effect of dependent component states on system reliability. In Examples 1 and 3, the true probabilities of system failure are close to the upper bounds of the probabilities of system failure that are from independent component assumption. This means that the effect of the dependency is not significant. For Example 1, the coefficients of correlation between component 1 and 2, 2 and 3, and 1 and 3 are 0.3025, 0.2727, and 0.2219, receptively. For Example 3, the coefficients of correlation between component 1 and 2 of concept 1, and component 3 and 4 of concept 2 are 0.3367 and 0.2207, receptively. These small coefficients of correlation indicate weak component dependency. Even so, it is risky for the designers of a new product to make decisions by treating components as independent states, because they may not know the weak dependency in advance during the conceptual design stage.

The result of Example 2 clearly shows the significant impact of dependent components on system reliability because the true probability of system failure is far away from the upper bound that is produced from the assumption of independent components. The coefficients of correlation between the three components are all 0.9230, which indicates the strong correlation between the components.

5. Discussions about the Uncertainty in Input Variables

The uncertainty in input variables will also affect the accuracy of reliability analysis [36, 37]. The proposed method can actually accommodate the uncertainty in some of its input variables, including the component factors of safety and coefficients of variation of component strengths. The system designers know neither their nominal values nor the uncertainty associated with these input variables. By treating the unknown variables as either design variables or constraints in the system reliability model in Eqs. (19) and (34), the proposed method can identify the likely values of the input variables corresponding to the minimum and maximum system reliabilities.

The uncertainty in other input variables is not considered in the proposed system reliability model. They include component probabilities of failure, the distribution of system load, and the types of component strength distributions. The uncertainty in these input variables may be in different forms due to different reasons. For example, if the samples for the system load are not sufficient, there might be several possible candidate distributions, and the distribution parameters themselves might also be random variables [37]. In an extreme case, if the data are too scarce, the load may be described by only an interval [38]. The component reliabilities may also be intervals because component suppliers may report percentage errors for their component reliabilities.

The proposed system reliability model in Eq. (19) can then be modified to account for the uncertainty in input variables. If several candidate distributions are possible for random input variables, the methodology for imprecise random variables [37] can be incorporated. If the uncertainty in the dependence between input variables has to be considered, the Bayesian approach [36] may be applied. If the uncertainty is in the form of intervals, denoted by $\mathbf{y}$, the system reliability in Eq. (19) can be modified as

$$
\begin{cases}
\min_{\mathbf{d}} \min_{\mathbf{y}} R_S(\mathbf{d}, \mathbf{y}; \mathbf{p}_L) \\
\text{subject to} \\
h_i(\mathbf{d}, \mathbf{y}; \mathbf{p}_L) = \iint_{w_i L < S_{R,i}} f_{L_i, S_{R,i}}(l, s)\, dl\, ds = R_i, \ i = 1, 2, \ldots, n \\
g_i(\mathbf{d}, \mathbf{y}; \mathbf{p}_L) = \min_{\mathbf{y}} n_{s,i}^{\min} - n_{s,i}(\mathbf{d}, \mathbf{y}; \mathbf{p}_L) \le 0, \\
g_{i+n}(\mathbf{d}, \mathbf{y}; \mathbf{p}_L) = n_{s,i}(\mathbf{d}, \mathbf{y}; \mathbf{p}_L) - \max_{\mathbf{y}} n_{s,i}^{\max} \le 0, \\
g_{i+2n}(\mathbf{d}, \mathbf{y}; \mathbf{p}_L) = c_i^{\min} - c_i(\mathbf{d}; \mathbf{p}_L) \le 0, \\
g_{i+3n}(\mathbf{d}, \mathbf{y}; \mathbf{p}_L) = c_i(\mathbf{d}; \mathbf{p}_L) - c_i^{\max} \le 0,
\end{cases}
\tag{54}
$$

In the above model, one more loop is added for identifying the extreme values with respect to interval variables. Due to the uncertainty in the input variables, the system reliability bounds produced will be wider, and the computational cost will also be higher. Efficient numerical algorithms are needed to solve the optimization model.

6. Conclusions

This work is concerned with the reliability prediction of a new product whose components are independently designed, tested, and manufactured by different suppliers. A system reliability method is developed to predict the reliability of the new product in the early design stage using the component reliabilities provided by component suppliers. The method is based on the strength-stress interference model that takes the dependence between components into consideration, thereby eliminating the assumption of independent component failures. As a result, the predicted system reliability bounds are much narrower than those from the assumption of independent component failures. This study has shown the feasibility of considering dependent component failures for predicting system reliability bounds in early design stage. The proposed method provides reliability predictions for decision making on eliminating or keeping design concepts during the conceptual design stage. It is useful if a concept selection method, for example, the Pugh Chart method, requires all design concepts be ranked with respect to performance criteria, including reliability. For some situations, however, designers of the new product are only interested in if the reliability requirement could be satisfied. Then the proposed method is not necessary once the minimum reliability (the lower bound) from the independent component assumption in Eq. (5) reaches the reliability target.

The proposed method is applicable for time invariant reliability problems. It can be extended to time variant problems in the future work. Time-dependent reliability could be addressed by considering time-dependent component stresses and strengths. The major research task is to obtain the autocorrelation function of the unknown stochastic processes of generalized component strengths. The ultimate goal is to evaluate the time-dependent system reliability for a given period of time.

As discussed in Sec. 5, uncertainty may also exist in the input variables required by the proposed system reliability method. The future work will be the development of computational methods that can efficiently solve the optimization models with the extra loop that accommodates the uncertainty in input variables.

This work assumes each component has only one failure mode. For a component with multiple failure modes, the component designers may use multiple limit-state functions to evaluate the reliability of the component. Although the component reliability may be reported to the designers of the new product, they however know neither the failure modes nor the limit-state functions of the component. A possible way to deal with this problem is to model the multiple failure modes of the component using a single equivalent limit-state function that can represent the limit-state functions of the multiple failure modes. Then the optimization models proposed in this work could be applied.

The proposed method is applied to series systems. Its application to parallel systems and mix systems is also a possible research task in the future work. Our future work will also deal with situations where a new product is subjected to multiple forces.

Acknowledgement

This material is based upon work supported by the National Science Foundation under grants CMMI 1300870 and CMMI 1234855, and the Intelligent Systems Center at the Missouri University of Science and Technology.

References

[1] Cross, N., 2008, Engineering Design Methods: Strategies for Product Design, John Wiley & Sons.
[2] Pahl, G., and Beitz, W., 2013, Engineering Design: A Systematic Approach, Springer Science & Business Media.
[3] Hsu, W., and Woon, I. M., 1998, "Current Research in the Conceptual Design of Mechanical Products," Computer-Aided Design, 30(5), pp. 377-389.
[4] Lewis, E. E., and Lewis, E., 1987, Introduction to Reliability Engineering, Wiley New York et al.
[5] Marini, V. K., 2013, "Information about Robustness, Reliability and Safety in Early Design Phases," Ph.D. thesis, Department of Management Engineering, Technical University of Denmark, Lyngby, Denmark.
[6] Parnell, G. S., Driscoll, P. J., and Henderson, D. L., 2011, Decision Making in Systems Engineering and Management, John Wiley & Sons.
[7] Ormon, S. W., Cassady, C. R., and Greenwood, A. G., 2002, "Reliability Prediction Models to Support Conceptual Design," IEEE Transactions on Reliability, 51(2), pp. 151-157.
[8] Nachtmann, H., and Chimka, J. R., 2003, "Fuzzy Reliability in Conceptual Design," Proc. Reliability and Maintainability Symposium. Annual, IEEE, pp. 360-364.
[9] Huang, Z., and Jin, Y., 2009, "Extension of Stress and Strength Interference Theory for Conceptual Design-for-Reliability," Journal of Mechanical Design, 131(7), pp. 071001-1-071001-11.
[10] Dhingra, A. K., 1992, "Optimal Apportionment of Reliability and Redundancy in Series Systems under Multiple Objectives," IEEE Transactions on Reliability, 41(4), pp. 576-582.
[11] Li, X. and Chen, J., 2004, "Aging Properties of the Residual Life Length of k-out-of-n Systems with Independent but Non-Identical Components," Applied Stochastic Models in Business and Industry, 20(2), pp. 143-153.
[12] Volkanovski, A., Čepin, M., and Mavko, B., 2009, "Application of the Fault Tree Analysis for Assessment of Power System Reliability," Reliability Engineering & System Safety, 94(6), pp. 1116-1127.

[13] Park, C., Kim, N. H., and Haftka, R. T., 2015, "The Effect of Ignoring Dependence between Failure Modes on Evaluating System Reliability," Structural and Multidisciplinary Optimization, pp. 1-18.
[14] Humphreys, P., and Jenkins, A., 1991, "Dependent Failures Developments," Reliability Engineering & System Safety, 34(3), pp. 417-427.
[15] Zhang, T. and Horigome, M., 2001, "Availability and Reliability of System with Dependent Components and Time-Varying Failure and Repair Rates," IEEE Transactions on Reliability, 50(2), pp. 151-158.
[16] Pozsgai, P., Neher, W., and Bertsche, B., 2002, "Models to Consider Dependence in Reliability Calculation for Systems Consisting of Mechanical Components," Proceedings of 3rd International Conference on Mathematical Methods in Reliability, Trondheim, Norwegen, pp. 539-542.
[17] Neil, M., Tailor, M., Marquez, D., Fenton, N., and Hearty, P., 2008, "Modelling Dependable Systems Using Hybrid Bayesian Networks," Reliability Engineering & System Safety, 93(7), pp. 933-939.
[18] Marriott, C. and Bate, P., 2010, "Dependent Failure Assessment in the Development of a Defuelling Facility for Nuclear Submarines," 2010, 5th IET International Conference on System Safety, pp.1-6.
[19] Youn, B. D., and Wang, P., 2009, "Complementary Intersection Method for System Reliability Analysis," Journal of Mechanical Design, 131(4), pp. 041004-1-041004-15.
[20] Nguyen, T. H., Song, J., and Paulino, G. H., 2010, "Single-loop System Reliability-based Design Optimization using Matrix-based System Reliability Method: Theory and Applications," Journal of Mechanical Design, 132(1), pp. 011005-1-011005-11.
[21] Wang, P., Hu, C., and Youn, B. D., 2011, "A Generalized Complementary Intersection Method (GCIM) for System Reliability Analysis," Journal of Mechanical Design, 133(7), pp. 071003-1-071003-13.
[22] Kahn, J., Linial, N., and Samorodnitsky, A., 1996, "Inclusion-Exclusion: Exact and Approximate," Combinatorica, 16(4), pp. 465-477.
[23] Boole, G., 1854, "Laws of Thought (American reprint of 1854 edition)," Dover, New York.
[24] Choi, S.-K., Grandhi, R., and Canfield, R. A., 2006, Reliability-Based Structural Design, Springer Science & Business Media.
[25] Hohenbichler, M., and Rackwitz, R., 1983, "First-order Concepts in System Reliability," Structural Safety, 1(3), pp. 177-188.
[26] Kounias, E. G., 1968, "Bounds for the Probability of a Union, with Applications," The Annals of Mathematical Statistics, 39(6), pp. 2154-2158.
[27] Hunter, D., 1976, "An Upper Bound for the Probability of a Union," Journal of Applied Probability, 13(3), pp. 597-603.
[28] Ditlevsen, O., 1979, "Narrow Reliability Bounds for Structural Systems," Journal of Structural Mechanics, 7(4), pp. 453-472.
[29] Zhang, Y. C., 1993, "High-Order Reliability Bounds for Series Systems and Application to Structural Systems," Computers & Structures, 46(2), pp. 381-386.
[30] Song, J., and Der Kiureghian, A., 2003, "Bounds on System Reliability by Linear Programming," Journal of Engineering Mechanics, 129(6), pp. 627-636.
[31] Ramachandran, K., 2004, "System Reliability Bounds: a New Look with Improvements," Civil Engineering and Environmental Systems, 21(4), pp. 265-278.
[32] Domyancic, L. C. and Millwater, H. R., 2012, "Advances in Bounding Techniques for Aircraft Structures," AIAA journal, 50(6), pp. 1307-1313.
[33] Høyland, A. and Rausand, M., 2009, System Reliability Theory: Models and Statistical Methods, John Wiley & Sons.
[34] Rausand, M., and Høyland, A., 2004, System reliability Theory: Models, Statistical Methods, and Applications, John Wiley & Sons.
[35] Du, X., and Huang, B., 2007, "Reliability-Based Design Optimization with Equality Constraints," International Journal for Numerical Methods in Engineering, 72(11), pp. 1314-1331.
[36] Noh, Y., Choi, K. K., and Lee, I., 2010, "Identification of Marginal and Joint CDFs using Bayesian Method for RBDO," Structural and Multidisciplinary Optimization, 40(1-6), pp.35-51.

[37] Hu, Z., Du, X., Kolekarb, N.S., and Banerjee, A., 2014, “Robust Design with Imprecise Random Variables and Its Application in Hydrokinetic Turbine Optimization,” Engineering Optimization, 46(3), pp. 393-419.
[38] Guo, J., and Du, X., 2010, “Reliability Analysis for Multidisciplinary Systems with Random and Interval Variables,” AIAA Journal, 48(1), pp. 82-91.

List of Table Captions

List of Figure Captions

Table 1 Information available to the designers of the new product

Known information	**Value**
Probability of component failure $p_{f,1}$	9.920×10^{-5}
Probability of component failure $p_{f,2}$	1.2696×10^{-4}
Probability of component failure $p_{f,3}$	3.87×10^{-6}
Distribution of system load L	$N(2000, 200^2)$ kN
Factor of safety for component 1 $n_{s,1}$	$[1.5, 2.5]$
Factor of safety for component 2 $n_{s,2}$	$[1.5, 2.5]$
Factor of safety for component 3 $n_{s,3}$	$[1.5, 2.5]$
Coefficient of variation of resistance of component 1 c_1	$[0.08, 0.20]$
Coefficient of variation of resistance of component 2 c_2	$[0.08, 0.20]$
Coefficient of variation of resistance of component 3 c_3	$[0.08, 0.20]$

Table 2 System reliability analysis results

Methods	**Bounds of $p_{f,s}$**	**Interval width**
Traditional method	$\left[1.2696,\ 2.3002\right]\times10^{-4}$	1.0306×10^{-4}
Proposed method	$\left[2.2891,\ 2.30\right]\times10^{-4}$	0.0109×10^{-4}
Exact	2.2950×10^{-4}	

Table 3 Information available to the designers of the new product

Known information	**Value**
Probability of component failure $p_{f,1}$	3.1789×10^{-4}
Probability of component failure $p_{f,2}$	3.1789×10^{-4}
Probability of component failure $p_{f,3}$	3.1789×10^{-4}
Distribution of system load L	$N(2400, 450^2)$ kN
Factor of safety for component 1 $n_{s,1}$	$[1.5, 2.2]$
Factor of safety for component 2 $n_{s,2}$	$[1.5, 2.2]$
Factor of safety for component 3 $n_{s,3}$	$[1.5, 2.2]$
Coefficient of variation of resistance of component 1 c_1	$[0.03, 0.15]$
Coefficient of variation of resistance of component 2 c_2	$[0.03, 0.15]$
Coefficient of variation of resistance of component 3 c_3	$[0.03, 0.15]$

Table 4 System reliability analysis results

Methods	**Bounds of** $p_{f,s}$	**Interval width**
Traditional method	$[3.1789,\ 9.5337]\times10^{-4}$	6.3548×10^{-4}
Proposed method	$[5.9094, 9.4721]\times10^{-4}$	3.5627×10^{-4}
Exact	6.1252×10^{-4}	

Table 5 Information available for design concept 1

Known information	**Value**
Probability of component failure $p_{f,1}$	2.2078×10^{-4}
Probability of component failure $p_{f,2}$	3.7709×10^{-4}
Distribution of system load L	$N(1500,160^2)$ kN
Factor of safety for component 1 $n_{s,1}$	$[1.5, 2.5]$
Factor of safety for component 2 $n_{s,2}$	$[1.5, 2.5]$
Coefficient of variation of resistance of component 1 c_1	$[0.08, 0.20]$
Coefficient of variation of resistance of component 2 c_2	$[0.08, 0.20]$

Table 6 Information available for design concept 2

Known information	**Value**
Probability of component failure $p_{f,3}$	1.9475×10^{-4}
Probability of component failure $p_{f,4}$	2.5523×10^{-4}
Distribution of system load L	$N(1500, 160^2)$ kN
Factor of safety for component 1 $n_{s,3}$	$[1.5, 2.5]$
Factor of safety for component 2 $n_{s,4}$	$[1.5, 2.5]$
Coefficients of variation of resistance of component 1 c_3	$[0.08, 0.20]$
Coefficients of variation of resistance of component 2 c_4	$[0.08, 0.20]$
Slope	30°
Coefficient of friction	$\mu_R = 0.2$

Table 7 System failure analysis results of the two concepts for the new system

Concepts	Methods	Bounds of $p_{f,s}$	Interval width	Exact value
Concept 1	Traditional method	$\left[3.7709,\ 5.9779\right]\times10^{-4}$	2.2070×10^{-4}	5.9498×10^{-4}
	Proposed method	$\left[5.8877,\ 5.9769\right]\times10^{-4}$	0.0892×10^{-4}	
Concept 2	Traditional method	$\left[2.5523,\ 4.4993\right]\times10^{-4}$	1.9470×10^{-4}	4.4931×10^{-4}
	Proposed method	$\left[4.4354,\ 4.4987\right]\times10^{-4}$	0.0633×10^{-4}	

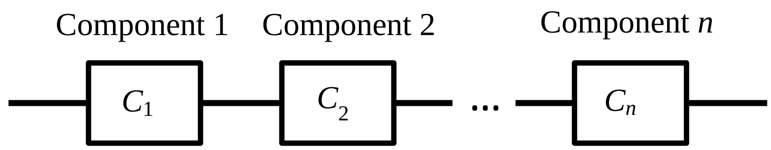


Figure 1. Series system

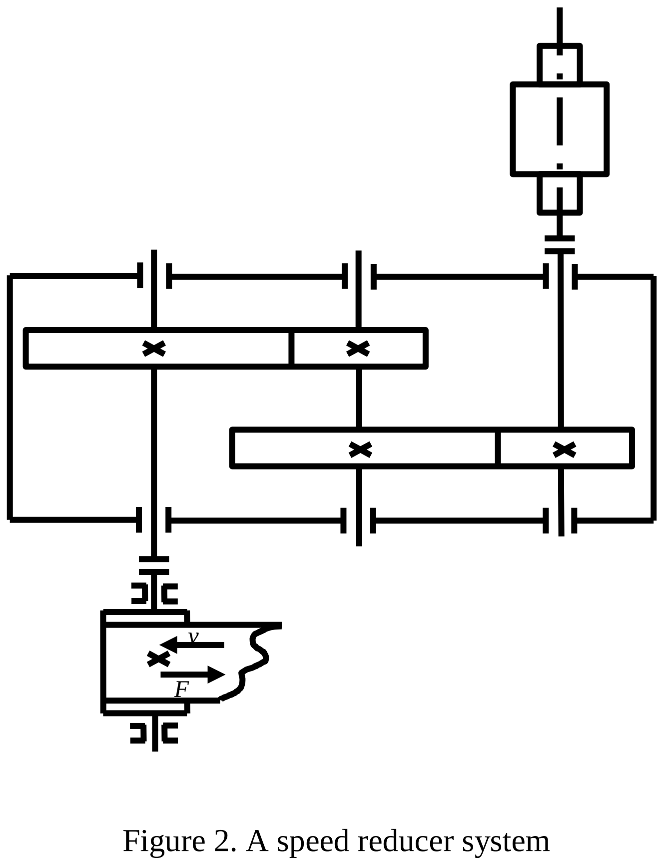


Figure 2. A speed reducer system

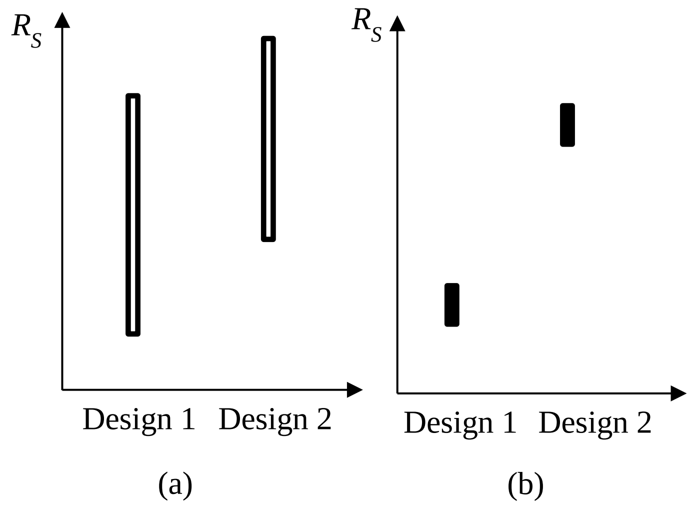


Figure 3. System reliability bounds of two designs

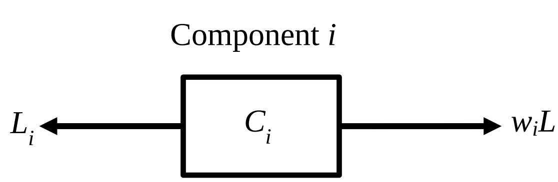


Figure 4. Simplified free-body diagram of component i

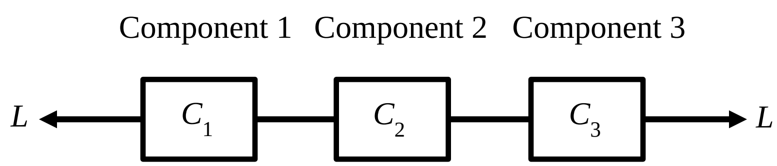


Figure 5. Three different components sharing same load

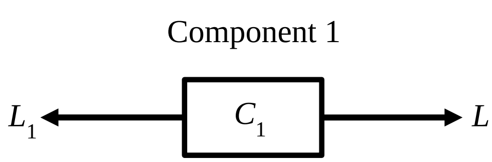


Figure 6. Simplified free-body diagram of component 1

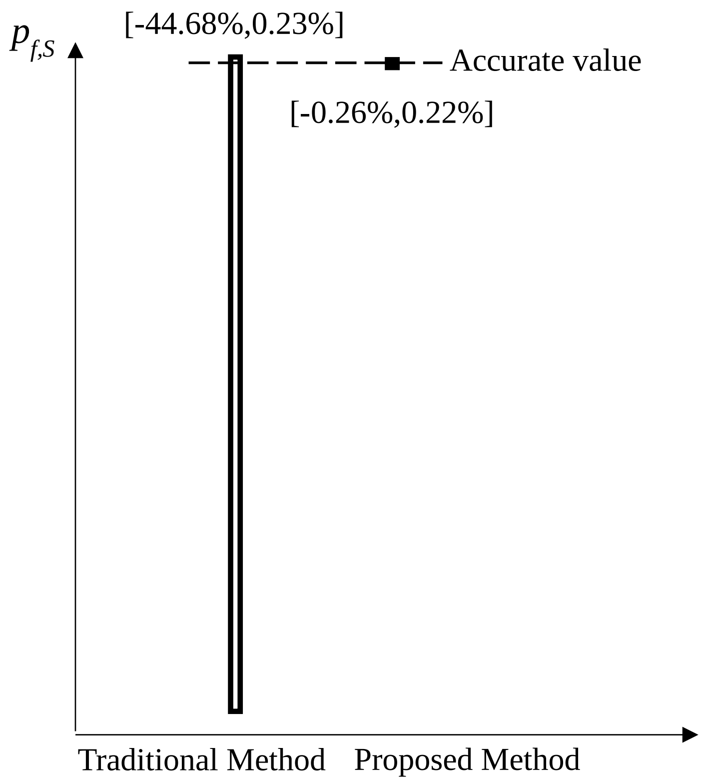


Figure 7. Bounds of probabilities of system failure

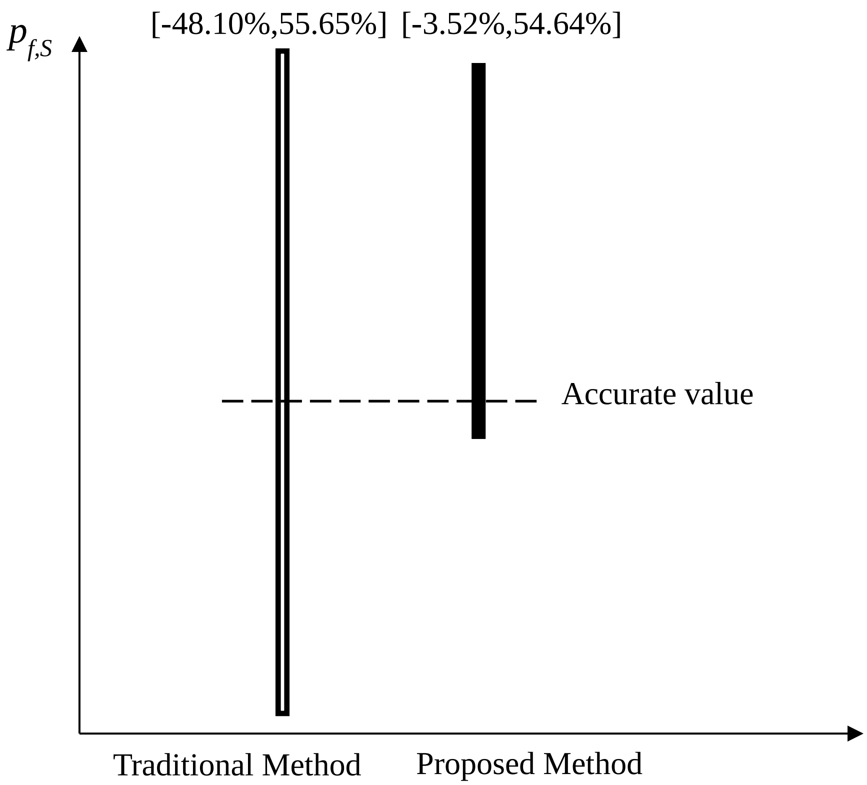
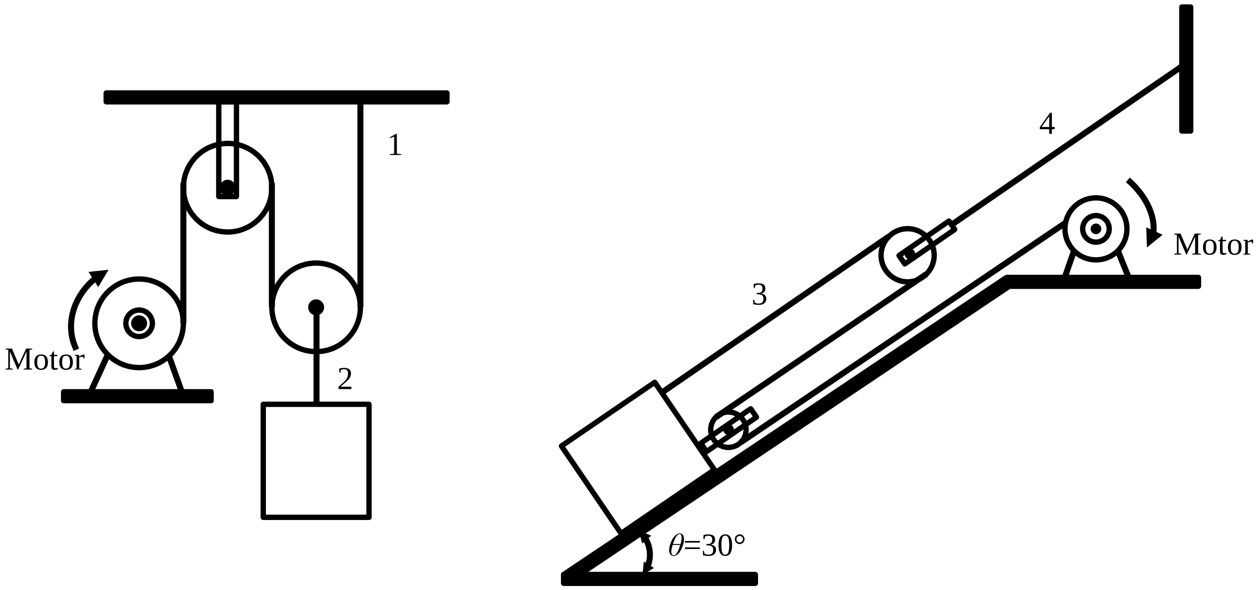


Figure 8. Bounds of probabilities of system failure



(a) Design concept 1 (b) Design concept 2

Figure 9. Two components sharing different loads

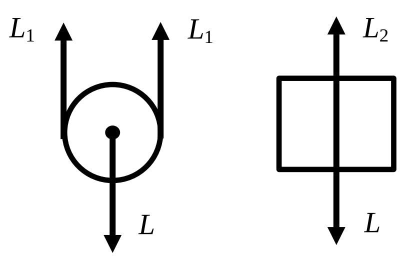


Figure 10. Simplified free-body diagrams of design concept 1

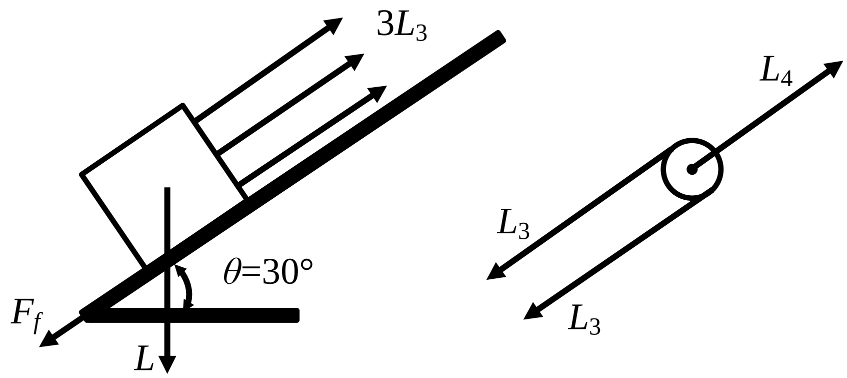


Figure 11. Simplified free-body diagrams of design concept 2

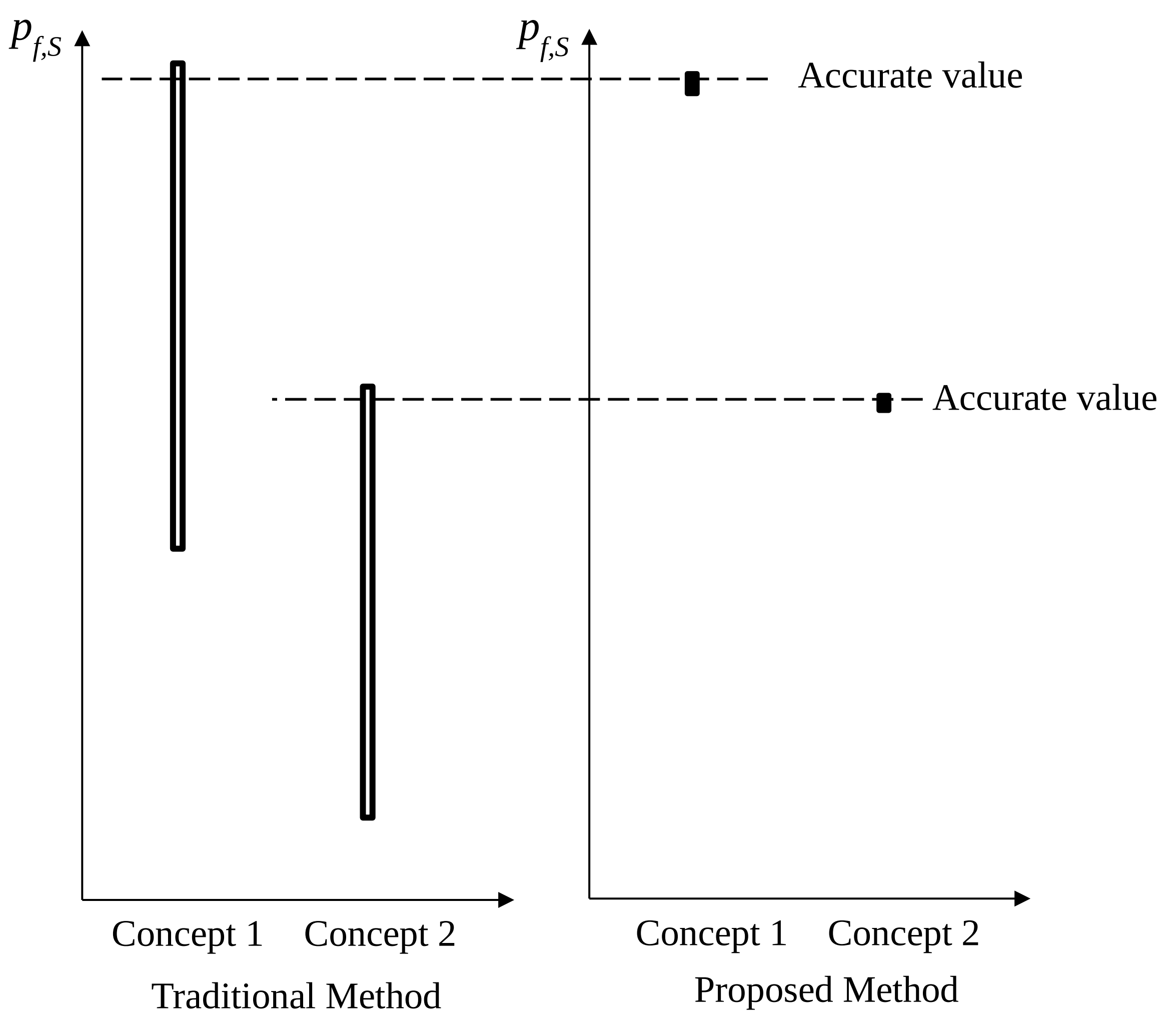


Figure 12. Bounds of probabilities of system failure