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Time-Dependent Reliability Analysis for Bivariate Responses

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Outline

- Objective
- Time-dependent System Reliability
- Proposed Method
- Examples
- Conclusions

Objective: Develop a new time-dependent system reliability method for bivariate responses that are general functions of random variables, stochastic processes and time.

Major contribution: Derivation of bivariate joint outcrossing rate.

Time-Dependent System Reliability

The limit-state function of failure mode i

$$G_i = g_i(\mathbf{X}, \mathbf{Y}(t), t)$$

$\mathbf{X}$: random variables; $\mathbf{Y}(t)$: stochastic processes.

Probability of failure

$$p_{f,i}(t_0, t_s) = \Pr\{g_i(\mathbf{X}, \mathbf{Y}(t), t) > e_i, \exists t \in [t_0, t_s]\}$$

e_i : failure threshold of component i

Time-Dependent System Reliability

Let Ω_s be the safe region for a system. For a series system

$$\Omega_s = \left\{ [\mathbf{X}, \mathbf{Y}(t)] \middle| \bigcap_{i=1}^{r} g_i(\mathbf{X}, \mathbf{Y}(t), t) < e_i, \forall t \in [t_0, t_s] \right\}$$

For a parallel system

$$\Omega_s = \left\{ [(\mathbf{X}, \mathbf{Y}(t)] \middle| \bigcup_{i=1}^{r} g_i(\mathbf{X}, \mathbf{Y}(t), t) < e_i, \forall t \in [t_0, t_s] \right\}$$

The time-dependent system reliability

$$R_s(t_0, t_s) = \Pr\{[\mathbf{X}, \mathbf{Y}(t)] \in \Omega_s, \forall t \in [t_0, t_s]\}$$

Time-Dependent Reliability for Bivariate Responses

Two limit-state functions:

$$G_i = g_i(\mathbf{X}, \mathbf{Y}(t), t) \text{ and } G_j = g_j(\mathbf{X}, \mathbf{Y}(t), t)$$

The joint time-dependent p_f

$$p_{f,ij}(t_0, t_s) = \Pr\{g_i(\mathbf{X}, \mathbf{Y}(\chi), \chi) > e_i \cap g_j(\mathbf{X}, \mathbf{Y}(\tau), \tau) > e_j, \exists \chi \text{ and } \tau \in [t_0, t_s]\}$$

$$p_{f,ij}(t_0, t_s) = p_{f,i}(t_0, t_s) + p_{f,j}(t_0, t_s) - p_{f,i \cup j}(t_0, t_s)$$

where

$$p_{f,i \cup j}(t_0, t_s) = \Pr\{g_i(\mathbf{X}, \mathbf{Y}(\chi), \chi) > e_i \cup g_j(\mathbf{X}, \mathbf{Y}(\tau), \tau) > e_j, \exists \chi \text{ and } \tau \in [t_0, t_s]\}$$

Part 1: Component Reliability

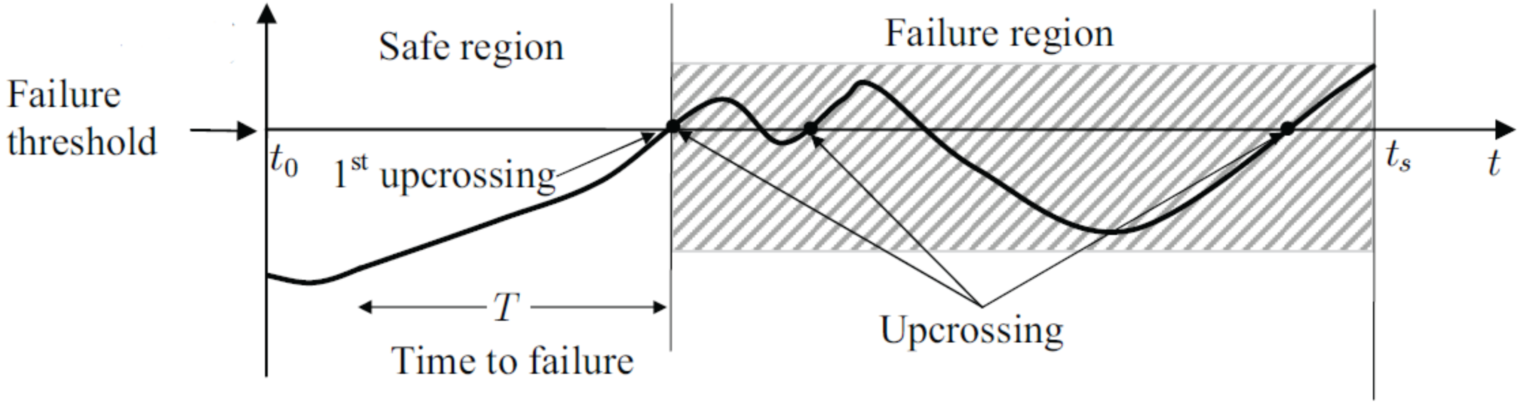
$$p_{f,ij}(t_0,t_s) = p_{f,i}(t_0,t_s) + p_{f,j}(t_0,t_s) - p_{f,i\cup j}(t_0,t_s)$$

Part 1 Part 2

Part 1: Upcrossing rate method for time-dependent component

$$p_{f,k}(t_0,t_s) = 1-[1-p_{f,k}(t_0)]\exp\left\{-\int_{t_0}^{t_s} v_k^+(t)dt\right\}$$

where $v_k^+(t)$ is the upcrossing rate.



Part 2: Joint Probability $p_{f,i\cup j}(t_0,t_s)$

For a series system

$$p_{f,i\cup j}(t_0,t_s)=1-R_{ij}(t_0)\exp\left\{-\int_{t_0}^{t_s} v_{i\cup j}^{+}(t)dt\right\}$$

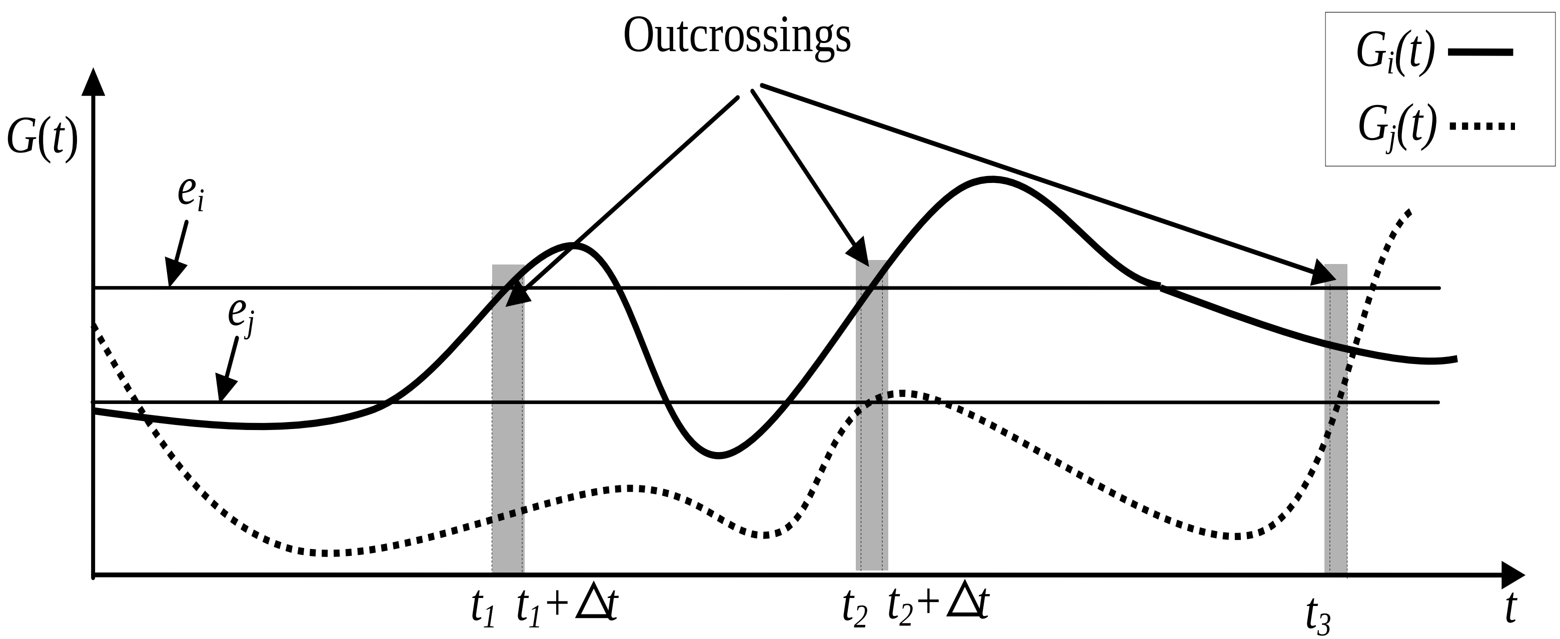
$R_{ij}(t_0)$: The probability that both components are safe at t0s

$$R_{ij}(t_0)=\Pr\{g_i(\mathbf{X},\mathbf{Y}(t_0),t_0)\le e_i \cap g_j(\mathbf{X},\mathbf{Y}(t_0),t_0)\le e_j\}$$

$v_{i\cup j}^{+}(t)$: Outcrossing rate of a series system with components *i* and *j* at time instant *t*

Joint Outcrossing Rate $v_{i \cup j}^{+}(t)$

$$v_{i \cup j}^{+}(t) = \lim_{\Delta t \to 0} \frac{\Pr\left\{ \begin{array}{l} \left[G_i(t) < e_i \cap G_j(t) < e_j \right] \cap \\ \left[G_i(t+\Delta t) > e_i \cup G_j(t+\Delta t) > e_j \right] \end{array} \right\}}{\Delta t}$$



Joint Outcrossing Rate $v_{i\cup j}^{+}(t)$

$$\Pr\left\{\begin{array}{l}\left[G_i(t)<e_i \cap G_j(t)<e_j\right] \\ \cap\left[G_i(t+\Delta t)>e_i \cup G_j(t+\Delta t)>e_j\right]\end{array}\right\}$$

$$= p_{ij}^{+-}(t)+p_{ij}^{-+}(t)+p_{ij}^{++}(t)$$

$$v_{i\cup j}^{+}(t)=v_{ij}^{+-}(t)+v_{ij}^{-+}(t)+v_{ij}^{++}(t)$$

$$v_{ij}^{+-}(t)=\lim_{\Delta t\to 0}\left(\frac{p_{ij}^{+-}(t)}{\Delta t}\right),\ v_{ij}^{-+}(t)=\lim_{\Delta t\to 0}\left(\frac{p_{ij}^{-+}(t)}{\Delta t}\right),\ v_{ij}^{++}(t)=\lim_{\Delta t\to 0}\left(\frac{p_{ij}^{++}(t)}{\Delta t}\right)$$

The problem now becomes to calculate the joint uprossing rates

Joint Upcrossing Rates

FORM transforms $(\mathbf{X},\mathbf{Y}(t))$ into standard normal random variables $\mathbf{U}(t)=(\mathbf{U_X},\mathbf{U_Y}(t))$. After linearization of the limit-state function at the MPP

$$G_k = g_k(\mathbf{X},\mathbf{Y}(t),t) > e_k,\ k=i \text{ or } j$$

$$\Downarrow$$

$$L_k(t) = \boldsymbol{\alpha}_k(t)\mathbf{U}(t)^T > \beta_k(t),\ k=i \text{ or } j$$

With the rice's formula

$$v_{ij}^{+-}(t) = \phi(\beta_i(t))\int_{-\infty}^{\beta_j(t)} \frac{\sigma_{\dot{L}_i|L_i=\beta_i(t),\,L_j=l_j}}{\sigma_{L_j|L_i=\beta_i(t)}}\phi\left(\frac{l_j-\mu_{L_j|L_i=\beta_i(t)}}{\sigma_{L_j|L_i=\beta_i(t)}}\right)Hdl_j$$

Where H is a function of reliability indexes and their derivatives

Joint Upcrossing Rates

After obtaining $v_{ij}^{+-}(t)$, $v_{ij}^{-+}(t)$ can be easily obtained by switch the subscripts i and j for above derived equations.

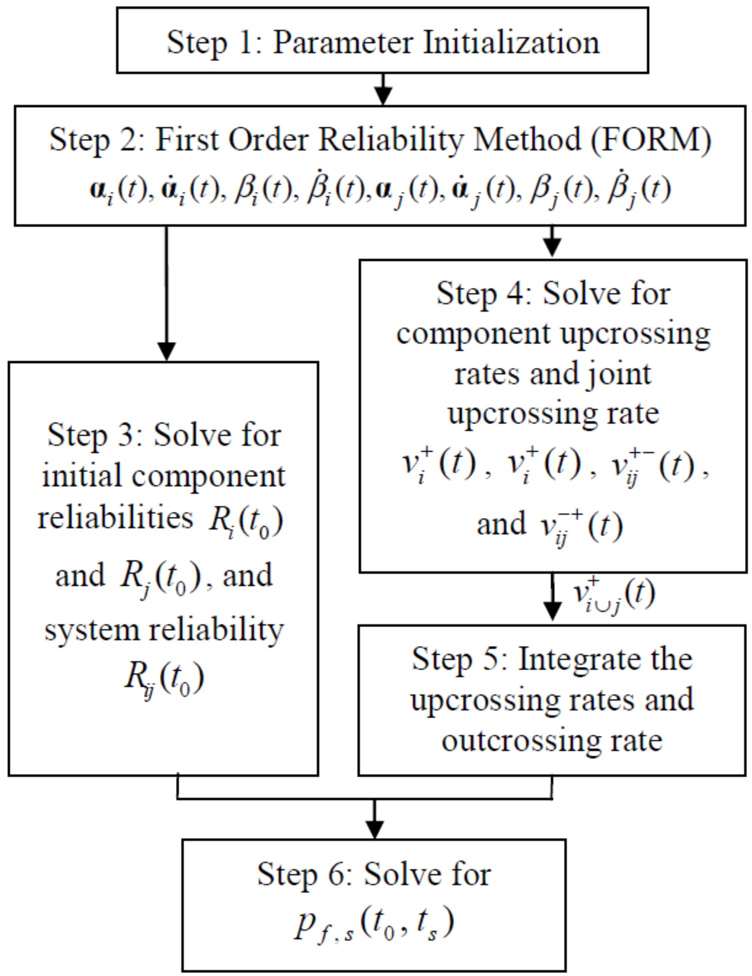
We proved that when Δt becomes infinitely small

$$v_{ij}^{++}(t) = 0$$

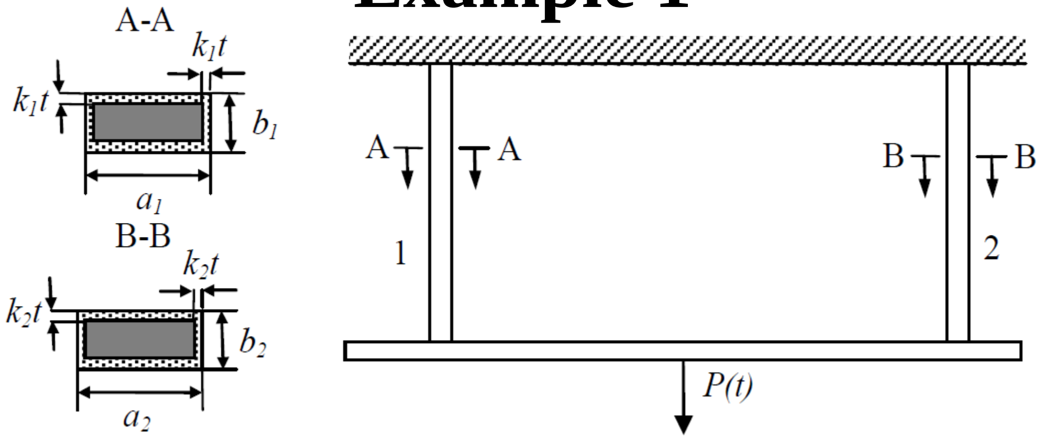
Therefore

$$v_{i \cup j}^{+}(t) = v_{ij}^{+-}(t) + v_{ij}^{-+}(t)$$

Procedures


Step 1: Parameter Initialization
Step 2: First Order Reliability Method (FORM)
$\boldsymbol{\alpha}_i(t), \dot{\boldsymbol{\alpha}}_i(t), \beta_i(t), \dot{\beta}_i(t), \boldsymbol{\alpha}_j(t), \dot{\boldsymbol{\alpha}}_j(t), \beta_j(t), \dot{\beta}_j(t)$
Step 3: Solve for initial component reliabilities $R_i(t_0)$ and $R_j(t_0)$, and system reliability $R_{ij}(t_0)$
Step 4: Solve for component upcrossing rates and joint upcrossing rate $v_i^+(t)$, $v_i^+(t)$, $v_{ij}^{+-}(t)$, and $v_{ij}^{-+}(t)$
$v_{i\cup j}^+(t)$
Step 5: Integrate the upcrossing rates and outcrossing rate
Step 6: Solve for $p_{f,s}(t_0, t_s)$

Example 1



A two-bar system

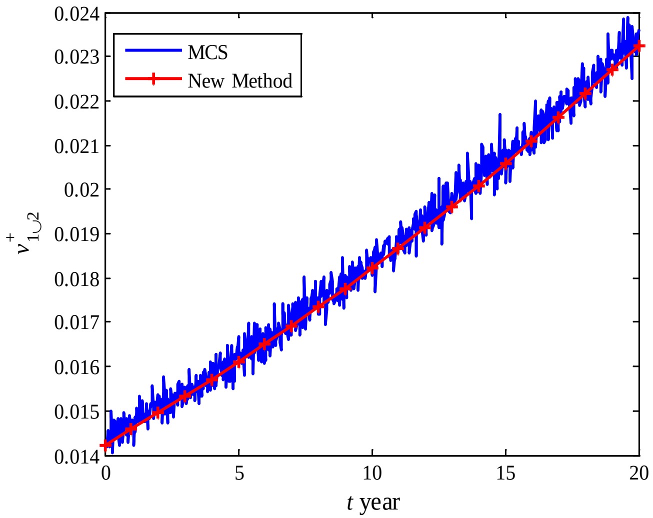
$$g_i(\mathbf{X}, \mathbf{Y}(t), t) = P(t)/2 - (a_i - 2k_i t)(b_i - 2k_i t)\sigma_{bi}$$

where $\mathbf{X} = [a_1, b_1, a_2, b_2, \sigma_{b1}, \sigma_{b2}]$, $\mathbf{Y}(t) = [P(t)]$.

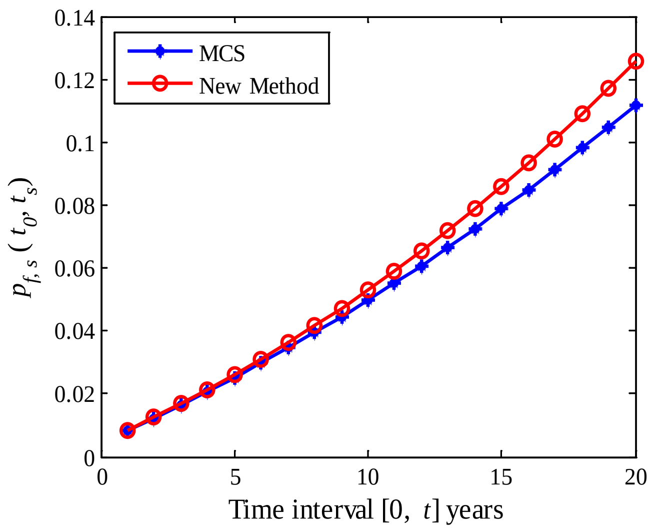
The auto-correlation function of $P(t)$

$$\rho^P(t_1, t_2) = \exp\left[-\frac{(t_2 - t_1)^2}{\zeta^2}\right], \zeta = 2 \text{ years}$$

Result of Example 1

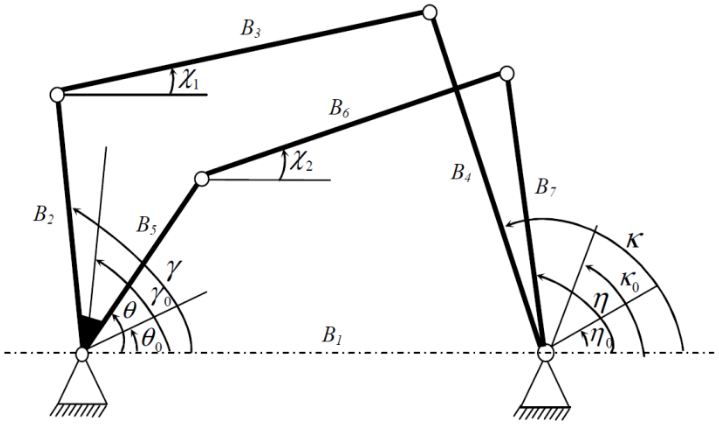


The curve of outcrossing rate from MCS is not smooth. The noise comes from the numerical discretization of stochastic process.

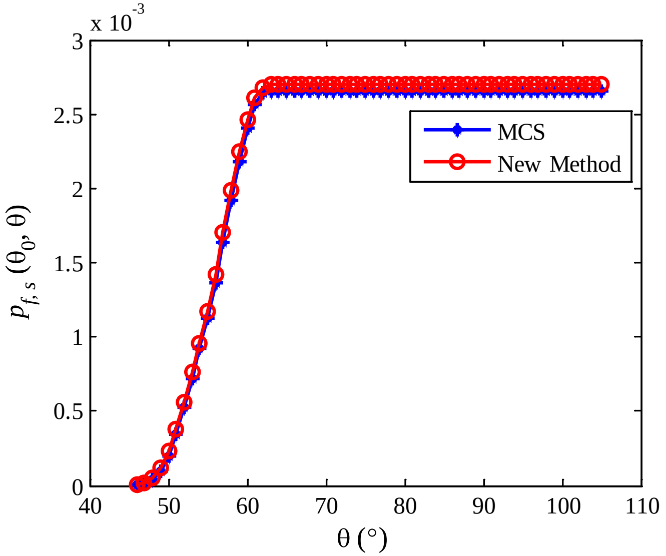


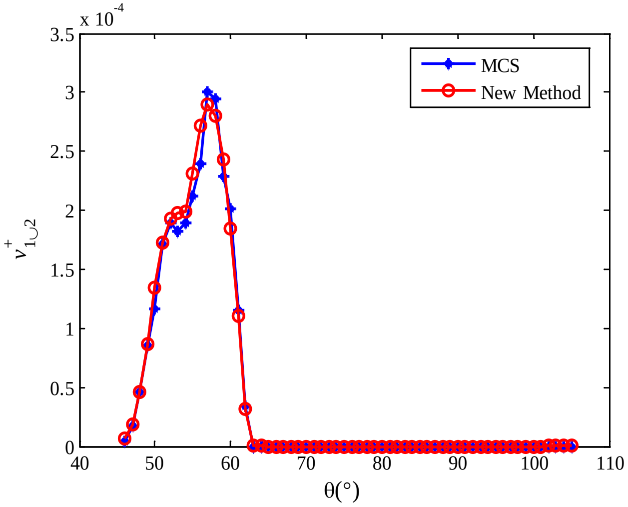
The error of the new method becomes large with a longer period of time or with a larger probability of failure. The error is mainly from the assumption of independent outcrossings. It is the intrinsic drawback of the outcrossing rate method.

Example 2



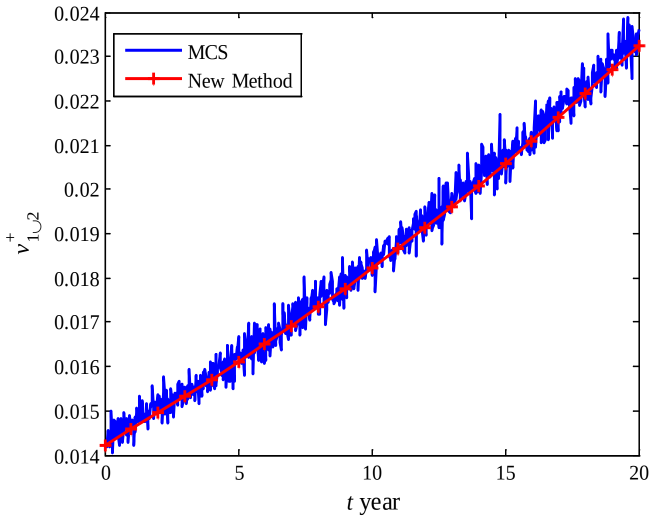
A function generator mechanism system

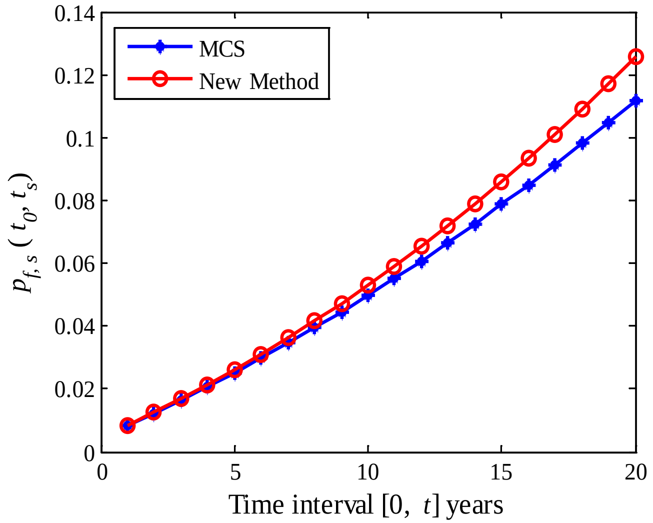




The results show good accuracy of the proposed method.

Result of Example 2





The curve of outcrossing rate from MCS is not smooth. The noise comes from the numerical discretization of stochastic process.

The error of the new method becomes large with a longer period of time or with a larger probability of failure. The error is mainly from the assumption of independent outcrossings. It is the intrinsic drawback of the outcrossing rate method.

Conclusions

- A reliability method is proposed for a system with two response variables that are functions of random variables, stochastic processes and time.
- When the dependency between upcrossings are weak, the method has good accuracy.

Future Work

- Improve the accuracy
- Extend the method to systems with multiple responses

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Thank You

Questions?