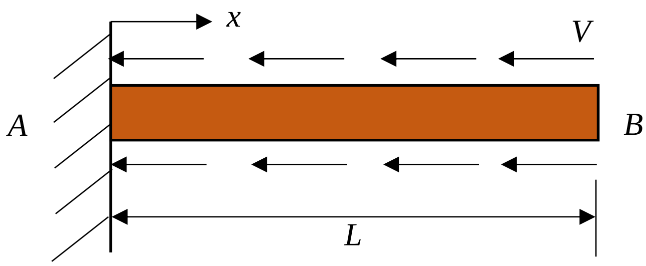
2-10. A wooden beam is subjected to a normally distributed load $v = Cx$ lb/ft, where $C \sim N(16, 1.2^2)$. The beam has a cross-sectional area $A = 12\,\text{in}^2$ and a modulus of elasticity $E = 35\times10^3$ ksi. The length is $L = 19$ ft. If the allowable displacement of B is $\delta_a = 0.1$ in, what is the probability of failure?



Solution:

The load is calculated by

$$V(x) = \int_0^x v\,dx = C\int_0^x x\,dx = \frac{1}{2}Cx^2$$

The displacement of B is then given by

$$\delta_B = \int_0^L \frac{V(x)}{EA}dx = \frac{C}{2EA}\int_0^{L(12)} x^2 dx = \frac{C}{2EA}\left(\frac{x^3}{3}\Big|_0^{L(12)}\right) = \frac{(12L)^3}{6EA}C$$

$$\mu_{\delta_B} = \frac{(12L)^3}{6EA}\mu_C$$

$$\sigma_{\delta_B} = \frac{(12L)^3}{6EA}\sigma_C$$

The probability of failure of the beam due to excessive deformation is

$$p_f = \Pr(\delta_B > \delta_a) = \Pr(Y = \delta_a - \delta_B < 0)$$

Y follows a normal distribution $Y \sim N(\mu_Y, \sigma_Y^2)$.

$$\mu_Y = \delta_a - \mu_{\delta_B} = \delta_a - \frac{(12L)^3}{6EA}\mu_C$$

$$\sigma_Y = \sigma_{\delta_B} = \frac{(12L)^3}{6EA}\sigma_C$$

Thus,

$$p_f = \Pr(Y<0) = \Pr\left(\frac{Y-\mu_Y}{\sigma_Y} < \frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\left(\delta_a - \frac{(12L)^3}{6EA}\mu_C\right)}{\left(\frac{(12L)^3}{6EA}\right)\sigma_C}\right) = \Phi\left(-4.3847\right) = 5.81\times10^{-6}$$

Ans.