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A DESIGN ORIENTED RELIABILITY METHODOLOGY FOR FATIGUE LIFE UNDER STOCHASTIC LOADINGS

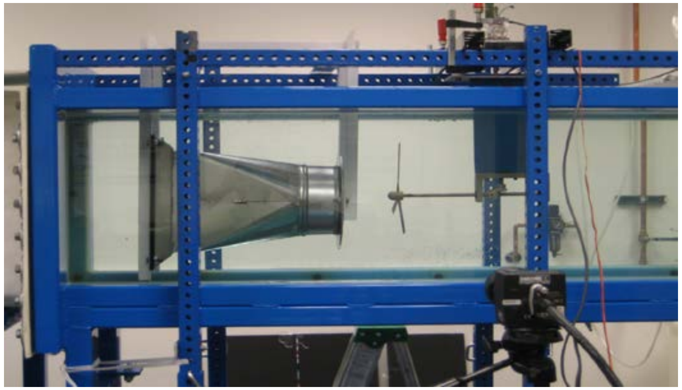
Zhen Hu, Xiaoping Du
Department of Mechanical & Aerospace Engineering
Missouri University of Science and Technology

Outline

- Background
- Fatigue reliability under stochastic loadings
- Examples
- Conclusions
- Future work

Introduction

Structures under stochastic loadings

Time dependent reliability

- Global extreme over time interval
- Investigated by Mahadevan, Mourelatos, Du, Wang, etc.

Fatigue reliability

- Local extremes
- Damages accumulation

Fatigue life analysis

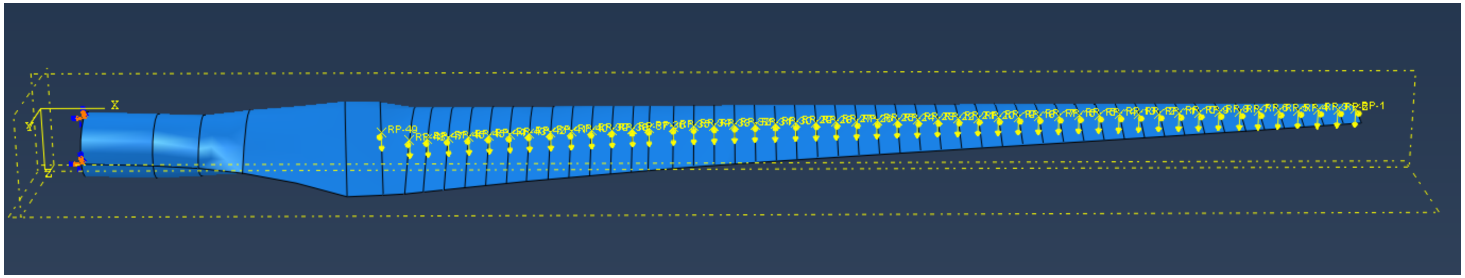
- Strain-life based method
  - Related to initial crack
- Stress-life based method
  - Based on material S-N curve
- Fracture mechanics method
  - Related to initiation and propagation of crack

Background

Current methods

- Monotonic structure responses to the stochastic loading
- Assumption of narrow-band or broad-band
- Based on field stress or strain data
- Monte Carlo Simulation
- Efficiency and accuracy may not be good for nonlinear functions w.r.t. stochastic loadings

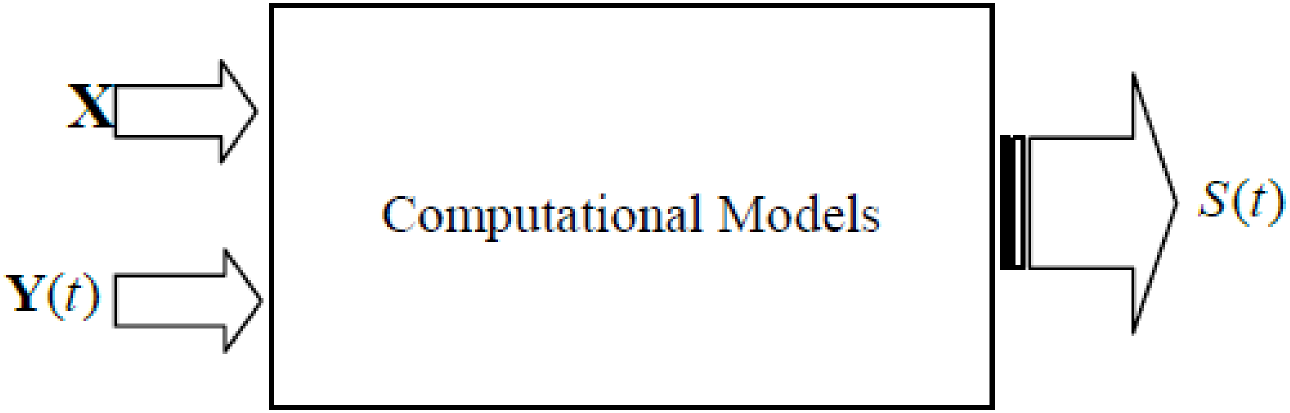
Examples



Random field loading: $Y(x,t) = \mu(x) + Z(x,t) = \mu(x) + \sum_{j=1}^{r} q_j(t)\boldsymbol{\varphi}_j$

Nonlinear combination of stochastic processes

Task



$$S(t) = g(\mathbf{X}, \mathbf{Y}(t))$$

- **X** is a vector of random variables
- **Y**(t) is a vector of stationary stochastic processes

$$p_f = \Pr\{T_F \le T\}$$

Predict fatigue reliability with less computational effort

Fatigue Reliability Analysis under Stochastic Loadings

First Order Reliability Method (FORM)

$$\begin{cases} \text{Min } \|\mathbf{u}_X\| \\ \text{s.t.} \\ D_F(T(\mathbf{u}_X)) \le 1 \end{cases}$$

Expected accumulated fatigue damage over a time interval conditioned on

$$\mathbf{x} = T(\mathbf{u}_X)$$

- Peak counting (PC)
- Level crossing counting (LCC)
- Range counting (RC)
- Rainflow counting (RFC)

Expected fatigue damage analysis

The Palmgren-Miner's rule

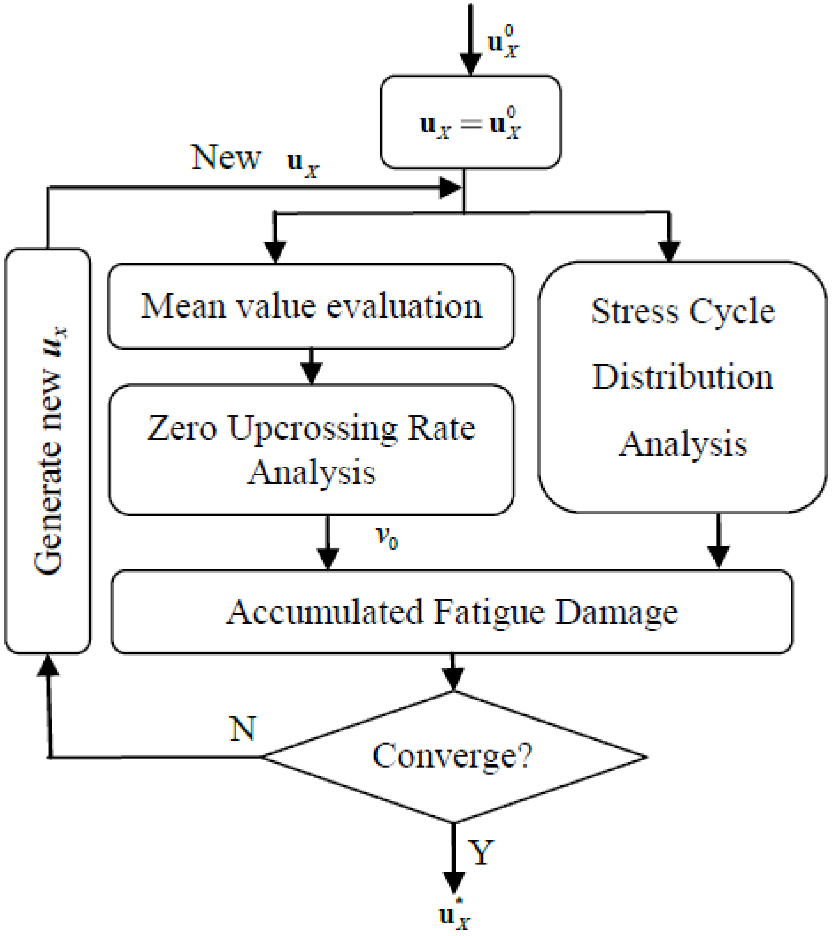
$$D_F = \sum_{i=1} \frac{n(s_i)}{N(s_i)} = \sum_{i} \frac{n(s_i)}{1/(\gamma s_i^{\kappa})}$$

$$D_F = v_0 T \int \gamma s^{\kappa} p(s) ds$$

- mean upcrossing rate of the stress process
- PDF of the stress cycle

Overview

- Calculate mean value upcrossing rate
- Obtain stress cycle distribution from stress response function
- Determine integration region for damage analysis



Mean Value Analysis

$$\mu_S = \int_0^{+\infty} s f(s) ds \quad \Rightarrow \quad \mu_S = \int_0^1 F^{-1}(P_S) dP_S$$

$$\mu_s = \int_0^1 F^{-1}(P_s) dP_s = \frac{1}{2} \sum_{j=1}^{r} w_j F^{-1}\left(\frac{\xi_j + 1}{2} \right)$$

$$P_S = F(s)$$

- CDF of stress response *S*
- Transform stress region to probability region (0,1)

Mean Value Analysis

$$\begin{cases} \underset{\mathbf{u}_{Y(t)}}{\text{Max}}\ s = F^{-1}\left(\dfrac{\xi_j+1}{2}\right) = g(T(\mathbf{u}_X), T(\mathbf{u}_{Y(t)})) \\ \text{s.t.} \\ \left\|\mathbf{u}_{Y(t)}\right\| = \begin{cases} -\Phi^{-1}\left(\dfrac{\xi_j+1}{2}\right), \text{ if } \dfrac{\xi_j+1}{2} \le 0.5 \\ \Phi^{-1}\left(\dfrac{\xi_j+1}{2}\right), \text{ otherwise} \end{cases} \end{cases}$$

Inverse MPP search is performed at each of the Gaussian quadrature points

Mean upcrossing rate analysis

Rice's formula:

$$v_0 = \omega_\mu(t)\phi\left(\beta_\mu(t)\right)\Psi\left(\dot{\beta}_\mu(t)/\omega_\mu(t)\right)$$

- The mean upcrossing rate is equivalent to the upcrossing rate of an Equivalent Gaussian process
- Derivation based on the Rice's formula got the similar expression as that from other methods
- Similar researches have been conducted by Mourelatos, Du, etc.

Stress cycle distribution analysis

For each peak larger than *s*, mathematically we have:

$$\begin{cases} S(t) > s \\ \dot{S}(t) = 0 \\ \ddot{S}(t) < 0 \end{cases}$$

$$f_{S,\dot{S},\ddot{S}}(S(t), \dot{S}(t), \ddot{S}(t))$$

$$S(t) = g(\mathbf{X}, \mathbf{Y}(t))$$

Computationally expensive

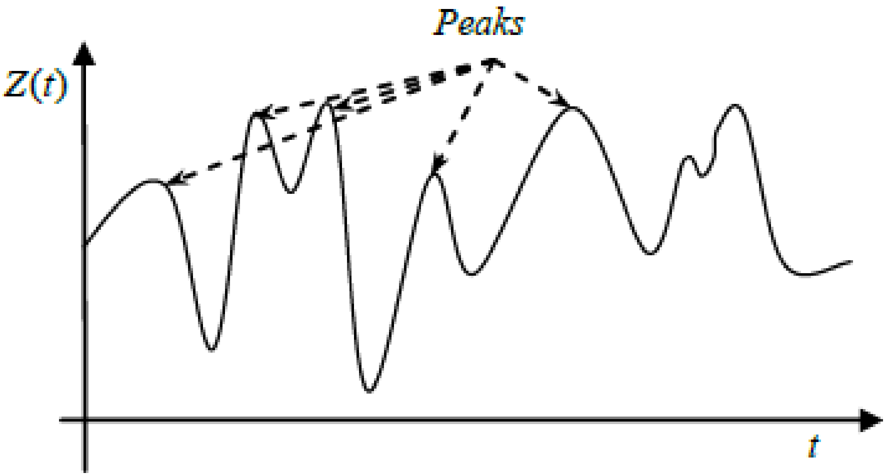


Fig. 1 Peaks of a stochastic process

Stress cycle distribution analysis

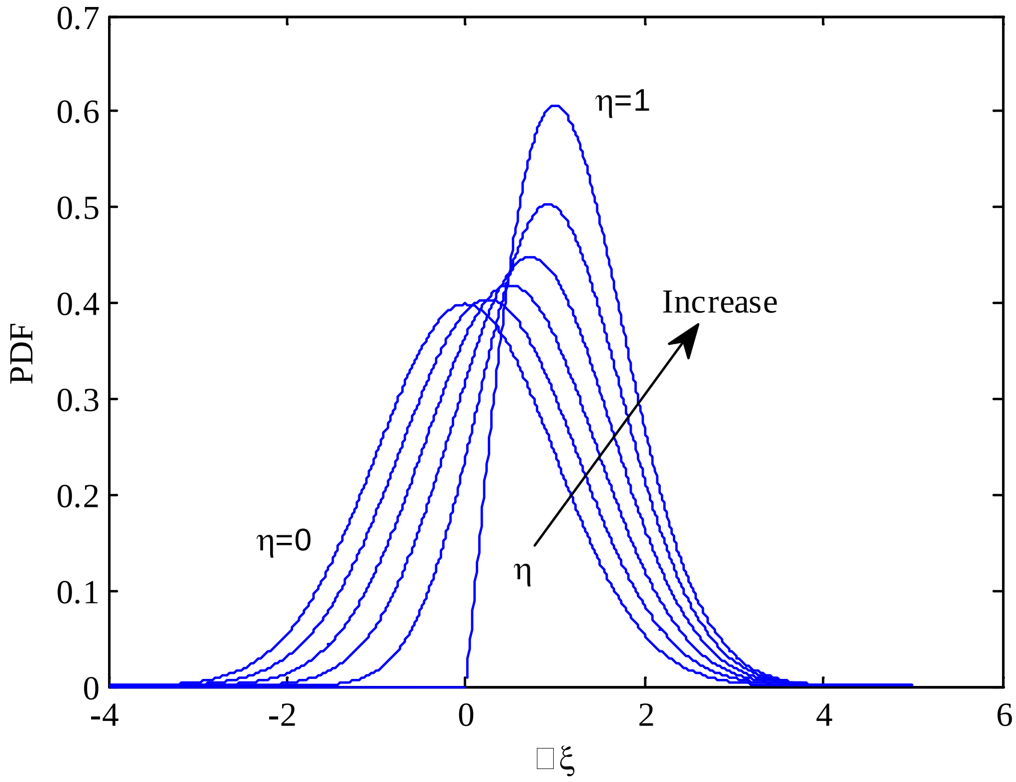
$$Z_i(t) = g(T(\mathbf{u}_X), T(\mathbf{U}_Y(t))) - s_i \xrightarrow{\text{FORM}} L(t) = \boldsymbol{\alpha}_i(t)\mathbf{U}_Y^T(t)$$

$$\Pr\{Z_i(t) = g(\mathbf{u}_X, T(\mathbf{U}_Y(t))) - s_i < 0\}$$
$$= \begin{cases} \Pr\{L(t) < -\beta_i\} = \Phi(-\beta_i), \text{ if } s_i < \mu_s \\ \Pr\{L(t) < \beta_i\} = \Phi(\beta_i), \text{ otherwise} \end{cases}$$

$$\Pr\{Z_p(t) < s_i\} = \begin{cases} \Pr\{L_p(t) < -\beta_i\}, \text{ if } s_i < \mu_s \\ \Pr\{L_p(t) < \beta_i\}, \text{ otherwise} \end{cases}$$

Stress cycle distribution analysis

For a Gaussian stochastic process with regularity factor η



Stress cycle distribution analysis

CDF: $$F_P(\xi) = \Phi\left(\frac{\xi}{\sqrt{1-\eta^2}}\right) - \eta e^{-\frac{\xi^2}{2}} \Phi\left(\frac{\eta\xi}{\sqrt{1-\eta^2}}\right)$$

PDF: $$f_p(\xi) = \sqrt{1-\eta^2}\,\phi\left(\frac{\xi}{\sqrt{1-\eta^2}}\right) + \eta\xi e^{-\frac{\xi^2}{2}} \Phi\left(\frac{\eta\xi}{\sqrt{1-\eta^2}}\right)$$

Regularity factor: $$\eta = \frac{\omega_0(t)}{\omega_p(t)} = \frac{\boldsymbol{\alpha}(t)\ddot{\mathbf{C}}_{12}(t,t)\boldsymbol{\alpha}(t)^T}{\sqrt{\boldsymbol{\alpha}(t)\ddot{\ddot{\mathbf{C}}}_{1122}(t_1,t_2)\Big|_{t_1=t_2=t}\boldsymbol{\alpha}(t)^T}}$$

Corresponding to the probability of one stress level *si*

Stress cycle distribution analysis

$$\int_{\mu_s}^{\infty} \gamma s^{\kappa} p(s) ds = \int_{P_{\mu_s}}^{1} \gamma (P^{-1}(P_{s_p}))^{\kappa} dP_{s_p}$$

$$\int_{P_{\mu_S}}^{1} \gamma (P^{-1}(P_{s_p}))^{\kappa} dP_{s_p} = \frac{1-P_{\mu_S}}{2} \sum_{i=1}^{r} w_i \gamma (P^{-1}(P_{pi}))^{\kappa}$$

$$\begin{cases} \underset{\mathbf{u}_{Y(t)}}{\text{Max}}\ s_p = P^{-1}(P_{pi}) = g(\mathbf{u}_X, \mathbf{u}_{Y(t)}, t) \\ \text{s.t.} \\ \boldsymbol{\alpha}_i(t) = \dfrac{\mathbf{u}_{Y(t)}}{\left\|\mathbf{u}_{Y(t)}\right\|} \\ \eta_i = \dfrac{\omega_0(t)}{\omega_m(t)} = \dfrac{\boldsymbol{\alpha}_i(t)\ddot{\mathbf{C}}_{12}(t,t)\boldsymbol{\alpha}_i(t)^T}{\sqrt{\boldsymbol{\alpha}_i(t)\ddot{\ddot{\mathbf{C}}}_{1122}(t_1,t_2)\big|_{t_1=t_2=t}\boldsymbol{\alpha}_i(t)^T}} \\ \beta_{obj} = F_P^{-1}(P_{pi})\big|\eta = \eta_i \\ \left\|\mathbf{u}_{Y(t)}\right\| = \beta_{obj} \end{cases}$$

- Transform stress peak region to the probability region
- Inverse peak distribution analysis is applied

Mathematical Example

$$S(t) = g(\mathbf{X}, \mathbf{Y}(t)) = Y_1(t) + Y_2(t) + Y_3(t)$$

Variable	Mean	Standard deviation	Process type
$Y_1(t)$	1	0.3	Gaussian
$Y_2(t)$	2	0.5	Gaussian
$Y_3(t)$	1.5	0.2	Lognormal

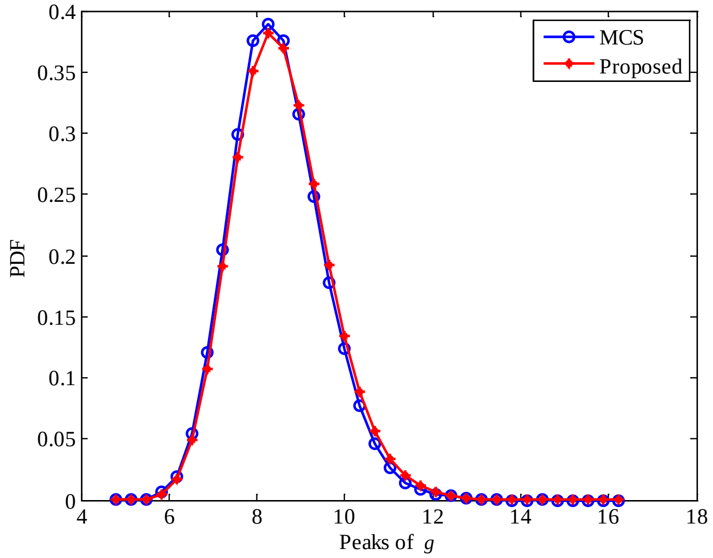
$$\rho_{Y_1}(t_1, t_2) = \exp[-(t_2 - t_1)^2]$$

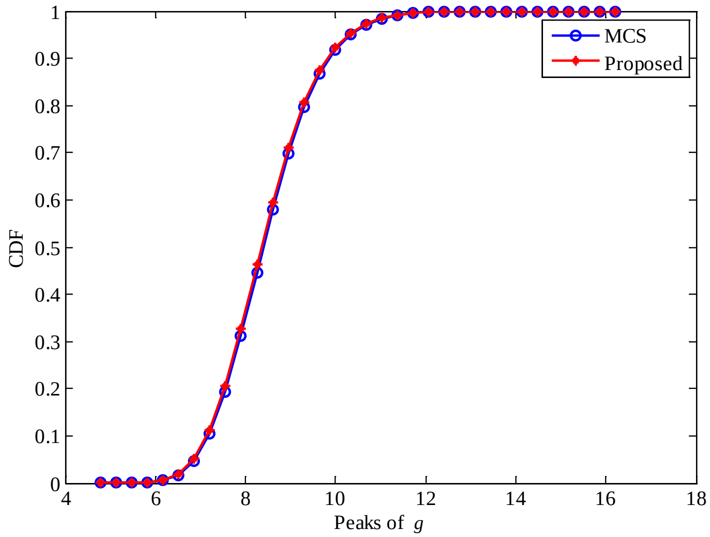
$$\rho_{Y_2}(t_1, t_2) = \cos[\pi(t_2 - t_1)]$$

$$Y_3(t) = \exp[0.2U_3(t) + 1.5]$$

$$\rho_{U_3}(t_1, t_2) = \exp[-(t_2 - t_1)^2 / 0.5^2]$$

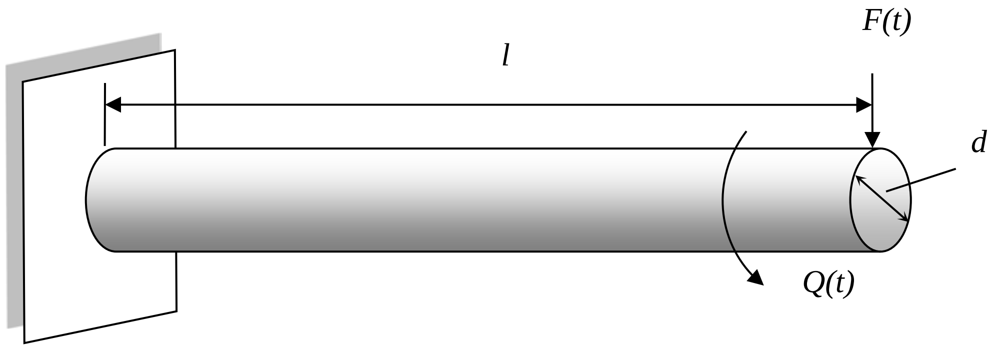
Results





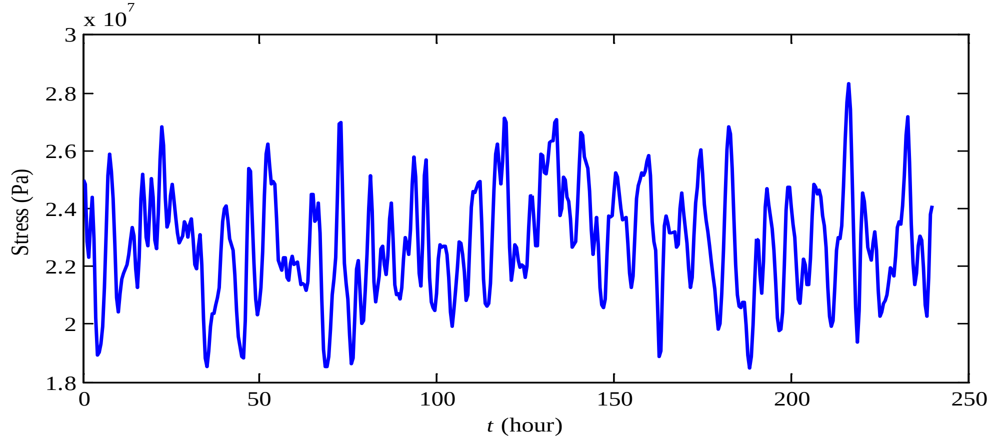
Variable	MCS	Proposed	Error (%)
Mean Value	7.5720	7.5423	0.39
Zero upcrossing rate	0.4420	0.4475	1.24
Function calls	5×10^7	488	-

Fatigue Reliability of A Shaft



Variable	Mean	Standard deviation	Distribution	Auto-correlation
d	59 mm	2.95 mm	Gaussian	N/A
l	200 mm	10 mm	Gaussian	N/A
γ	2.87×10^{-13}	5.74×10^{-15}	Gaussian	N/A
κ	1.0693	0.0214	Gaussian	N/A
$\ln(F(t))$	7.59 N	0.11 N	Lognormal	Eq. (71)
$Q(t)$	250 Nm	25 Nm	Gaussian	Eq. (72)

Results



	Probability of failure	Error (%)	Function call	Time
Proposed	0.0822	0.24	1.84×10^{4}	About 1 min
MCS	0.0824	N/A	3×10^{8}	About 113 hours

Variable	MCS	Proposed	Error (%)
Mean Value	2.475×10^{7} Pa	2.470×10^{7} Pa	0.2
Zero upcrossing rate	0.1826	0.1839	0.71
$\int \gamma s^{\kappa} p(s) ds$	3.1216×10^{-5}	3.104×10^{-5}	0.56

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Conclusions

- The method is for fatigue reliability under stochastic loadings.
- Stationary process assumption is made.
- Peak counting method is employed.
- Both Gaussian and non-Gaussian stochastic loadings can be addressed.

Future work

- Consider non-stationary stochastic loadings.
- Study the other cycle counting methods, such as RC, LCC and RFC.
- Further improve the efficiency.

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