# Uncertainty Quantification in Time-Dependent Reliability Analysis

**Zhen Hu[a], Sankaran Mahadevan[a], and Xiaoping Du[b]**

a. Department of Civil and Environmental Engineering
Vanderbilt University, Nashville, TN, USA
b. Department of Mechanical and Aerospace Engineering
Missouri University of Science and Technology, Rolla, MO, USA

# Outline

VANDERBILT UNIVERSITY

- **Background**
  - Time-dependent reliability analysis
  - Epistemic uncertainty
- **Uncertainty quantification in time-dependent reliability analysis**
  - Modeling of epistemic uncertainty in random variables
  - Modeling of epistemic uncertainty in stochastic processes
  - Uncertainty quantification of time-dependent reliability
- **Numerical examples**
- **Conclusions and future work**

# Time-dependent reliability analysis

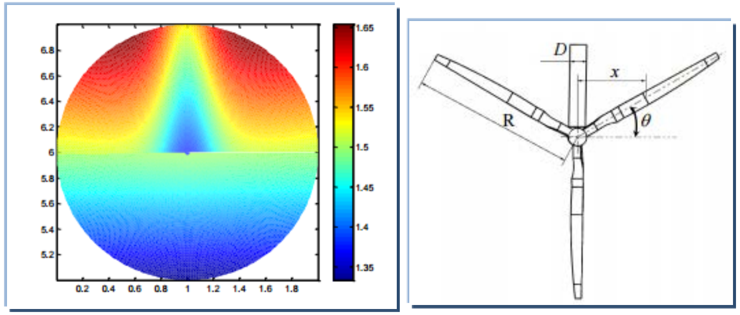
VANDERBILT UNIVERSITY

Failure: $g(\mathbf{X}, \mathbf{Y}(t), t) > 0$ — response

Reliability: $R(t_0, t_e) = \Pr\{g(\mathbf{X}, \mathbf{Y}(t), t) \le 0, \forall t \in [t_0, t_e]\}$

Random variables — Stochastic processes

$$p_f(t_0, t_e) = \Pr\{G(\tau) = g(\mathbf{X}, \mathbf{Y}(\tau), \tau) > 0, \exists \tau \in [t_0, t_e]\}$$




An example

- The ability that a system performs its intended function over a time period of interest.
- The probability that the first time to failure (FTTF) is larger than a time period of interest.
- The longer the time period, the lower the reliability.

# Time-dependent reliability analysis

VANDERBILT UNIVERSITY

## ▪State of the art

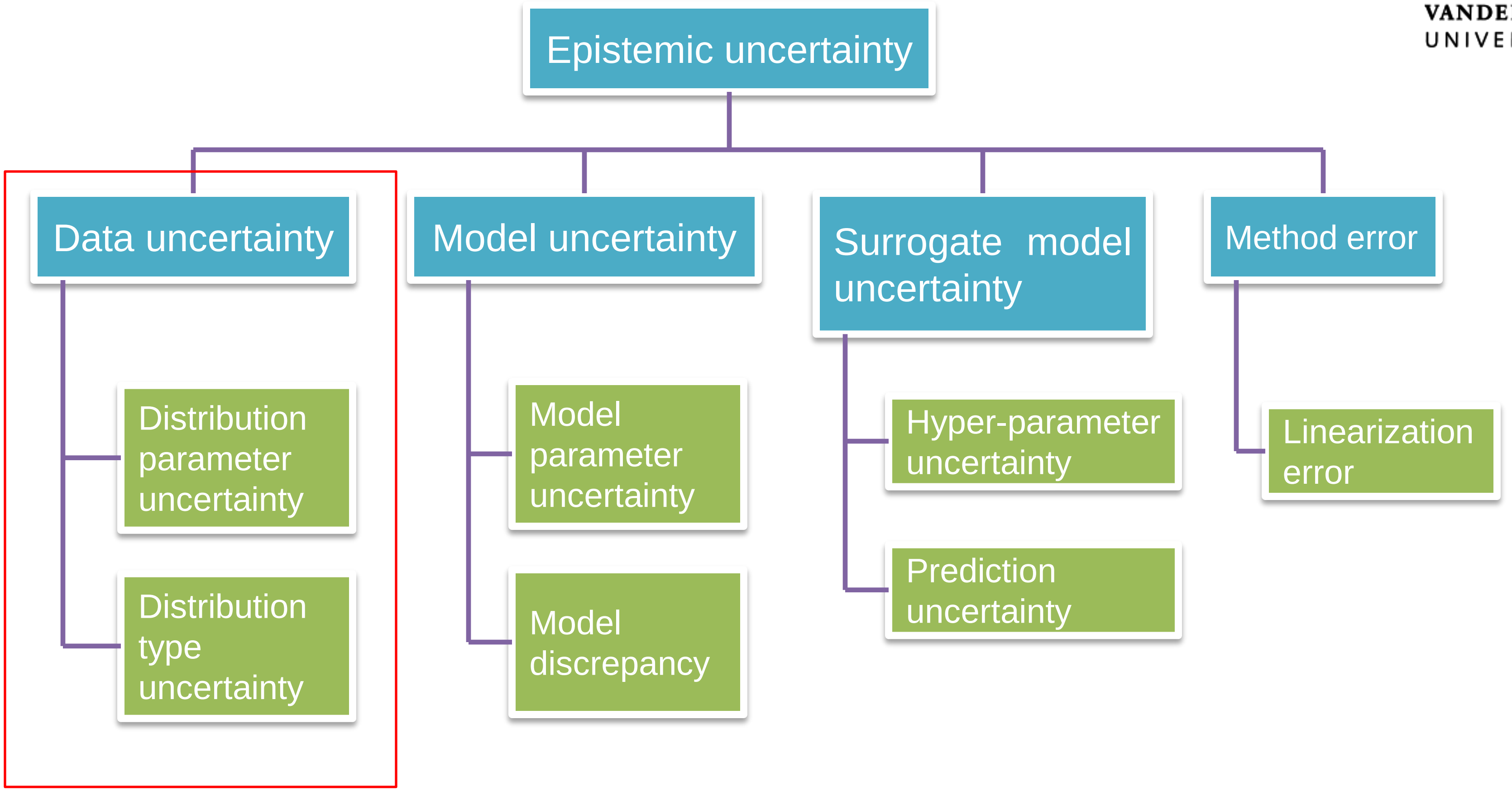
- Composite limit-state function method (*Singh and Mourelatos, 2010, Du, 2013*)
- Importance sampling approach (*Mori and Ellingwood, 1993, Dey and Mahadevan, 2000, Singh and Mourelatos, 2011*)
- Upcrossing rate-based methods (*Madsen, 1990, Zhang and Du, 2011, Hu and Du, 2013*)
- Extreme value response method (*Wang and Wang, 2012, Hu and Du, 2015*)
- Other sampling-based methods (*Hu and Du, 2014, Jiang, et al., 2014, Wang and Mourelatos, 2014*)

Only aleatory uncertainty

## ▪Challenges

- Representation of epistemic uncertainty in time-dependent reliability analysis
- Time-dependent reliability analysis including epistemic uncertainty

# Epistemic uncertainty




- Epistemic uncertainty in random variables
- Epistemic uncertainty in stochastic processes
- How to efficiently perform uncertainty quantification in time-dependent reliability analysis?

# Modeling of Epistemic Uncertainty in Random Variables

VANDERBILT UNIVERSITY

**Data uncertainty**

- Distribution type uncertainty
- Distribution parameter uncertainty

Bayes' theorem

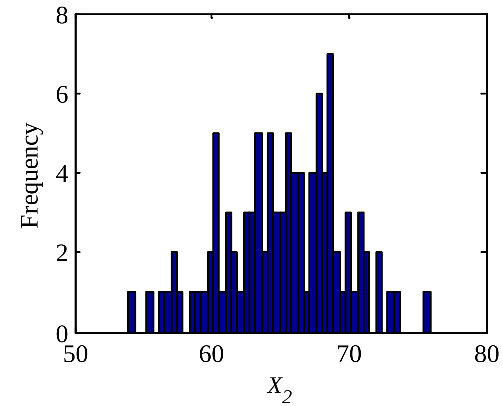
$$p(\boldsymbol{\theta}|\mathbf{x}) = \frac{L(\mathbf{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})}{\int L(\mathbf{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})d\boldsymbol{\theta}}$$

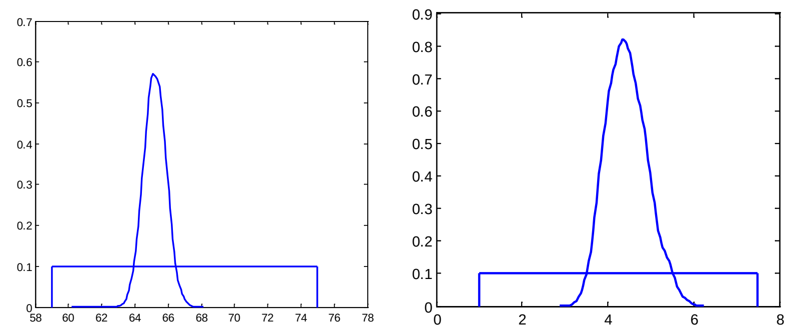
$$p(\boldsymbol{\theta}|\mathbf{x}) \propto L(\mathbf{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})$$

Posterior distribution

Prior distribution

An example




Mean

Standard deviation

Distribution parameters

# Modeling of Epistemic Uncertainty in Stochastic Processes

VANDERBILT UNIVERSITY

**Modeling of stochastic processes**

- Karhunen-Loeve (KL) expansion method
- Polynomial Chaos Expansion (PCE)
- Linear Expansion (LE) method
- Orthogonal Series Expansion (OSE) method
- Expansion Optimal Linear Estimation (EOLE) method
- ARMA or ARIMA model

**ARMA (*p*, *q*)**

$$Y_j(t_i) = \varphi_j^{(0)} + \varphi_j^{(1)} Y_j(t_{i-1}) + \varphi_j^{(2)} Y_j(t_{i-2}) + \varphi_j^{(p)} Y_j(t_{i-p}) + \varepsilon(t_i) - \omega_j^{(1)} \varepsilon(t_{i-1}) - \cdots - \omega_j^{(q)} \varepsilon(t_{i-q})$$

Mean

$$\mu_{Y_j} = \frac{\varphi_j^{(0)}}{1 - \varphi_j^{(1)} - \cdots - \varphi_j^{(p)}}$$

Auto-covariance

$$\gamma_k - \varphi_j^{(1)} \gamma_{k-1} - \cdots - \varphi_j^{(p)} \gamma_{k-p} = \begin{cases} (1 - \omega_j^{(1)} \psi_1 - \cdots - \omega_j^{(q)} \psi_q) \sigma_\varepsilon^2, & \text{for } k = 0 \\ -(\omega_j^{(k)} + \omega_j^{(k+1)} \psi_1 + \cdots + \omega_j^{(q)} \psi_{q-k}) \sigma_\varepsilon^2, & \text{for } k = 1, \cdots, q \\ 0 & \text{for } k \geq q+1 \end{cases}$$

Auto-correlation

$$\rho_{Y_j}(k) = \varphi_j^{(1)} \rho_{Y_j}(k-1) + \varphi_j^{(2)} \rho_{Y_j}(k-2) + \cdots + \varphi_j^{(p)} \rho_{Y_j}(k-p)$$

Required in time-dependent reliability analysis

# Modeling of Epistemic Uncertainty in Stochastic Processes

VANDERBILT UNIVERSITY

## Bayesian ARMA model

For given time series coefficients $\boldsymbol{\varphi}_k, \boldsymbol{\omega}_k$ , the standard deviation in the noise term:

$$\sigma_\varepsilon^2 = \frac{1}{n_{ts}(n-p-1)} \sum_{j=1}^{n_{ts}} \sum_{i=p+1}^{n_t} [Y_k^j(t_i) - \hat{Y}_k(t_i)]^2$$

Number of trajectories ($n_{ts}$); Number of time instants ($n_t$); Observations ($Y_k^j(t_i)$); Estimations of the time series ($\hat{Y}_k(t_i)$)

The likelihood function $L(\mathbf{D}_k | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k)$

$$L(\mathbf{D}_k | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k) = \prod_{i=1}^{n_{ts}} L(\mathbf{D}_k^i | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k)$$ ← All trajectories

$$L(\mathbf{D}_k^i | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k) = L(Y_k^i(t_1), Y_k^i(t_2), \cdots, Y_k^i(t_{nt}) | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k)$$ ← One trajectory

$$L(Y_k^i(t_1), Y_k^i(t_2), \cdots, Y_k^i(t_{nt}) | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k) = L(Y_k^i(t_{nt}) | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k, \mathbf{Y}_{-t_{nt}}) L(\mathbf{Y}_{-t_{nt}} | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k)$$ ← Time instants

⇓

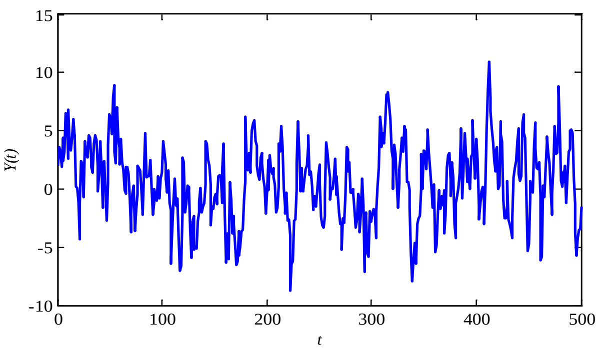
$$p(\boldsymbol{\varphi}_k, \boldsymbol{\omega}_k | \mathbf{D}_k) \propto L(\mathbf{D}_k | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k) \pi(\boldsymbol{\varphi}_k) \pi(\boldsymbol{\omega}_k)$$

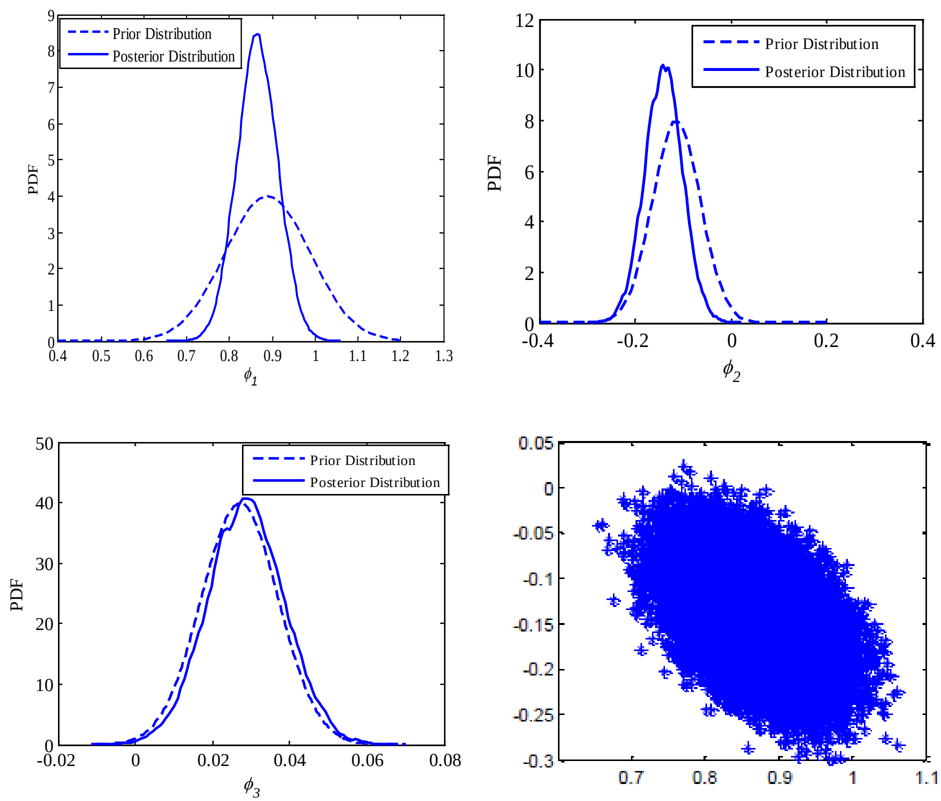
# Modeling of Epistemic Uncertainty in Stochastic Processes

VANDERBILT UNIVERSITY

An example

$$Y(t_i) = 0.8231Y(t_{i-1}) - 0.1256Y(t_{i-2}) + 0.0812Y(t_{i-3}) + N(0, 2^2)$$

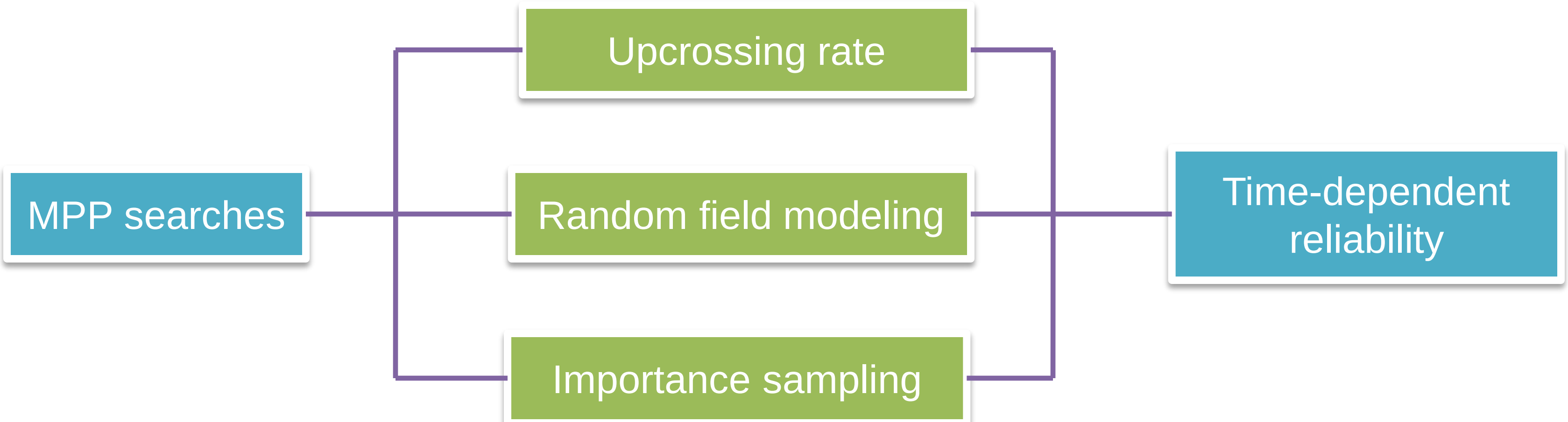



- Model the time series model based on collected data
- MCMC is used for calibration
- Data from MCMC is recorded to preserve the correlation between posterior distributions

VANDERBILT UNIVERSITY

## MPP-based time-dependent reliability analysis




- The effects of epistemic uncertainty on MPP-based time-dependent reliability analysis
- Problems with stationary stochastic processes

- Two key elements: $\beta$ and $\mathbf{u}_{\mathbf{Y}}^{*}$

Affects failure threshold in *U* space and correlation over time

Affects Correlation over time

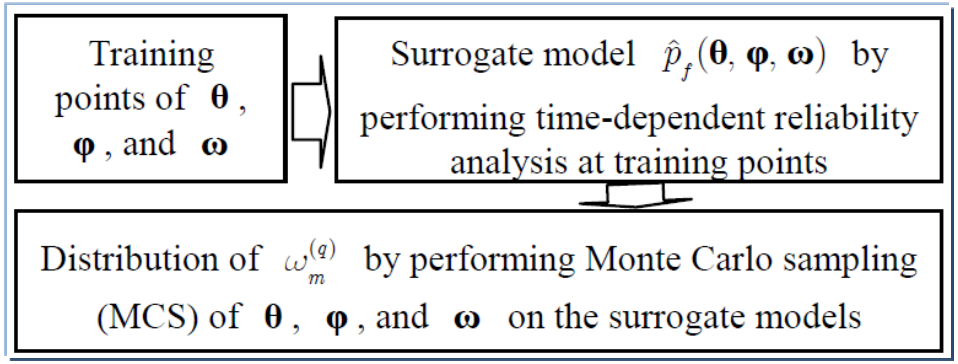
$$
\begin{aligned}
\rho_L(t_0, t) &= \boldsymbol{\alpha}_{\mathbf{X}}\boldsymbol{\alpha}_{\mathbf{X}}^T + \boldsymbol{\alpha}_{\mathbf{Y}}\boldsymbol{\rho}(t_0, t)\boldsymbol{\alpha}_{\mathbf{Y}}^T \\
&= \boldsymbol{\alpha}_{\mathbf{X}}\boldsymbol{\alpha}_{\mathbf{X}}^T + \boldsymbol{\alpha}_{\mathbf{Y}}\boldsymbol{\alpha}_{\mathbf{Y}}^T + \boldsymbol{\alpha}_{\mathbf{Y}}\boldsymbol{\rho}(t_0, t)\boldsymbol{\alpha}_{\mathbf{Y}}^T - \boldsymbol{\alpha}_{\mathbf{Y}}\boldsymbol{\alpha}_{\mathbf{Y}}^T \\
&= 1 + \frac{1}{\beta^2}\left(\mathbf{u}_{\mathbf{Y}}^{*}\boldsymbol{\rho}(t_0, t)\mathbf{u}_{\mathbf{Y}}^{*\mathrm{T}} - \mathbf{u}_{\mathbf{Y}}^{*}\mathbf{u}_{\mathbf{Y}}^{*\mathrm{T}}\right)
\end{aligned}
$$

# Uncertainty Quantification in Time-Dependent Reliability Analysis

VANDERBILT UNIVERSITY

**Quantify the uncertainty in the time-dependent reliability analysis result**

<u>A straightforward way</u>




Number of stochastic processes

Number of random variables

- Dimensionality of the surrogate model may be high: $m(1+p+q)+n\times nr$
- Computationally expensive

VANDERBILT UNIVERSITY

New approach

**Main task: how to efficiently obtain the key elements $\beta$ and $\mathbf{u}_{\mathbf{Y}}^{*}$ for given value of the epistemic uncertainty**

$$\begin{cases} \min\ \beta(t_0) = \sqrt{\mathbf{u}_{\mathbf{X}^a}^2 + \mathbf{u}_{\tilde{\mathbf{X}}}^2 + \mathbf{u}_{\mathbf{Y}^{(\varphi,\,\omega)}(t_0)}^2} \\ g(T(\mathbf{u}_{\mathbf{X}^a}),\, T(\mathbf{u}_{\tilde{\mathbf{X}}}),\, T(\mathbf{u}_{\mathbf{Y}^{(\varphi,\,\omega)}(t_0)})) = e \end{cases}$$

Classification of random variables and stochastic processes:

- Group one: $\mathbf{X}^a$ -- random variables and stochastic processes that are exactly modeled
- Group two: $\tilde{\mathbf{X}}$, $\mathbf{Y}^{(\varphi,\,\omega)}(t)$ -- random variables and stochastic processes with epistemic uncertainty

For given value of epistemic parameters → $f(\mathbf{x})$ and $f(\mathbf{y})$:

$$p_f(t_0) = \int\int (p_f(t_0)\big|\mathbf{x},\, \mathbf{y}) f(\mathbf{x}) f(\mathbf{y}) d\mathbf{x}\, d\mathbf{y}$$

For given realization of variables with epistemic uncertainty:

$$p_f(t_0)\big|\tilde{\mathbf{x}},\, \mathbf{y} = \Pr\{G(t_0) = g(\mathbf{X}^a,\, \tilde{\mathbf{x}},\, \mathbf{y}) > e\}$$

# Uncertainty Quantification in Time-Dependent Reliability Analysis

VANDERBILT UNIVERSITY

## New approach (Cont.)

Introducing an auxiliary variable:

$$\Phi(u_{p_f}) = \Pr\{U_a < u_{p_f}\} = p_f(t_0) \big| \mathbf{x}, \mathbf{y}$$

$$\downarrow$$

$$p_f(t_0) = \Pr\{U_a - \Phi^{-1}(p_f(t_0) \big| \tilde{\mathbf{X}}, \mathbf{Y}^{(\boldsymbol{\varphi}, \boldsymbol{\omega})}(t_0)) \leq 0\}$$

MPP search

$$\begin{cases} \min\limits_{\mathbf{u}} \beta(t_0) = \|\mathbf{u}\| \\ \mathbf{u} = [u_a, \mathbf{u}_{\tilde{\mathbf{X}}}, \mathbf{u}_{\mathbf{Y}^{(\boldsymbol{\varphi}, \boldsymbol{\omega})}(t_0)}] \\ \Phi(u_a) = p_f(t_0) \big| T(\mathbf{u}_{\tilde{\mathbf{X}}}), T(\mathbf{u}_{\mathbf{Y}^{(\boldsymbol{\varphi}, \boldsymbol{\omega})}(t_0)}) \end{cases}$$

$$\begin{cases} \min\limits_{\mathbf{u}} \beta(t_0) = \|\mathbf{u}\| \\ \mathbf{u} = [u_a, \mathbf{u}_{\tilde{\mathbf{X}}}, \mathbf{u}_{\mathbf{Y}^{(\boldsymbol{\varphi}, \boldsymbol{\omega})}(t_0)}] \\ u_a = -\beta^C \big| T(\mathbf{u}_{\tilde{\mathbf{X}}}), T(\mathbf{u}_{\mathbf{Y}^{(\boldsymbol{\varphi}, \boldsymbol{\omega})}(t_0)}) \end{cases}$$

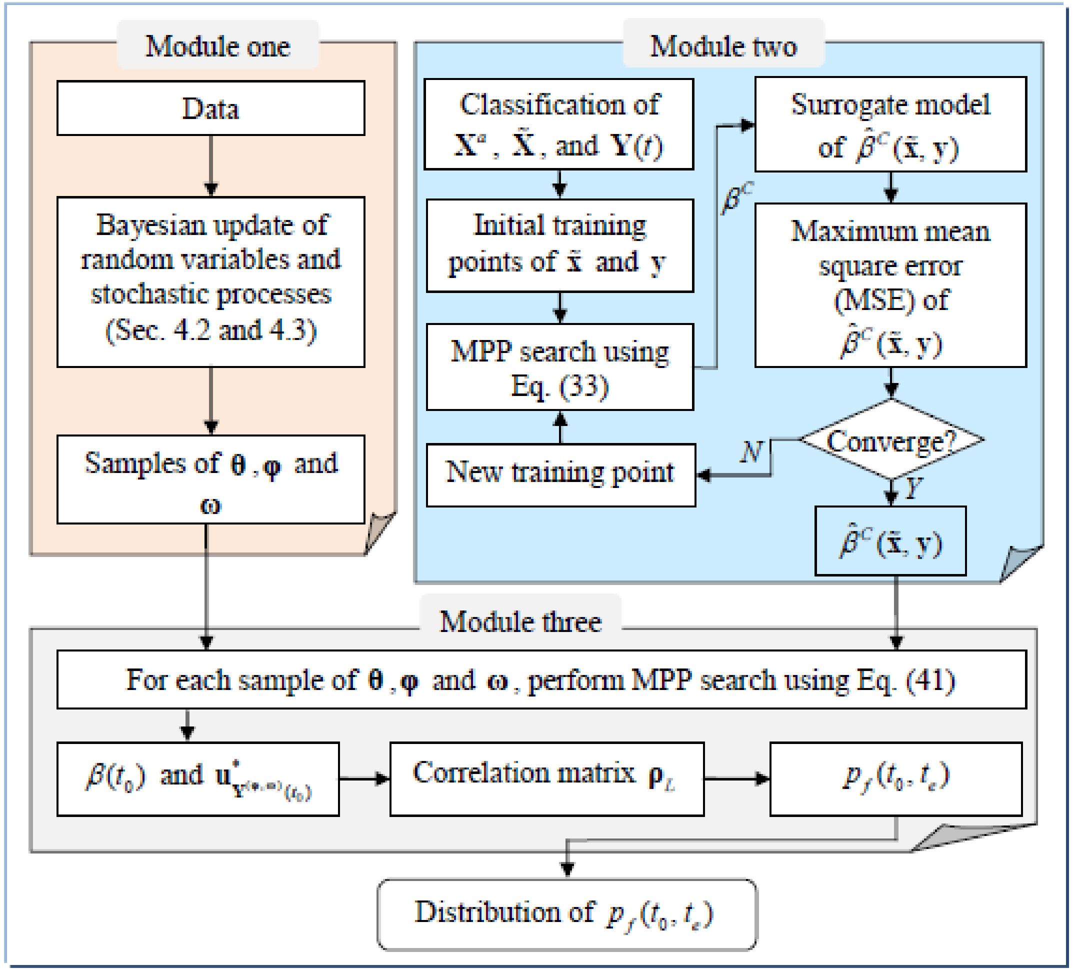
Conditional reliability index

Surrogate model of conditional reliability index

- Independent from the distribution of variables with epistemic uncertainty
- Dependent on value of the variables with epistemic uncertainty
- Still applicable when the distribution changes
- Dimension: $m(1+p+q)+n \times nr \rightarrow m+n$

# Implementation procedure

New approach (Cont.)



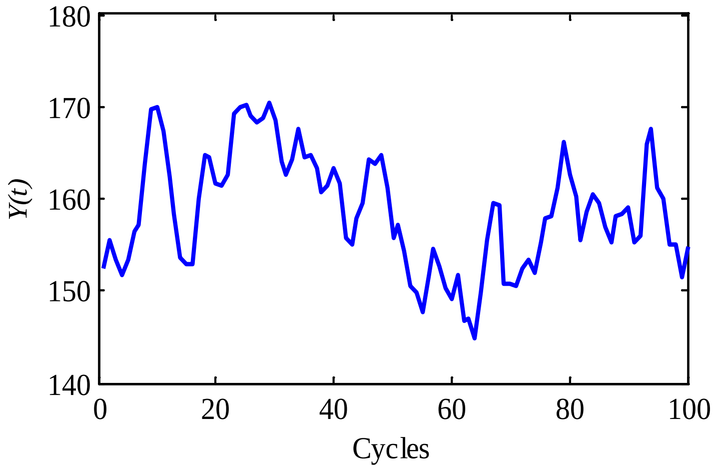
## A mathematical example

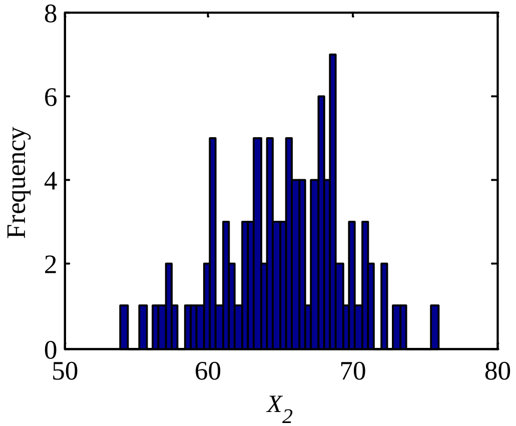
$$g(t) = X_1 + X_2 - Y_1(t)$$

$$p_f(t_0, t_e) = \Pr\{g(\tau) = X_1 + X_2 - Y_1(\tau) > 0, \exists \tau \in [t_0, t_e]\}$$

$X_1$ is exactly modeled , $X_2$ and $Y_1(t)$ are modeled based on data.

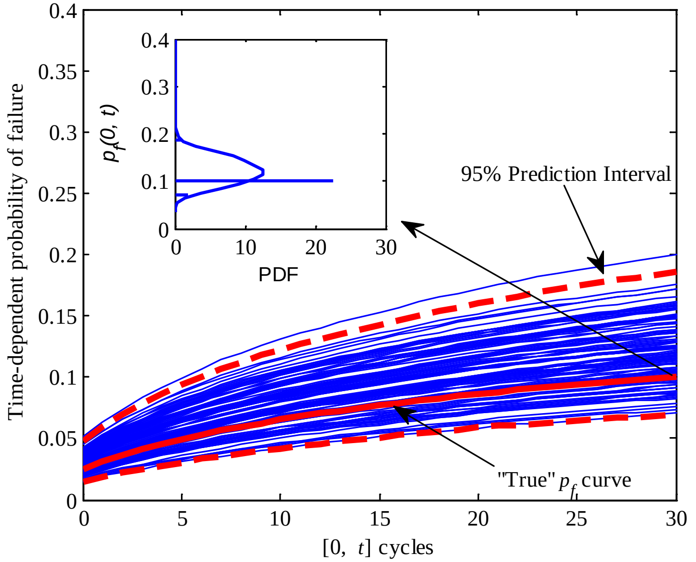
Experimental data



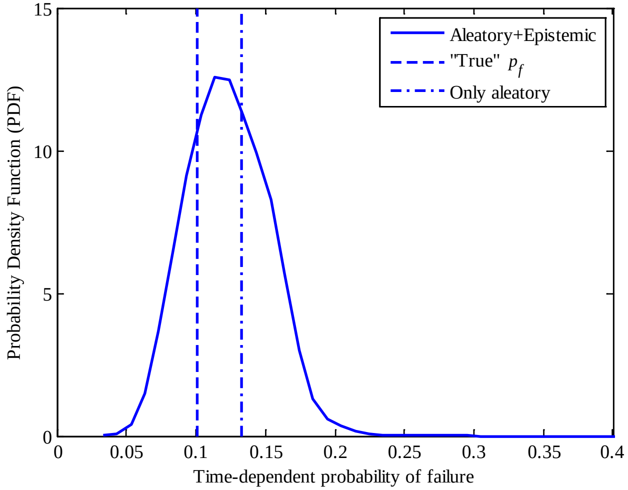
VANDERBILT UNIVERSITY

## A mathematical example (Cont.)

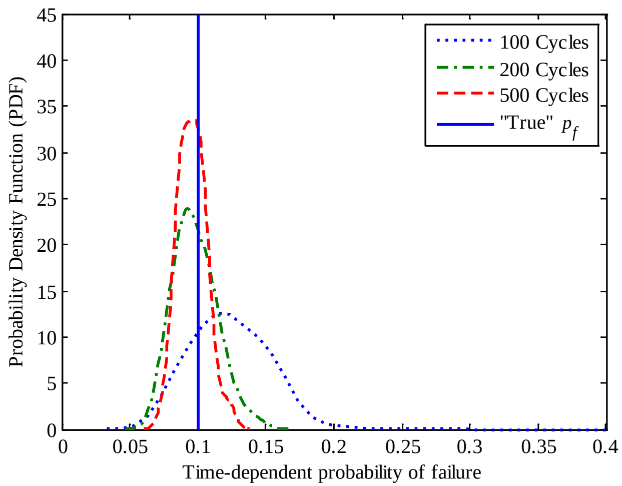



UQ results of time-dependent reliability analysis

| Method | Only aleatory | Aleatory+Epistemic |
|---|---|---|
| NOF | 70 | 106 |




Last time instant

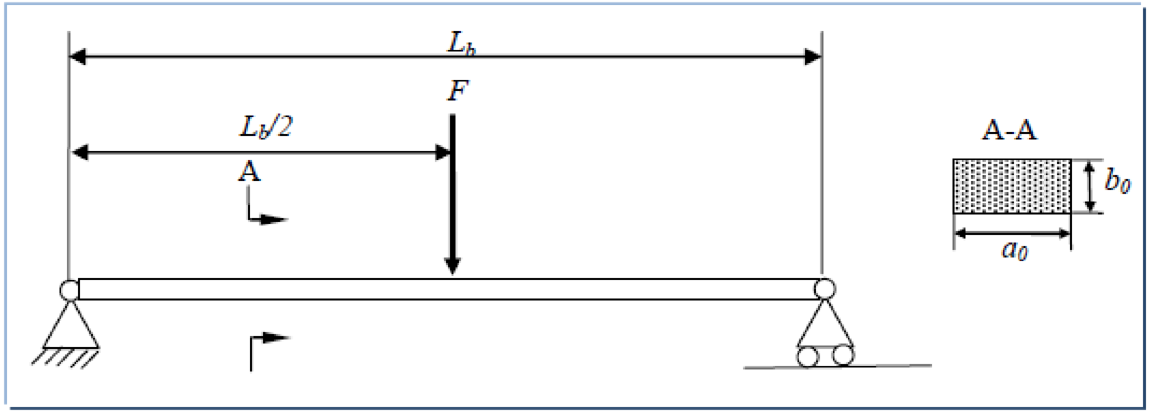



More data

# Numerical Examples

VANDERBILT UNIVERSITY

## A beam example

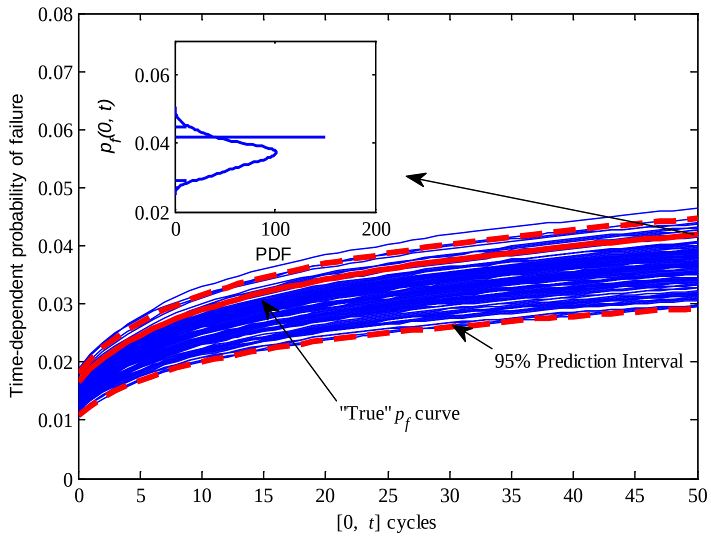


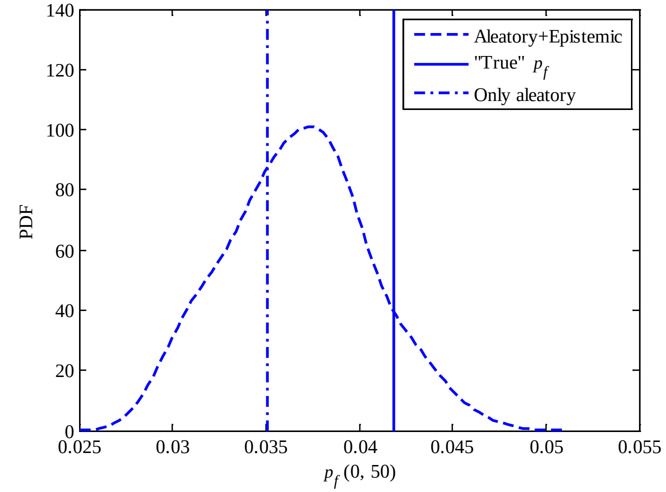

$$g(\mathbf{X}, \mathbf{Y}(t)) = \left(\frac{F(t)L_b}{4} + \frac{\rho_{st} a_0 b_0 L_b^2}{8}\right) - \frac{1}{4} a_0 b_0^2 \sigma_u$$

| Variable | Mean | Standard deviation | Distribution |
|---|---|---|---|
| $a_0$ | 0.22 m | 0.005 m | Normal |
| $b_0$ | 0.042 m | $2\times10^{-3}$ m | Normal |
| $\sigma_u$ | $2.4\times10^8$ Pa | $1.1\times10^7$ Pa | Normal |
| $F(t)$ | Stochastic loading constructed from historical data | | |
| $L_b$ | 6 m | 0 | Deterministic |
| $\rho_{st}$ | 78.5 kN/m$^3$ | 0 | Deterministic |

# Numerical Examples

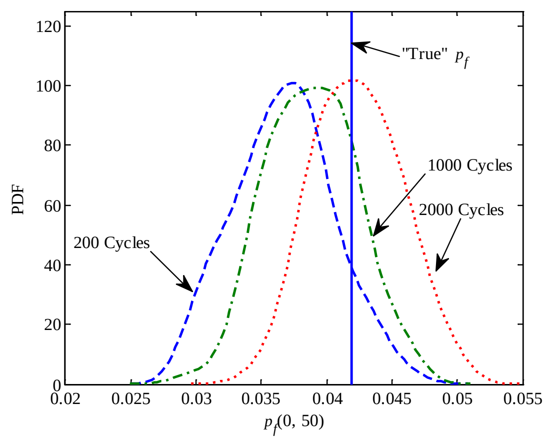
## A beam example (Cont.)



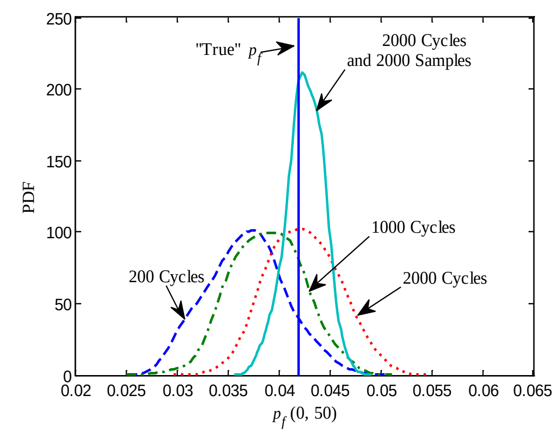




| Method | Only aleatory | Aleatory+Epistemic |
|---|---|---|
| NOF | 182 | 412 |

- Uncertainty in the reliability analysis result is more sensitive to the epistemic uncertainty in the random variable




More stochastic load data




More random variable data

# Conclusions

- A framework is developed for time-dependent reliability analysis in the presence of data uncertainty
- A new method is proposed to improve the efficiency of uncertainty quantification in time-dependent reliability analysis
- Numerical examples demonstrated the effectiveness of the proposed method

## Future work

- Inclusion of other sources of epistemic uncertainty in time-dependent reliability analysis (i.e. model uncertainty)
- Time-dependent sensitivity analysis to quantify contributions of epistemic uncertainty sources

# Acknowledgement

VANDERBILT UNIVERSITY


- Air Force Office of Scientific Research (Grant No. FA9550-15-1-0018, Technical Monitor: Dr. David Stargel)
- National Science Foundation through grant CMMI 1234855
- The Intelligent Systems Center at the Missouri University of Science and Technology.

VANDERBILT UNIVERSITY

# Thank You