ASME DETC 2010

A Second-Order Reliability Method with First-Order Efficiency

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Outline

- Objective
- Background
- Methodology
- Examples
- Conclusions

Objective

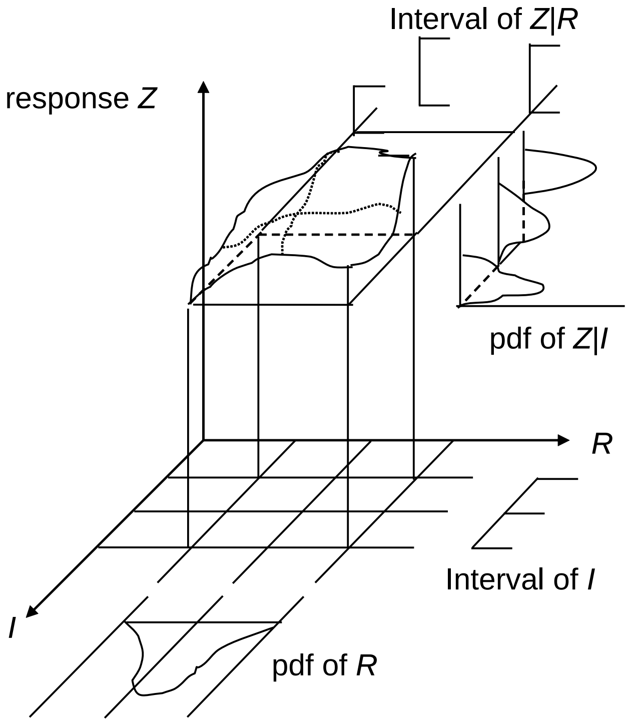
- Improve the efficiency of the FORM (First Order Reliability Method)
- Preserve the efficiency of the FORM
- Avoid sampling-based method

Background - FORM

Robustness with **R** and **I**

- Robustness measured by standard deviation (std) of *Z*, σ_Z
- Because of **I**,

$$\sigma_Z^{\min} \le \sigma_Z \le \sigma_Z^{\max}$$



Task

- Given
  - Distributions of **R**
  - Width of **I**
- Find
  - Effect of **R**: average std

$$\overline{\sigma}_Z = \frac{1}{2}\left(\sigma_Z^{\max} + \sigma_Z^{\min}\right)$$

  - Effect of **I**, std difference

$$\delta\sigma_Z = \sigma_Z^{\max} - \sigma_Z^{\min}$$

Challenges

- $Z = f(\mathbf{X}) = f(\mathbf{R},\mathbf{I})$
  - f is nonlinear and expensive
  - Large dimension of (**R**,**I**)
  - Probabilistic analysis (PA) for σ_Z
  - Interval analysis (IA) for $\sigma_Z^{\max}$ and $\sigma_Z^{\min}$
- Solution: Taylor expansion

Problem with the 1st Order Expansion

- $Z = f(\mathbf{R},\mathbf{I})$

$$\Delta Z = f(\mathbf{R},\mathbf{I}) - f(\bar{\mathbf{R}},\bar{\mathbf{I}}) \approx \sum_{i=1}^{nR} \left.\frac{\partial f}{\partial R_i}\right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} \Delta R_i + \sum_{j=1}^{nI} \left.\frac{\partial f}{\partial I_j}\right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} = \nabla_{\mathbf{R}}^T \Delta\mathbf{R} + \nabla_{\mathbf{I}}^T \Delta\mathbf{I}$$

- Assume R_i are independent

$$\sigma_Z = \sqrt{(\nabla_{\mathbf{R}}^T)^2 \sigma_{\mathbf{R}}^2} = \sqrt{\sum_{i=1}^{nR} \left(\left.\frac{\partial g}{\partial R_i}\right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} \right)^2 \sigma_{Ri}^2}$$

- Problem: no $\sigma_Z^{\min}$ and $\sigma_Z^{\max}$

Semi-2nd-Order Approximation

$$\Delta f(\mathbf{R}\ ,\mathbf{I}) = \nabla_{\mathbf{R}}^{T}\Delta\mathbf{R} + \nabla_{\mathbf{I}}^{T}\Delta\mathbf{I} + \frac{1}{2}\Delta\mathbf{X}^{T}\nabla^{2}\Delta\mathbf{X}$$

Interaction terms of **R** and **I** are kept.

$$\nabla^2 = \begin{pmatrix} & 0 & & \frac{\partial^2 g}{\partial R_1 \partial I_1} & \cdots & \frac{\partial^2 g}{\partial R_1 \partial I_{nI}} \\ & & & \cdots & \cdots & \cdots \\ & & & \frac{\partial^2 g}{\partial R_{nR} \partial I_1} & \cdots & \frac{\partial^2 g}{\partial R_{nR} \partial I_{nI}} \\ \frac{\partial^2 g}{\partial R_1 \partial I_1} & \cdots & \frac{\partial^2 g}{\partial R_1 \partial I_{nI}} & & & \\ \cdots & \cdots & \cdots & & 0 & \\ \frac{\partial^2 g}{\partial R_{nR} \partial I_1} & \cdots & \frac{\partial^2 g}{\partial R_{nR} \partial I_{nI}} & & & \end{pmatrix}$$

σ_z Bounds

$$\sigma_z = \sqrt{\sum_{i=1}^{nR}\left(\left.\frac{\partial f}{\partial R_i}\right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} + \sum_{j=1}^{nI}\left.\frac{\partial^2 f}{\partial R_i \partial I_j}\right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} \Delta I_j\right)^2 \sigma_{Ri}^2}$$

Calls f function $(nR + 3 \times nR \times nI + 1)$ times

For $\sigma_Z^{\min}$

$$\min_{\mathbf{I}} \sum_{i=1}^{nR}\left(\left.\frac{\partial f}{\partial R_i}\right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} + \sum_{m=1}^{nI}\left.\frac{\partial^2 f}{\partial R_i \partial I_j}\right|_{\bar{\mathbf{R}},\bar{\mathbf{I}}} \Delta I_j\right)^2 \sigma_i^2$$

$$s.t.\ \ \mathbf{I}^l \le \mathbf{I} \le \mathbf{I}^u$$

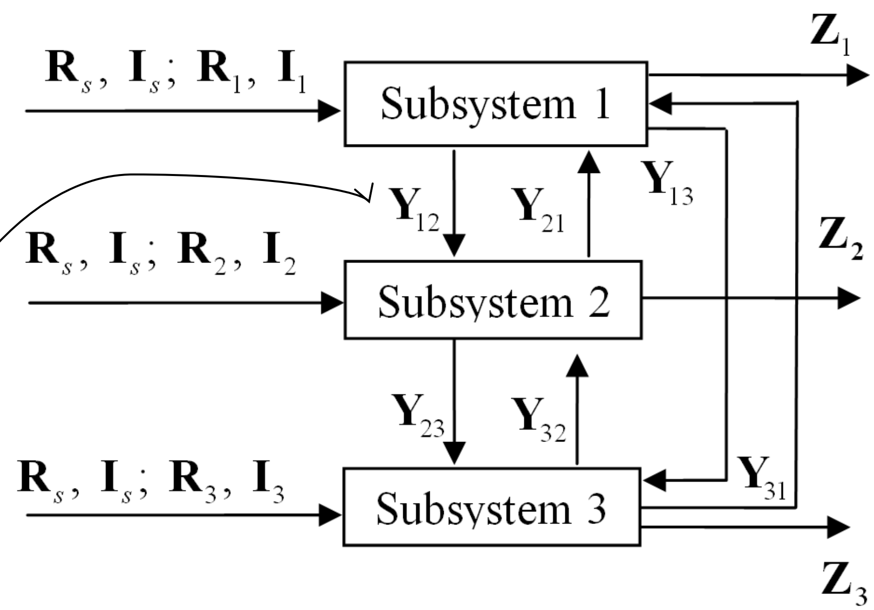
No f function calls

Multidisciplinary Systems

Multidisciplinary analysis (MDA)

- Given: **R, I**
- Find: **Z**
- Step 1: solve **Y**
  - $\mathbf{Y}_{ij} = \mathbf{Y}_{ij}(\mathbf{R}_s, \mathbf{I}_s, \mathbf{R}_i, \mathbf{I}_i, \mathbf{Y}_{\bullet i})$
  - Iterative process
- Step 2: solve Z_i

$$\mathbf{Z}_i = \mathbf{Z}_i(\mathbf{R}_s, \mathbf{I}_s, \mathbf{R}_i, \mathbf{I}_i, \mathbf{Y}_{\bullet i})$$



Linking variables

Robustness assessment needs to call MDA repeatedly.

Robustness Assessment

- Given: $(\mathbf{R}_s, \mathbf{I}_s, \mathbf{R}_i, \mathbf{I}_i)$
- Find: $\sigma_Z^{\min}$ and $\sigma_Z^{\max}$
- Step 1: Solve **Y**
  - Semi-2nd-Taylor expansion

$$\Delta\mathbf{Y} = \mathbf{C}^{-1}\mathbf{A}\Delta\mathbf{R}_s + \mathbf{C}^{-1}\mathbf{B}\Delta\mathbf{R} + \mathbf{C}^{-1}\mathbf{M}$$

$$\mathbf{B}_i^{\mathbf{Y}} = \begin{pmatrix} \mathbf{B}_{i1}^{\mathbf{Y}} & & & & & \\ & \cdots & & & \mathbf{0} & \\ & & \mathbf{B}_{i(i-1)}^{\mathbf{Y}} & & & \\ & & & \mathbf{B}_{i(i+1)}^{\mathbf{Y}} & & \\ & \mathbf{0} & & & \cdots & \\ & & & & & \mathbf{B}_{inSi}^{\mathbf{Y}} \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} \mathbf{U}_1 & -\frac{\partial \mathbf{Y}_{1\cdot}}{\partial \mathbf{Y}_{2\cdot}} & \cdots & -\frac{\partial \mathbf{Y}_{1\cdot}}{\partial \mathbf{Y}_{nS\cdot}} \\ -\frac{\partial \mathbf{Y}_{2\cdot}}{\partial \mathbf{Y}_{1\cdot}} & \mathbf{U}_2 & \cdots & -\frac{\partial \mathbf{Y}_{2\cdot}}{\partial \mathbf{Y}_{nS\cdot}} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{\partial \mathbf{Y}_{nS\cdot}}{\partial \mathbf{Y}_{1\cdot}} & -\frac{\partial \mathbf{Y}_{nS\cdot}}{\partial \mathbf{Y}_{2\cdot}} & \cdots & \mathbf{U}_{nS} \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} A_1^{\mathbf{Y}} \\ A_2^{\mathbf{Y}} \\ \cdots \\ A_{nS}^{\mathbf{Y}} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} B_1^{\mathbf{Y}} & & \mathbf{0} & \\ 0 & B_2^{\mathbf{Y}} & & \\ & & \cdots & \\ & \mathbf{0} & & B_{nS}^{\mathbf{Y}} \end{pmatrix}$$

$$\mathbf{A}_i^{\mathbf{Y}} = \begin{pmatrix} \mathbf{A}_{i1}^{\mathbf{Y}} & & & & & \\ & \cdots & & & \mathbf{0} & \\ & & \mathbf{A}_{i(i-1)}^{\mathbf{Y}} & & & \\ & & & \mathbf{A}_{i(i+1)}^{\mathbf{Y}} & & \\ & \mathbf{0} & & & \cdots & \\ & & & & & \mathbf{A}_{inSi}^{\mathbf{Y}} \end{pmatrix}$$

Robustness Assessment

- Step 2 – Solve Z_i
  - Semi-2nd-Taylor expansion

$$\Delta \mathbf{Z}_i = \mathbf{A}_i^{\mathbf{Z}} \Delta \mathbf{R}_s + \mathbf{B}_i^{\mathbf{Z}} \Delta \mathbf{R}_i + \frac{\partial \mathbf{Z}_i}{\partial \mathbf{Y}_{\cdot i}} \Delta \mathbf{Y}_{\cdot i} + \mathbf{M}_i^{\mathbf{Z}}(\mathbf{I}_s, \mathbf{I}_i)$$

$$\mathbf{A}_i^{\mathbf{Z}} = \begin{pmatrix} \frac{\partial Z_{i,1}}{\partial R_{s,1}} + \sum_{k=1}^{nSI} \frac{\partial^2 Z_{i,1}}{\partial R_{s,1} \partial I_{s,k}} & \cdots & \frac{\partial Z_{i,1}}{\partial R_{s,nSR}} + \sum_{k=1}^{nS_I} \frac{\partial^2 Z_{i,1}}{\partial R_{s,nSR} \partial I_{s,k}} \\ \cdots & \cdots & \\ \frac{\partial Z_{i,n_{ij}}}{\partial R_{s,1}} + \sum_{k=1}^{nSI} \frac{\partial^2 Z_{i,n_{ij}}}{\partial R_{s,1} \partial I_{s,k}} & \cdots & \end{pmatrix}$$

$$\mathbf{B}_i^{\mathbf{Z}} = \begin{pmatrix} \frac{\partial Z_{i,1}}{\partial R_{i,1}} + \sum_{k=1}^{nI_i} \frac{\partial^2 Z_{i,1}}{\partial R_{i,1} \partial I_{i,k}} & \cdots & \frac{\partial Z_{i,1}}{\partial R_{i,nR_i}} + \sum_{k=1}^{nI_i} \frac{\partial^2 Z_{i,1}}{\partial R_{i,nR_i} \partial I_{i,k}} \\ \cdots & \cdots & \cdots \\ \frac{\partial Z_{i,n_{ij}}}{\partial R_{i,1}} + \sum_{k=1}^{nI_i} \frac{\partial^2 Z_{i,n_{ij}}}{\partial R_{i,1} \partial I_{i,k}} & \cdots & \frac{\partial Z_{i,n_{ij}}}{\partial R_{i,nR_i}} + \sum_{k=1}^{nI_i} \frac{\partial^2 Z_{i,n_{ij}}}{\partial R_{i,nR_i} \partial I_{i,k}} \end{pmatrix}$$

$$\frac{\partial \mathbf{Z}_i}{\partial \mathbf{Y}_{ki}} = \begin{pmatrix} \frac{\partial \mathbf{Z}_{i,1}}{\partial Y_{ki,1}} & \frac{\partial \mathbf{Z}_{i,1}}{\partial Y_{ki,2}} & \cdots & \frac{\partial \mathbf{Z}_{i,1}}{\partial Y_{ki,n_{ki}}} \\ \frac{\partial \mathbf{Z}_{i,2}}{\partial Y_{ki,1}} & \frac{\partial \mathbf{Z}_{i,2}}{\partial Y_{ki,2}} & \cdots & \frac{\partial \mathbf{Z}_{i,2}}{\partial Y_{ki,n_{ki}}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathbf{Z}_{i,n_{ij}}}{\partial Y_{ki,1}} & \frac{\partial \mathbf{Z}_{i,n_{ij}}}{\partial Y_{ki,2}} & \cdots & \frac{\partial \mathbf{Z}_{i,n_{ij}}}{\partial Y_{ki,n_{ki}}} \end{pmatrix}$$

$$\frac{\partial \mathbf{Z}_i}{\partial \mathbf{Y}_{\cdot i}} = \left(\frac{\partial \mathbf{Z}_i}{\partial \mathbf{Y}_{1i}}, \frac{\partial \mathbf{Z}_i}{\partial \mathbf{Y}_{2i}}, \cdots \frac{\partial \mathbf{Z}_i}{\partial \mathbf{Y}_{(i-1)i}}, \frac{\partial \mathbf{Z}_i}{\partial \mathbf{Y}_{(i+1)i}}, \cdots \frac{\partial \mathbf{Z}_i}{\partial \mathbf{Y}_{nS}} \right)$$

Robustness Assessment

- Step 3: calculate $\sigma_Z^{\min}$ and $\sigma_Z^{\max}$

$$\boldsymbol{\sigma}^2{}_Z = \mathbf{Q}(\mathbf{I})\boldsymbol{\sigma}^2{}_{Rs} + \mathbf{T}(\mathbf{I})\boldsymbol{\sigma}^2{}_R$$

$\mathbf{Q}(\mathbf{I}), \mathbf{T}(\mathbf{I})$: functions of $\mathbf{I}$,

including 1st and 2nd derivatives

$$\begin{aligned} &\min_{\mathbf{I}} \ \sigma_Z(\mathbf{I}) \\ &s.t. \ \ \mathbf{I}^l \le \mathbf{I} \le \mathbf{I}^u \end{aligned}$$

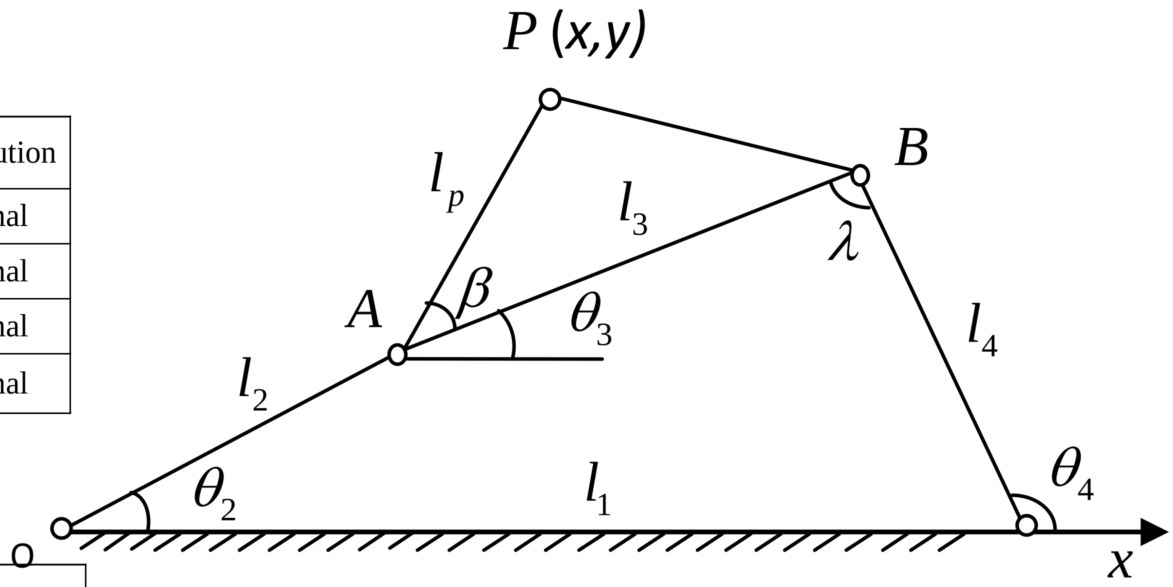
Example 1: 4-Bar Linkage Single Discipline

Random variables

Variable	Mean	Standard Deviation	Distribution
$R_1(l_2)$	51.68mm	0.20 mm	Normal
$R_2(l_3)$	297.06mm	0.20 mm	Normal
$R_3(l_4)$	197.08mm	0.20 mm	Normal
$R_4(l_p)$	99.11mm	0.20 mm	Normal

Interval variables

Variable	I^L	I^U
$I_1(l_1)$	$\bar{l}_1 - 0.5$ mm	$\bar{l}_1 + 0.5$ mm
$I_2(\beta)$	$\bar{\beta} - 1^\circ$	$\bar{\beta} + 1^\circ$



Output: P(x,y)

4-Bar Linkage - Results

	GRA	MCS
at 10°	(0.178,0.203) mm	(0.172,0.207) mm
at 10°	(0.177 ,0.203) mm	(0.171,0.207) mm
at 60°	(0.149,0.260) mm	(0.148,0.261) mm
at 60°	(0.148 ,0.259) mm	(0.146 ,0.259) mm

GRA: General robustness assessment

Multidisciplinary Cylinder Problem

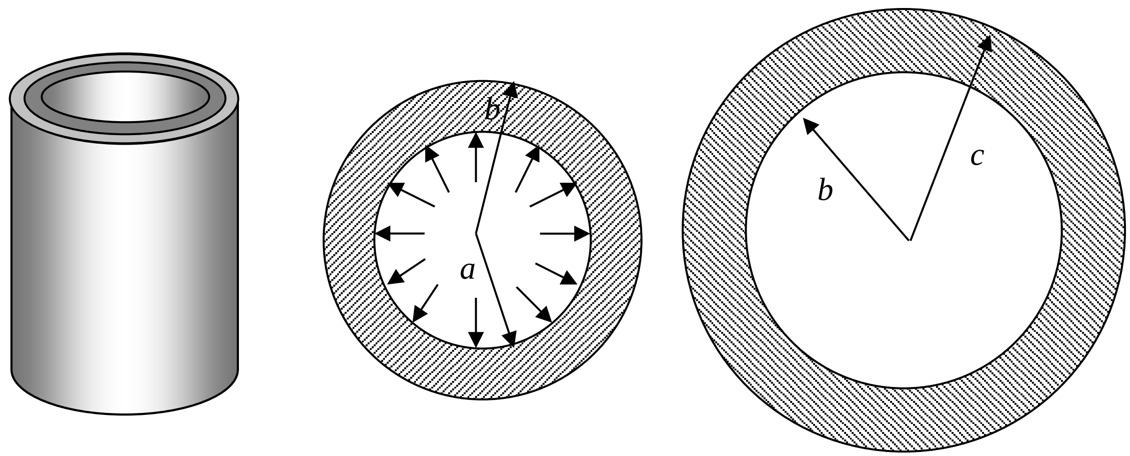
Output: P(*x*,*y*)

$$Z_1^{(1)} = \sigma_a - S$$

$$Z_1^{(2)} = \sigma_b - S$$

σ_a and σ_b: stresses at a and b

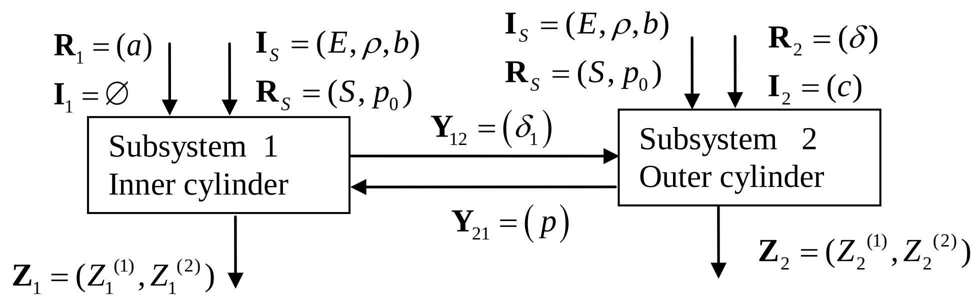
S: strength



Compound cylinder system

Subsystem 1: Inner cylinder

Subsystem 2: Outer cylinder



Cylinder Problem

Random variables

Variables	Mean	Standard Deviation	Distribution
S	10.0×10^3 psi	1.0×10^3 psi	Normal
p_0	20.0×10^3 psi	2.0×10^3 psi	Normal
a	75 in	0.1 in	Normal
δ	0.004 in	0.0004 in	Normal

Interval variables

Variables	I^L	I^U
E	$30\times10^6\times(1-2\%)$ psi	$30\times10^6\times(1+2\%)$ psi
ρ	$0.3\times(1-2\%)$	$0.3\times(1+2\%)$
b	9.95 in	10.05 in
c	14.95 in	15.05 in

Result

	GRA	MCS
$\sigma_{\mathbf{z}_1}^{\max}(psi)$	$[3.485\times10^3\ 2.444\times10^3]$	$[3.597\times10^3\ 2.570\times10^3]$
$\sigma_{\mathbf{z}_1}^{\min}(psi)$	$[3.484\times10^3\ 2.419\times10^3]$	$[3.562\times10^3\ 2.515\times10^3]$
$\sigma_{\mathbf{z}_2}^{\max}(psi)$	$[2.40\times10^3\ 1.667\times10^3]$	$[2.544\times10^3\ 1.750\times10^3]$
$\sigma_{\mathbf{z}_2}^{\min}(psi)$	$[2.373\times10^3\ 1.667\times10^3]$	$[2.50\times10^3\ 1.724\times10^3]$

Conclusions

- The robustness assessment is efficient.
- The accuracy is good when standard deviations and widths are small.
- The method is not applicable when the performance function is expanded at a saddlepoint.
- Future work will be the use of the method for robust design.

Acknowledgement

- NSF CMMI 0400081.
- China Scholarship Council.