Unified Uncertainty Analysis by the First Order Reliability Method

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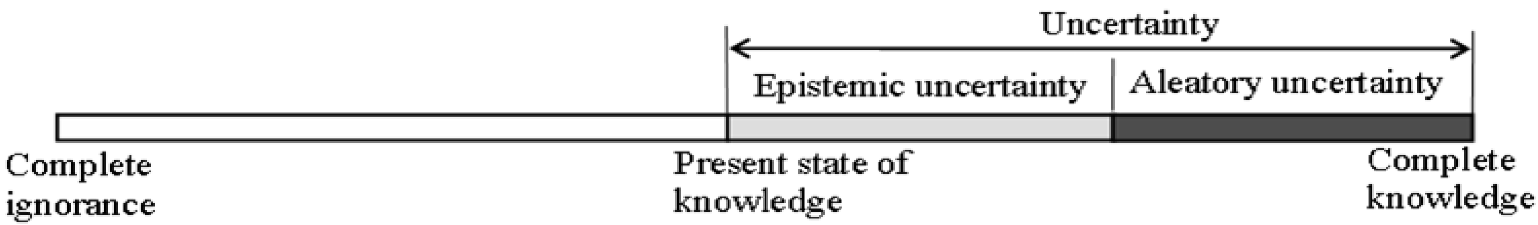
Outline

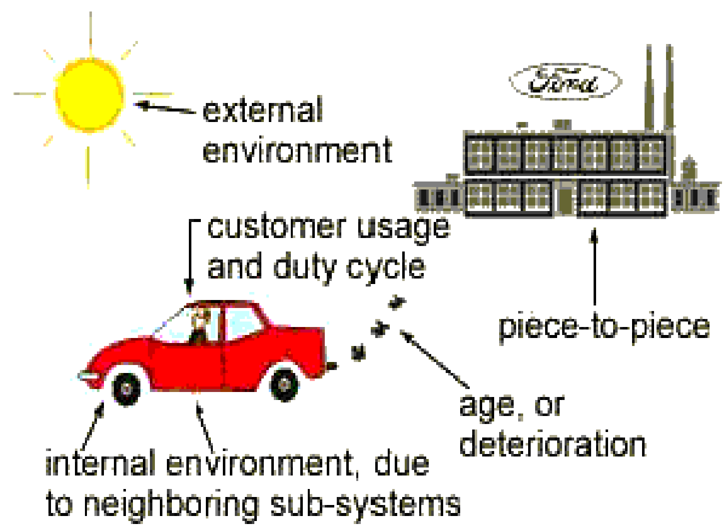
- Why epistemic uncertainty
- Framework of unified uncertainty analysis
- FORM based approach
- Example
- Conclusions

Uncertainty

- The difference between the present state of knowledge and the complete knowledge

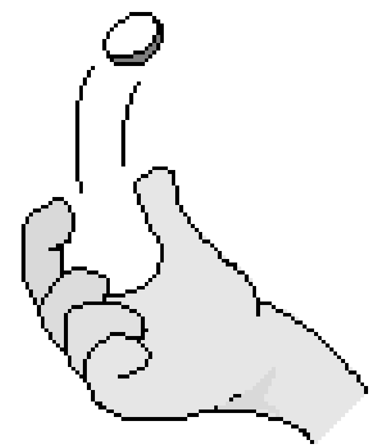


- Piece-to-piece variation
- Customer usage and duty cycle
- External operating environment (climate, road conditions, etc.)
- Internal operating environment (interaction with neighboring subsystems)



Uncertainty Types

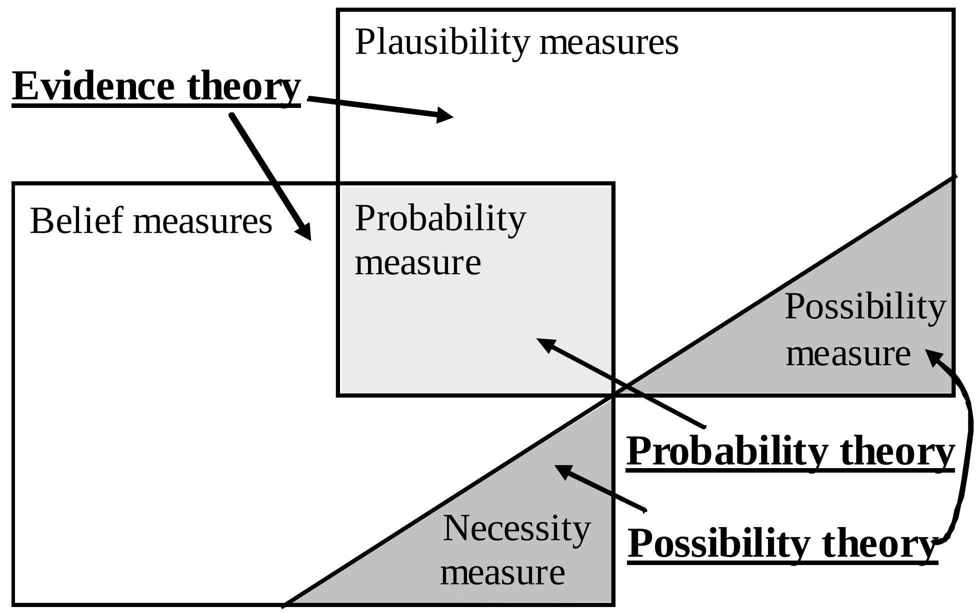
- Aleatory type – inherent variation due to the nature of randomness
- Epistemic type - lack of knowledge
- Uniform and unbent coin
  - Pr(heads) =0.5
  - Aleatory: the chance of heads



- Bent coin
  - Epistemic: Pr(heads)=?
  - Aleatory: the chance of heads

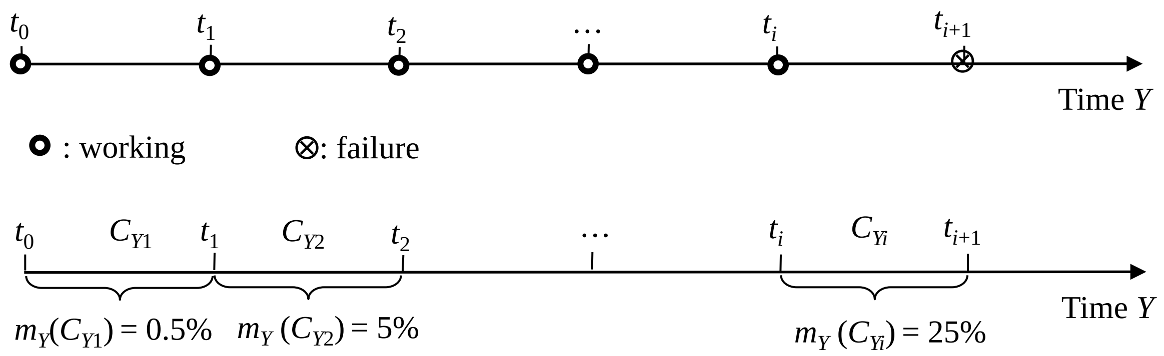
Model Epistemic Uncertainty

- Probability theory
  - Distributions
- Evidence theory
  - Intervals
- Fuzzy set
  - Membership functions
- We will focus on evidence theory in this preliminary study.



Intervals in Evidence Theory

- Example: periodical condition monitoring



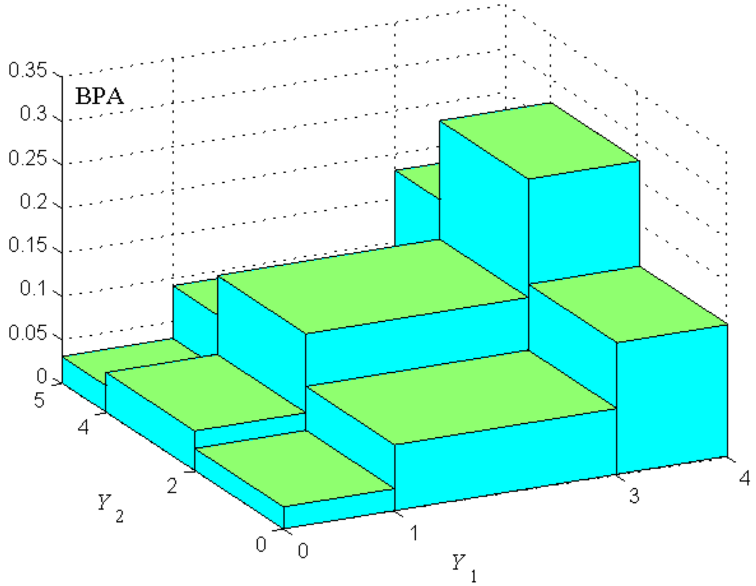
***m*: Basic Probability Assignment**

- The CAE simulation of an application has a 10% error.
- The diameter is 10±0.01 mm.

Basic Probability Assignment

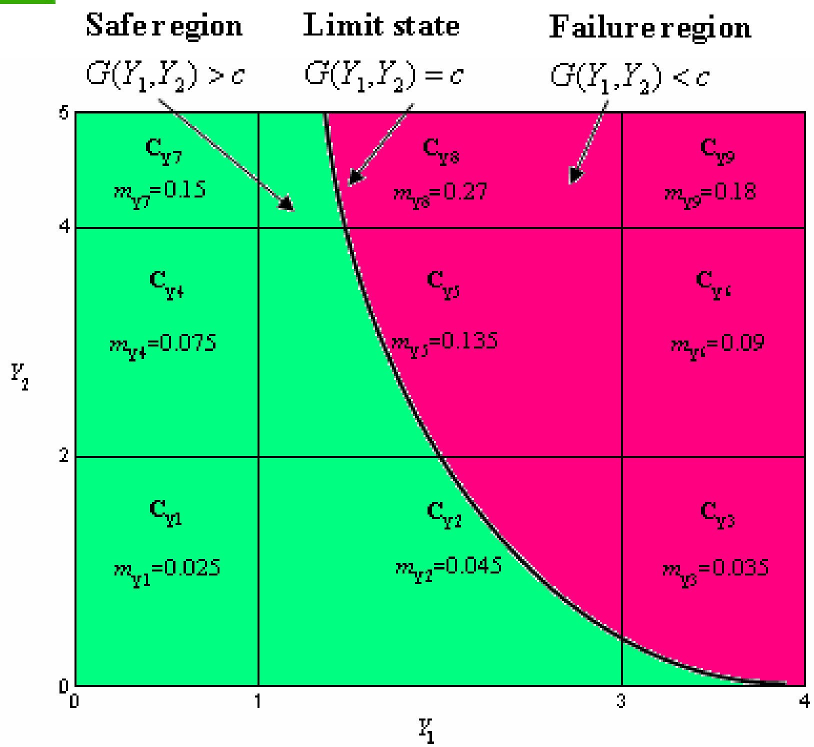
- BPA can be obtained from statistical data, multiple sources, and expert opinions.
- Joint BPA

Y_1 \ Y_2	$m_{Y2}([0,2]) = 0.25$	$m_{Y2}([2,4]) = 0.45$	$m_{Y2}([4,5]) = 0.3$
$m_{Y1}([0,1]) = 0.1$	0.025	0.045	0.03
$m_{Y1}([1,3]) = 0.3$	0.075	0.135	0.09
$m_{Y1}([3,4]) = 0.6$	0.15	0.27	0.18



Belief and Plausibility

- Performance function Z=G(**Y**)
  - **Y** - Parameters with epistemic uncertainty
- Failure event: F = {Z=G(**Y**)<c}



- Belief

$$Bel(F) = \sum_{A \in F} m_{\mathbf{Y}}(A) = 0.09 + 0.18 = 0.27$$

- Plausibility

$$Pl(F) = \sum_{A \cap F \neq \emptyset} m_{\mathbf{Y}}(A) =$$

$$0.045 + 0.135 + 0.27 + 0.03 + 0.09 + 0.18 = 0.75$$

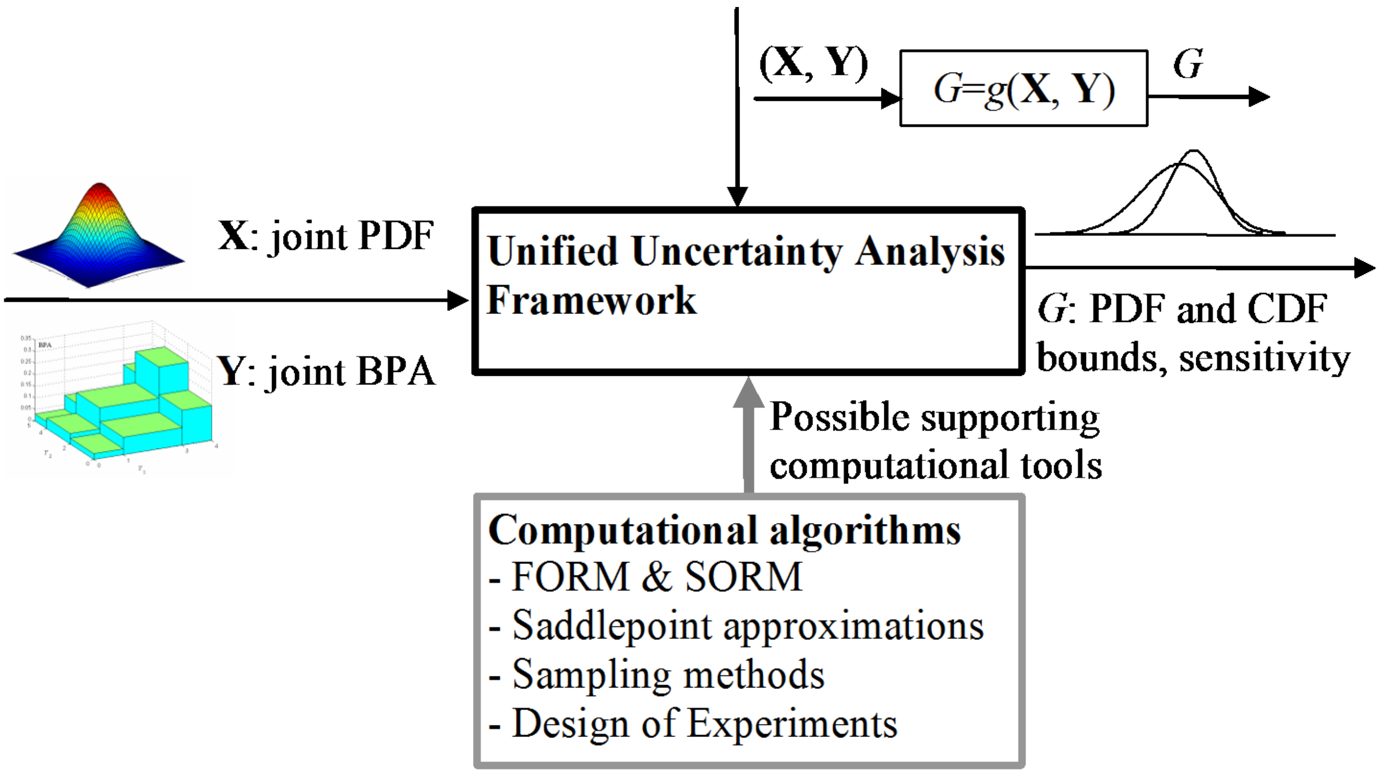
- Probability of failure

$$P_f = \Pr\{G(\mathbf{Y}) < 0\}$$

$$Bel(F) \le p_f \le Pl(F)$$

Unified Uncertainty Analysis

- Performance function Z=G(**X**, **Y**)
  - **X** – Aleatory parameters with distributions
  - **Y** – Epistemic parameters with intervals



Probability Bounds (Belief and Plausibility)

$$p_f = \sum_{i=1}^{n} m_{\mathbf{Y}}(\mathbf{C}_{\mathbf{Y}i}) \Pr\left\{ G(\mathbf{X}, \mathbf{Y}) < c \middle| \mathbf{Y}_i \in \mathbf{C}_{\mathbf{Y}i} \right\}$$

$$Bel(F) = (p_f)_{\min}$$

$$= \sum_{i=1}^{n} m_{\mathbf{Y}}(\mathbf{C}_{\mathbf{Y}i}) \Pr\left\{ G_{\max}(\mathbf{X}, \mathbf{Y}) < c \middle| \mathbf{Y} \in \mathbf{C}_{\mathbf{Y}i} \right\}$$

$$Pl(F) = (p_f)_{\max}$$

$$= \sum_{i=1}^{n} m_{\mathbf{Y}}(\mathbf{C}_{\mathbf{Y}i}) \Pr\left\{ G_{\min}(\mathbf{X}, \mathbf{Y}) < c \middle| \mathbf{Y} \in \mathbf{C}_{\mathbf{Y}i} \right\}$$

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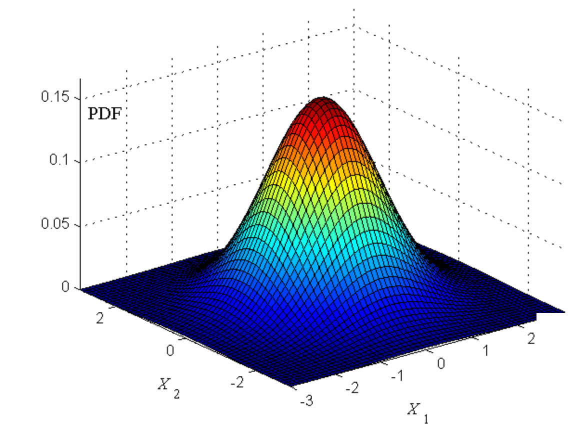
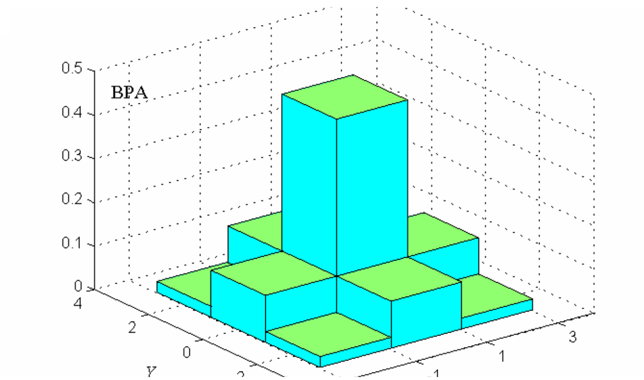
A Simple Example

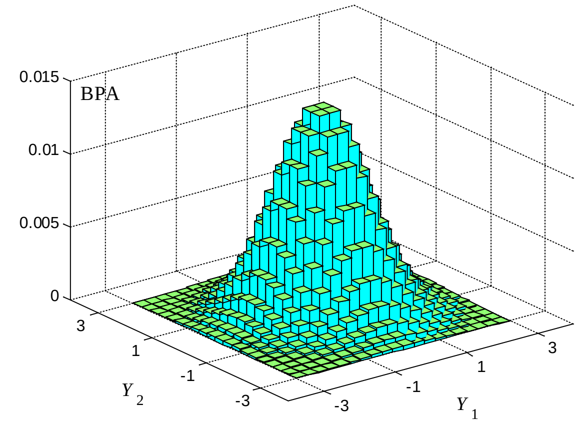
$$G = g(\mathbf{X},\mathbf{P}) = X_1^2 + X_2^2 + Y_1^2 + Y_2^2$$

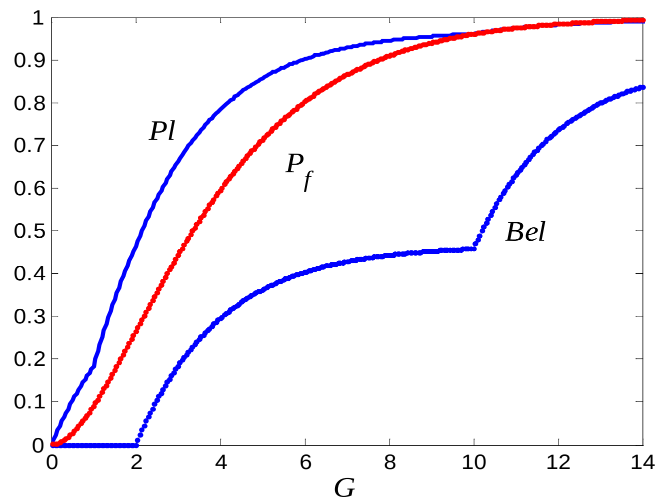
X_1, X_1, Y_1, Y_2 ~ N(0, 1);

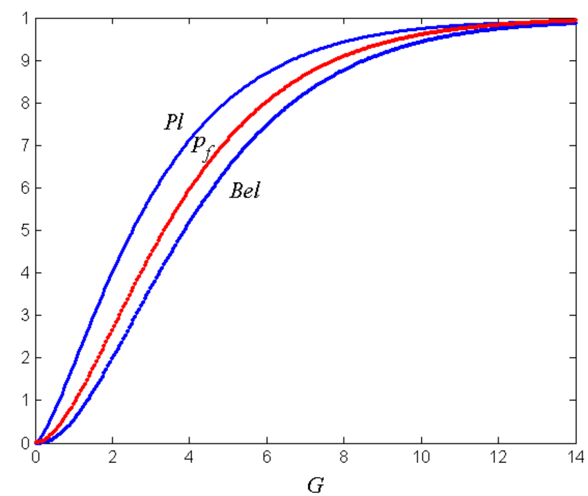
Only the joint BPA of Y_1, Y_2 are know











Computational Issues

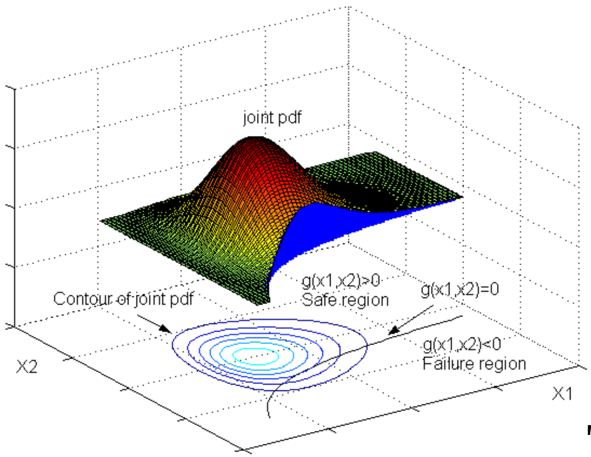
- Calculate the probability in each set of interval variables.
- Search the maximum and minimum performance function values in each set.
- Monte Carlo simulation may not be applicable.
- The analysis is computationally intensive.

$$Bel(F) = \sum_{i=1}^{n} m_{\mathbf{Y}}(\mathbf{C}_{\mathbf{Y}i}) \Pr\left\{ G_{\max}(\mathbf{X}, \mathbf{Y}) < c \,\middle|\, \mathbf{Y} \in \mathbf{C}_{\mathbf{Y}i} \right\}$$

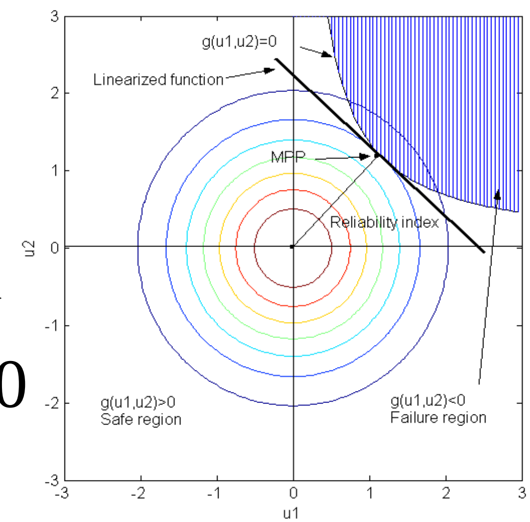
A FORM-Based Approach

First Order Reliability Method

$$p_f = \Pr\{G(\mathbf{X}) < 0\}$$



X -> **U** (standard normal)

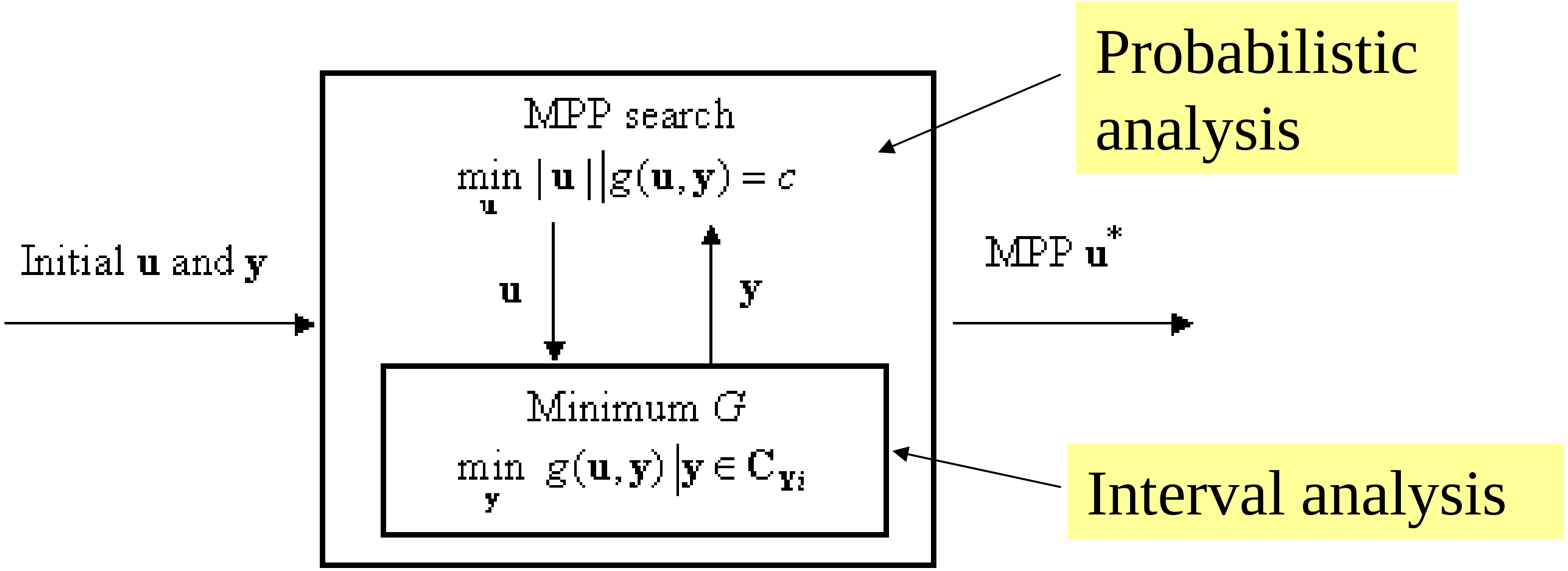


$$\min_{\mathbf{u}} |\mathbf{u}| \,\big|\, G(\mathbf{u}) = 0$$

The most Probable Point **u*** and the shortest distance (reliability index) β

$$p_f = \Phi(-\beta)$$

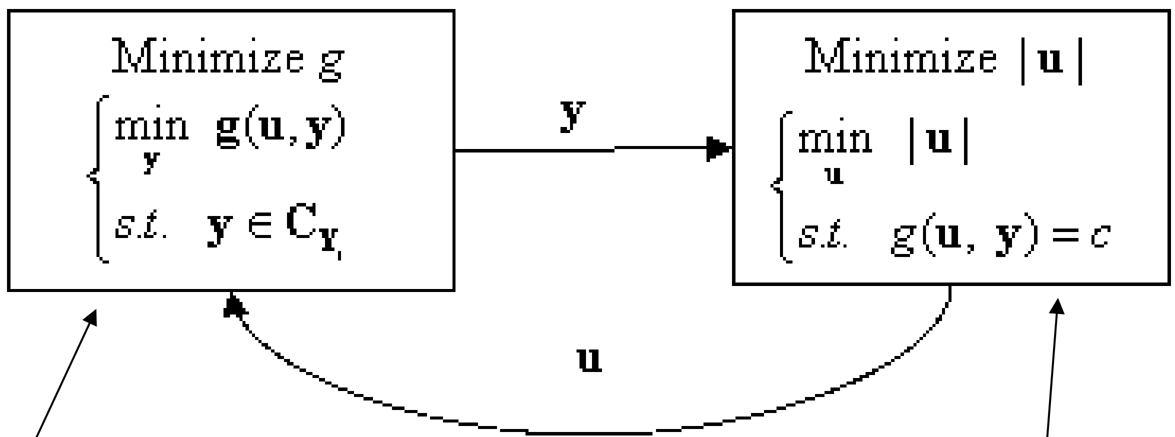
Expensive Double-Loop Method



$$\Pr\left\{ G_{\max}(\mathbf{X},\mathbf{Y}) < c \,\middle|\, \mathbf{Y} \in \mathbf{C}_{\mathbf{Y}i} \right\} = \Phi(-\beta)$$

$$Bel(F) = \sum_{i=1}^{n} m_{\mathbf{Y}}(\mathbf{C}_{\mathbf{Y}i}) \Pr\left\{ G_{\max}(\mathbf{X},\mathbf{Y}) < c \,\middle|\, \mathbf{Y} \in \mathbf{C}_{\mathbf{Y}i} \right\}$$

New Sequential Single Loops Method



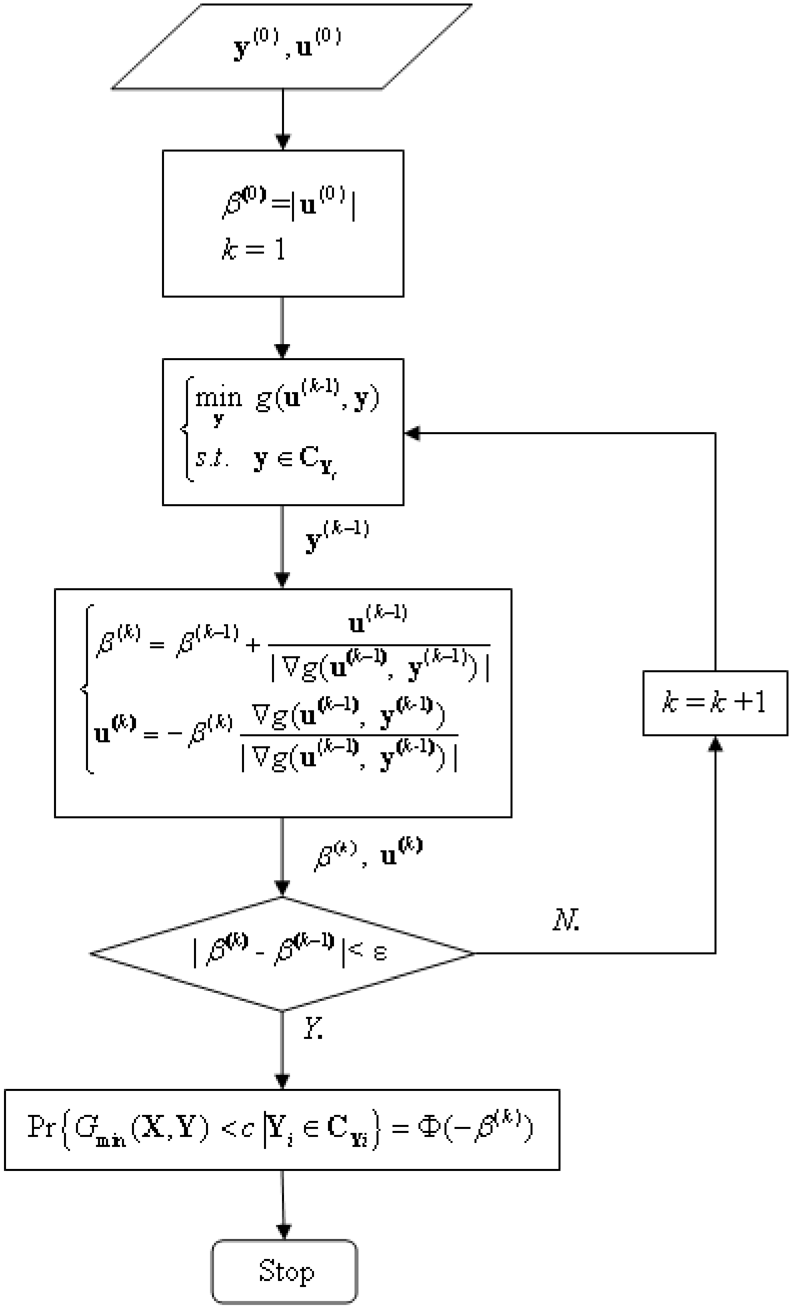
IA: Optimization algorithms

PA

$$\begin{cases} \beta^{(k)} = \beta^{(k-1)} + \dfrac{\mathbf{u}^{(k-1)}}{|\nabla g(\mathbf{u}^{(k-1)}, \mathbf{y}^{(k-1)})|} \\ \mathbf{u}^{(k)} = -\beta^{(k)} \dfrac{\nabla g(\mathbf{u}^{(k-1)}, \mathbf{y}^{(k-1)})}{|\nabla g(\mathbf{u}^{(k-1)}, \mathbf{y}^{(k-1)})|} \end{cases}$$

Flowchart

The new method is more efficient.



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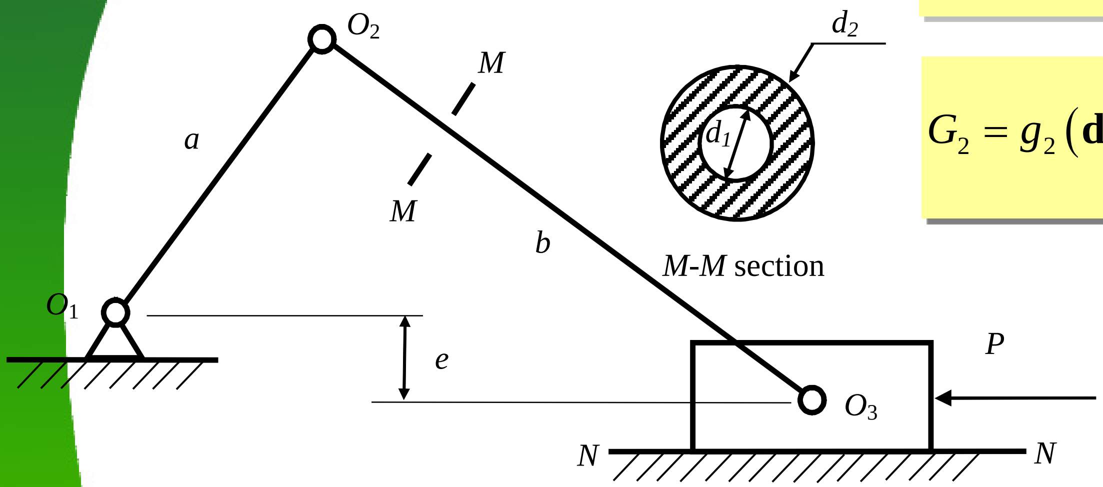
Example

$$G_1 = g_1(\mathbf{X},\mathbf{Y}) = S - \frac{4P(a+b)}{\pi\left(\sqrt{(a+b)^2 - e^2} - \mu e\right)\left(d_2^2 - d_1^2\right)}$$

$$G_2 = g_2(\mathbf{d},\mathbf{X},\mathbf{Y}) = \frac{\pi^3 E\left(d_2^4 - d_1^4\right)}{64b^2} - \frac{P(a+b)}{\sqrt{(a+b)^2 - e^2} - \mu e}$$

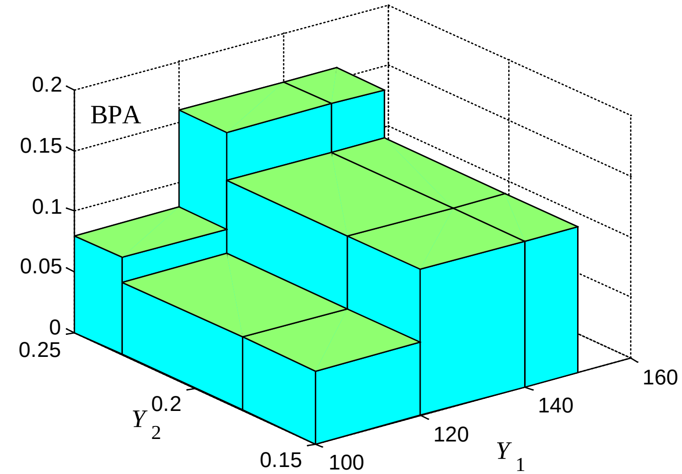


Epistemic variables

	Variable	Intervals	BPA
Y_1	Offset e	[100, 120]	0.2
		[120, 140]	0.4
		[140, 150]	0.4
Y_2	Coefficient of friction μ	[0.15, 0.18]	0.3
		[0.18, 0.23]	0.3
		[0.23, 0.25]	0.4

Aleatory variables

Variable	Symbols	Mean	Std	Distribution
X_1	a	100 mm	0.01 mm	Normal
X_2	b	300 mm	0.01 mm	Normal
X_3	P	250 kN	25 kN	Normal
X_4	E	200 GPa	30 GPa	Normal
X_5	S	290 MPa	29 MPa	Normal



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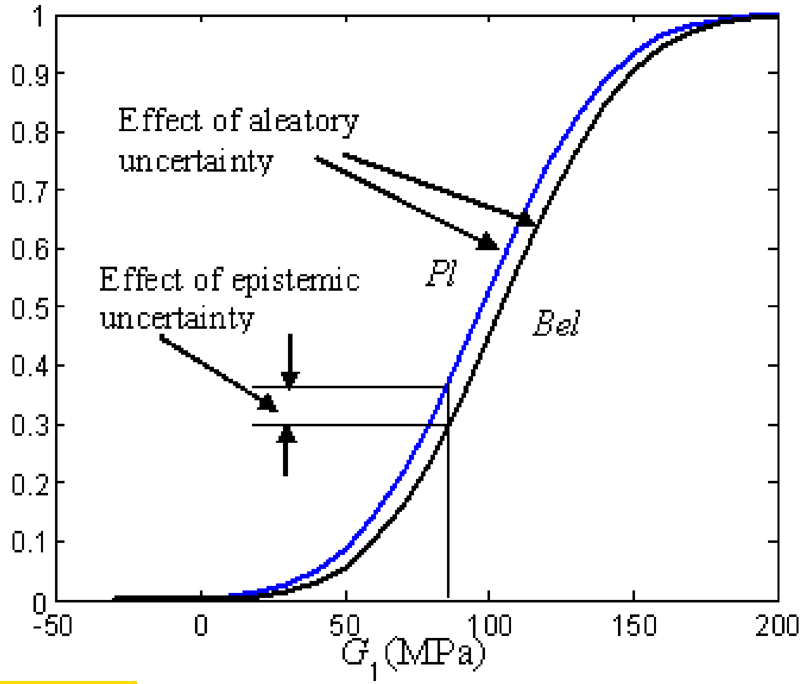
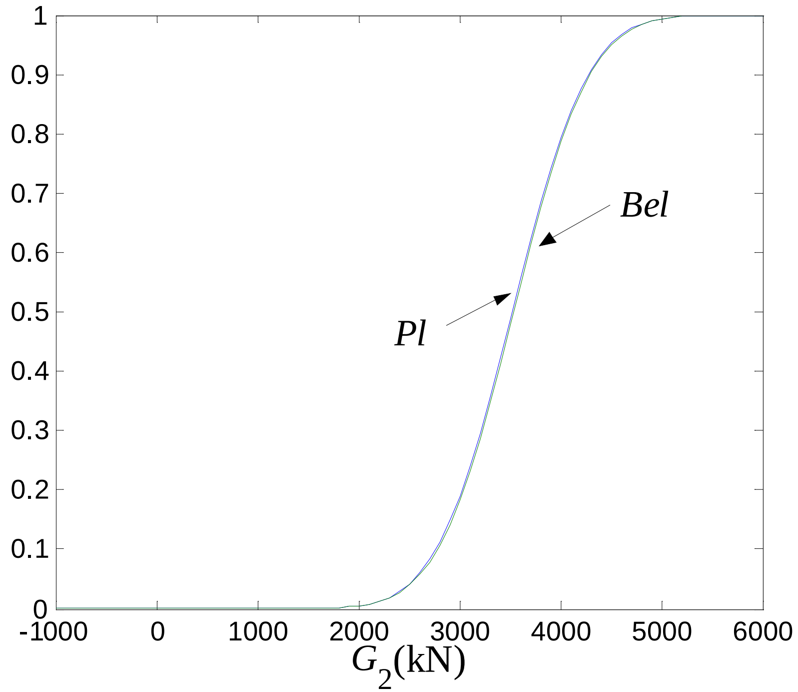
Results

	Belief	Plausibility	N_1	N_2
$F_1 = \{\mathbf{X},\mathbf{Y} \mid g_1(\mathbf{X},\mathbf{Y}) < 0\}$	$Bel(F_1) = 1.3309\times10^{-3}$	$Pl(F_1) = 2.7226\times10^{-3}$	8657	984
$F_2 = \{\mathbf{X},\mathbf{Y} \mid g_2(\mathbf{X},\mathbf{Y}) < 0\}$	$Bel(F_2) = 5.068\times10^{-11}$	$Pl(F_2) = 5.331\times10^{-11}$	4752	972

N_1 - # of function calls by double-loop method

N_2 - # of function calls by sequential single loop method



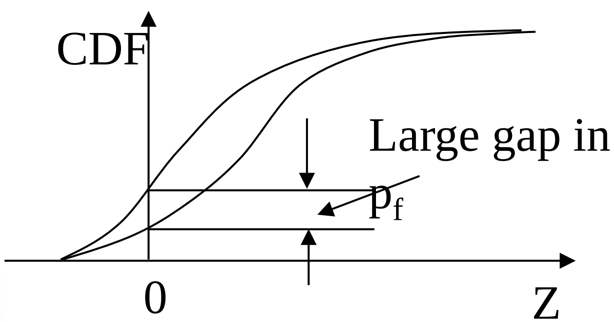


Future Work

- Sensitivity: identify the most important uncertain variables

Most important **Y'**

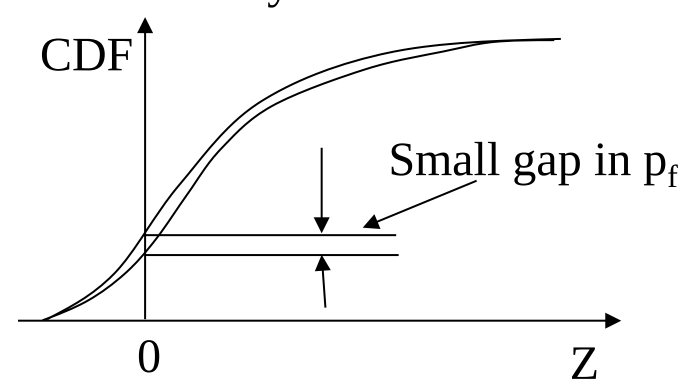
Large effect of epistemic uncertainty



Sensitivity analysis

Collect more Information on **Y'**

Reduced effect of epistemic uncertainty



- Integrate the method With optimization

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