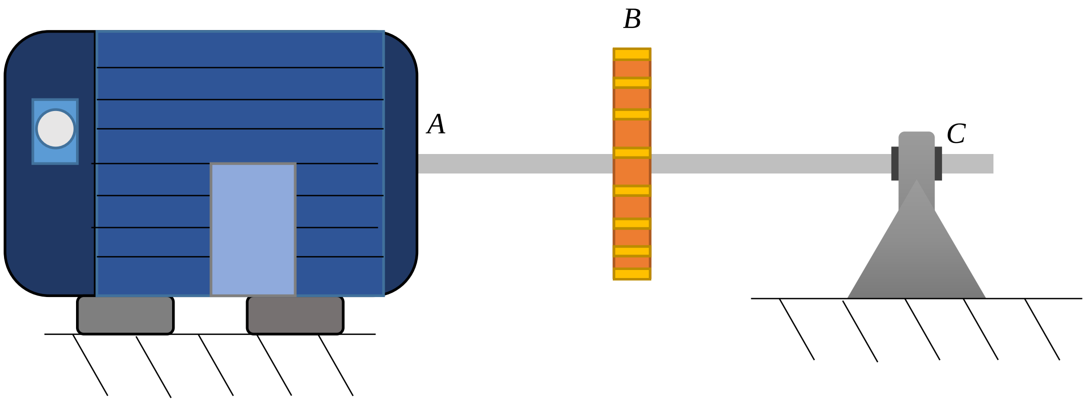
3-9. A shaft is supported by a smooth bearing at C and has a diameter $d = 10$ mm. The motor operates at an angular velocity $\omega = 20$ rad/s and delivers $P \sim N\left(4, 0.2^2\right)$ kW power. What is the probability that the shaft will fail if the allowable torsion shear stress is $\tau_a \sim N\left(1500, 150^2\right)$ MPa? Assume P and τ_a are independent.



Solution:

Torque:

$$T = \frac{P}{\omega}$$

$$\mu_T = \frac{\mu_P}{\omega} = \frac{4000}{20} = 200 \text{ N}\cdot\text{m} \quad \textbf{(1)}$$

$$\sigma_T = \frac{\sigma_T}{\omega} = \frac{200}{20} = 10 \text{ N}\cdot\text{m} \quad \textbf{(2)}$$

$T \sim N\left(200, 10^2\right)$ N·m due to unit balance.

Torsion shear stress: The polar moment of inertia is $J = \dfrac{\pi}{32} d^4 = \dfrac{\pi}{32}\left(10^4\right) = 981.75 \text{ mm}^4$.

$$\tau = \frac{Tc}{J}$$

$$\mu_\tau = \frac{\mu_T\left(\frac{d}{2}\right)}{J} = 1018.59 \text{ MPa} \tag{3}$$

$$\sigma_\tau = \frac{\sigma_T\left(\frac{d}{2}\right)}{J} = 50.93 \text{ MPa} \tag{4}$$

$\tau \sim N\left(1018.59, 50.93^2\right)$ MPa

Probability of failure:

$$p_f = \Pr(\tau > \tau_a) = \Pr(Y = \tau_a - \tau < 0) \tag{5}$$

Since $P \sim N\left(4, 0.2^2\right)$ kW, and $\tau_a \sim N\left(1500, 150^2\right)$ MPa are independent, Y also follows a normal distribution of $Y \sim N(\mu_y, \sigma_y^2)$.

$$\mu_y = \mu_{\tau_a} - \mu_\tau = 481.41 \text{ MPa} \tag{6}$$

$$\sigma_y = \sqrt{\sigma_{\tau_a}^2 + \sigma_\tau^2} = \sqrt{150^2 + 50.93^2} = 158.41 \text{ MPa} \tag{7}$$

Equation (5) can be written as

$$p_f = \Pr\left(Y < 0\right) = \Pr\left(\frac{Y - \mu_y}{\sigma_y} < \frac{-\mu_y}{\sigma_y}\right) = \Phi\left(\frac{-\mu_y}{\sigma_y}\right) = \Phi\left(-3.0390\right) = 1.1868\left(10^{-3}\right)$$ **Ans.**