ASME 2004 DETC Conferences

A Saddlepoint Approximation Method for Uncertainty Analysis - Second Order Approximation

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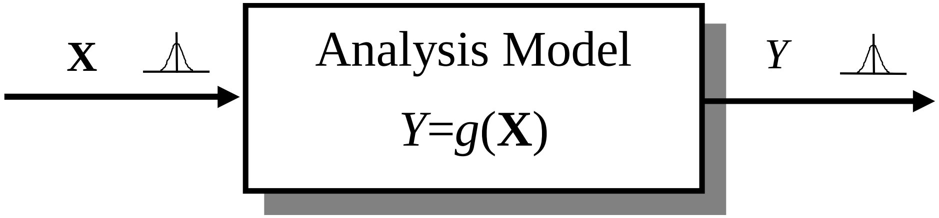
The Name. The Degree. The Difference.

Outline

- Review of reliability analysis method
- Introduction to Saddlepoint Approximations
- Saddlepoint approximation method for uncertainty analysis
- Examples
- Conclusion

Uncertainty Analysis

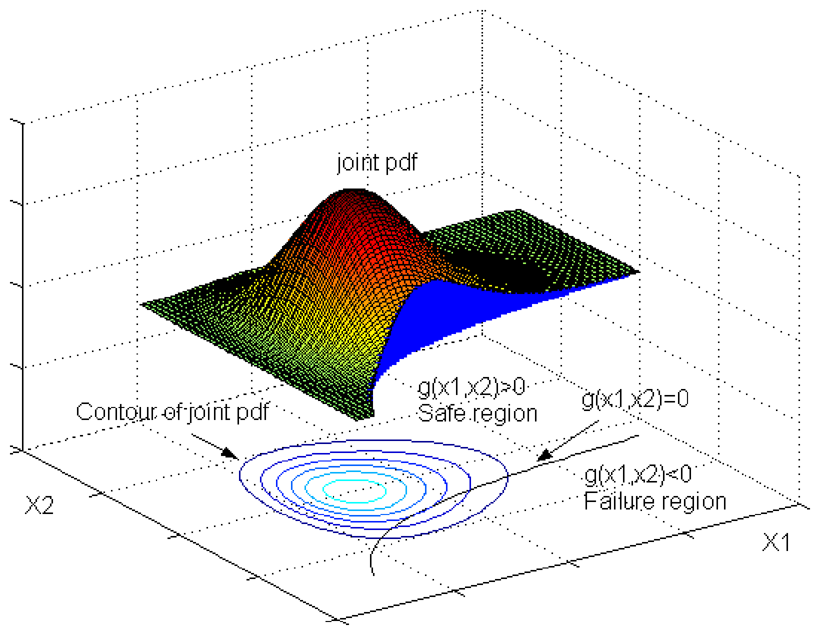
- *Given*: distributions of input variables **X**
  - Joint *pdf* $f_{\mathbf{X}}(\mathbf{x})$
- *Find*: *cdf* of *Y*
  - $F_Y(y)=\Pr\{g(X)<y\}$



Probability Integration

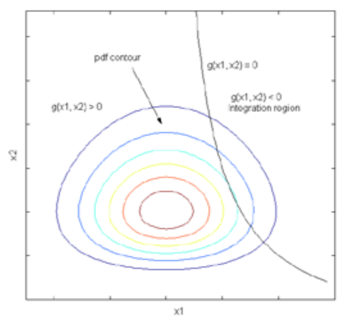
$$F_Y(y) = \Pr\{g(\mathbf{X}) < y\} = \int_{g(\mathbf{x})<y} f_{\mathbf{x}}(\mathbf{x})d\mathbf{x}$$

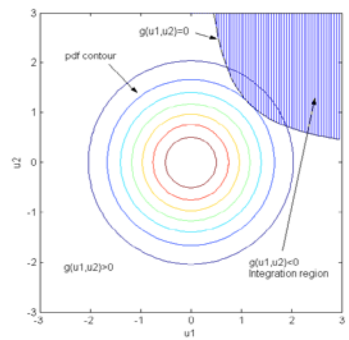
- Multidimensional integral
- Nonlinear integrand $f_X(\mathbf{x})$
- Nonlinear Integration boundary $g(\mathbf{x})=0$
- It is difficult or even impossible to obtain a theoretic or numerical solution.
- Approximation is needed.

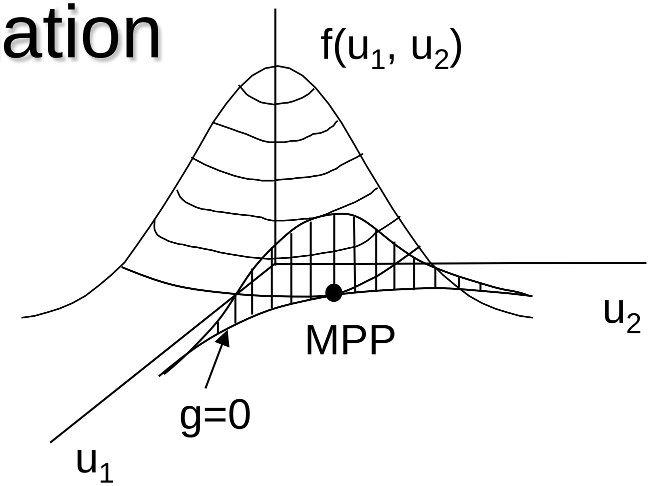


FORM and SORM

- Transformation and approximation



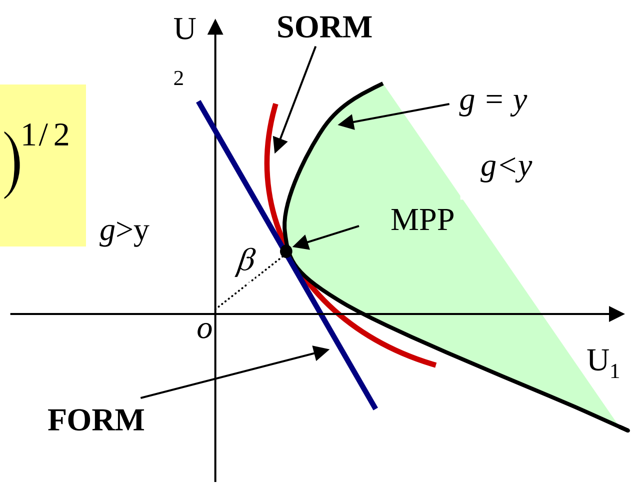




- SORM – more accurate

$$F_Y(y) = \Pr\{g(\mathbf{X}) < y\} = \Phi(-\beta)\prod_{i=1}^{n-1}(1+\beta\kappa_i)^{1/2}$$

when β is very large.



Saddlepoint Approximation (SPA)

- Proposed in 1954
- Used for approximating the distribution of sum of random variables
- *pdf* of Y=g(X)

$$f(y) = \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} \exp\{K(\xi) - \xi y\} d\xi$$

- K – Cumulant Generation Function (CGF)

SPA (Cont.)

$$f(y) = \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} \exp\{K(\xi) - \xi y\} d\xi$$

- Approximate $K(\xi) - \xi y$ at the saddlepoint *t* to the second order

$$\frac{\partial\left[K(\xi) - \xi y\right]}{\partial \xi} = 0$$

$$K'(\xi) - \xi y = 0$$

t - saddlepoint

$$f(y) = \left\{\frac{1}{2\pi K''(t)}\right\}^{\frac{1}{2}} e^{K(t) - ty}$$

cdf

$$F_Y = P\{Y \le y\} = \Phi(w) + \phi(w)\left(\frac{1}{w} - \frac{1}{v}\right)$$

$$w = sign(t)\left\{2\left[ty - K(t)\right]\right\}^{1/2}$$

$$v = t\left\{K''(t)\right\}^{1/2}$$

Features of SPA

- Accurate probability estimation
  - especially in the *tail area of a distribution* (*where a reliability resides*!)
  - Goutics and Casella, 1999.
- Small sample asymptotics
  - *besides the theoretical reasons, one empirical reason for the excellent small sample behavior is that the Saddlepoint Approximations are density-like objects and do not show the polynomical-like waves*.
  - Field and Ronchetti, 1990

First Order SPA

- Linearize the performance function in the original random space
- More accurate than FORM
- Sometime more accurate than SORM (presented in MAO 04)

Second Order SPA (SOSPA)

- Same procedure as SORM to approximate the performance function at the MPP

$$Y \approx V_n - \left(\beta + \frac{1}{2} \mathbf{V}'^{T} \mathbf{D} \mathbf{V}' \right)$$

- CGF of Y

$$K(t) = -\beta t + \frac{1}{2} t^2 - \frac{1}{2} \sum_{i}^{n-1} \log(1 - 2t\lambda_i)$$

$$K'(t) = -\beta + t + \sum_{i}^{n-1} \frac{\lambda_i}{1 - 2t\lambda_i} = y \longrightarrow \text{Saddlepoint}$$

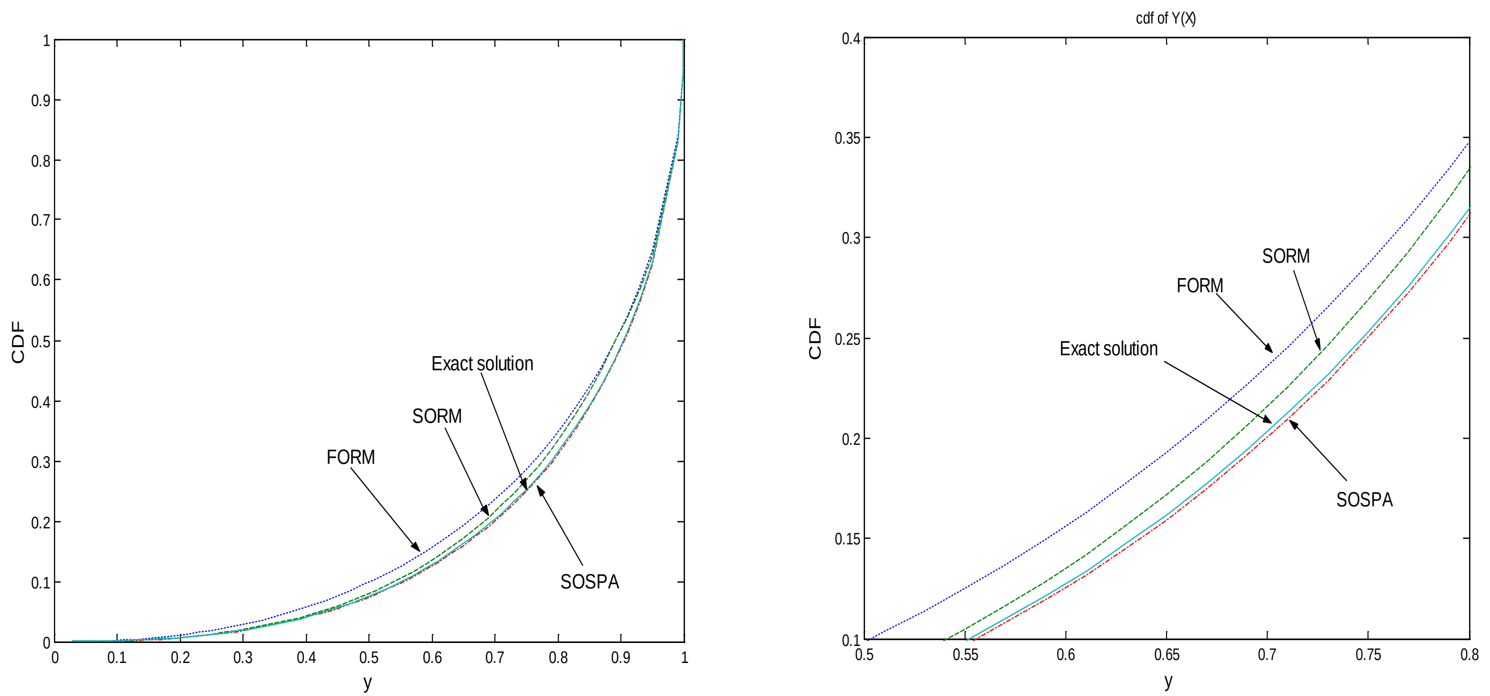
Example

- Performance function

$$Y = g(\mathbf{X}) = \frac{X_1 + X_2}{X_1 + X_2 + X_3}$$

- X_i follows distributions with degree of freedom 2, 3, 1

Result



SOSPA is the more accurate method than SORM for this example.

Comparison

FORM	Transformation from X to U May increase nonlinearity 1st approximation of g(U)
SORM	Transformation from X -> U May increase nonlinearity 2nd approximation of g(U)
FOSPA	No transformation from X to U Doesn’t increase nonlinearity 1st approximation of g(X)
SOSPA	Transformation from X -> U May increase nonlinearity 2nd approximation of g(U)

Accuracy (generally)

SOSPA>=SORM

SORM > FORM

FOSPA >= FORM

(FOSPA>=SORM) or (FOSPA<=SORM)

Efficiency (generally)

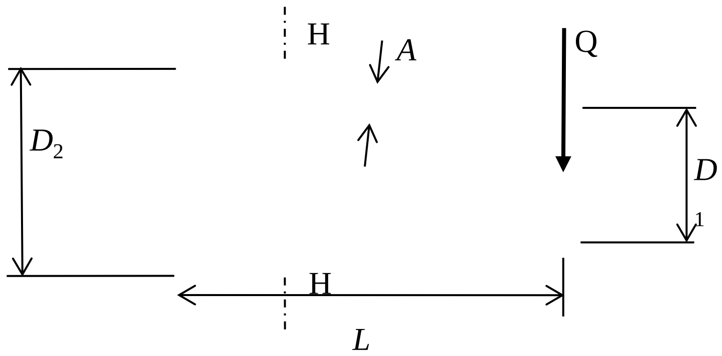
FOSPA >= FORM>SOSPA and SORM

SOSPA=SORM

>: better than; =: the same as

Example

X	Mean	STD	Distribution
a	10.0	0.01	Normal
d_1	40	0.01	Normal
d_2	58	0.01	Normal
L	120	0.01	Normal
Q	2700	50	Gumbel
S_y	200 N/mm^2	20 N/mm^2	Normal



$$g(\mathbf{X}) = S_y - \frac{Q(2l - d_2)d_2}{4I}$$

Probability of failure

$$P_f = \Pr\{g(\mathbf{X}) < 0\}$$

Results

Method	P_f	Function Calls
FORM	0.0149×10^{-4}	66
SORM	0.0149×10^{-4}	122
FOSPA	0.1423×10^{-4}	51
SOSPA	0.0149×10^{-4}	122
MCS	0.1300×10^{-4}	10^6

For this example

Accuracy: FOSPA>SOSPA=SORM>FORM

Efficiency: FOSPA>FORM>SOSPA=SORM

Future Work

- Sampling based SPA for large scale problems
  - Use fewer samples to generate CGF of performance function
- Extension to design under uncertainty
  - Reliability-based design
  - Robust design
  - Design for Six Sigma

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