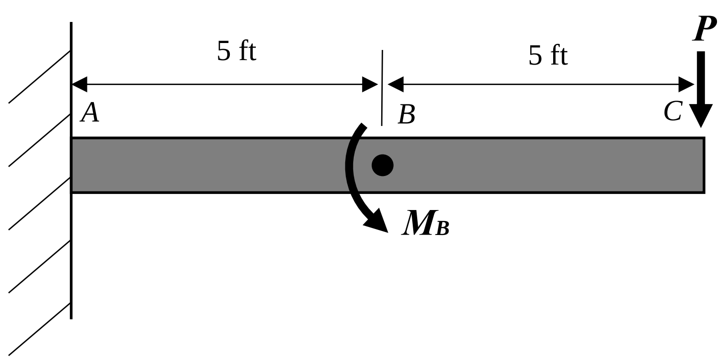
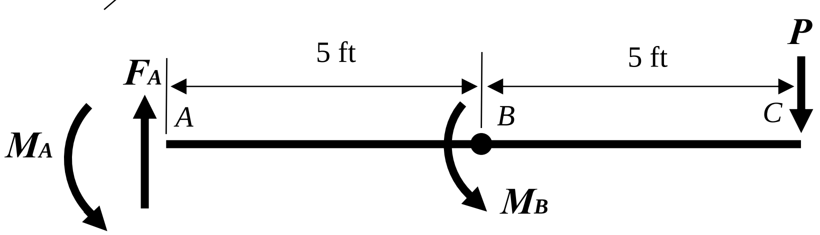
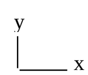
4-10. A beam is subjected to a normally distributed load $P \sim N\left(400, 50^2\right)$ lb and a normally distributed moment $M_B \sim N\left(20, 4^2\right)$ in·lb as shown. If the maximum allowable shear stress is $\tau_a \sim N\left(20, 4^2\right)$ psi, what is the minimum cross sectional area of the beam with a probability of failure of 10^{-4} due to excessive shear stress? If the maximum allowable bending stress is $S_a \sim N\left(1900, 200^2\right)$ psi, then what is the probability of failure due to excessive normal stress if you assume the beam has a square cross section. Assume P, M_B, τ_a, and S_a are independent.





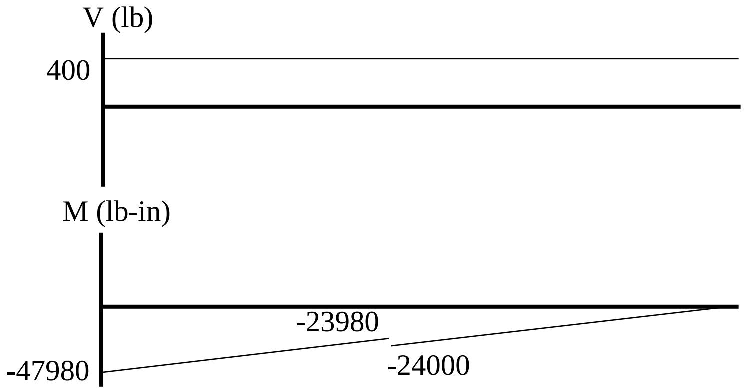


Solution:

Sum the forces: Find the maximum shear stress that is developed by summing the forces in the y direction and take the moment about A.

$$+\uparrow \sum F_y = 0; \qquad F_A - P = 0; \qquad F_A = P$$

$$+\circlearrowleft \sum M_A = 0; \qquad M_A + M_B - (120)P = 0; \qquad M_A = (120)P - M_B$$



The maximum shear is P and the maximum moment is at point A with magnitude M_A.

Shear Stress: Calculate the maximum shear stress in the beam due to P.

$$\tau_{\max} = \frac{V}{A}$$

$$\mu_\tau = \frac{\mu_V}{A} = \frac{400}{A} \text{ psi} \quad \textbf{(1)}$$

$$\sigma_\tau = \frac{\sigma_V}{A} = \frac{50}{A} \text{ psi} \quad \textbf{(2)}$$

$$\tau_{\max} \sim N\left(\frac{400}{A}, \frac{50}{A}\right) \text{ psi}$$

Probability of failure:

$$p_f = \Pr(\tau_{\max} > \tau_a) = \Pr(Y = \tau_a - \tau_{\max} < 0) \quad \textbf{(3)}$$

Since $P \sim N\left(400, 50^2\right)$ lb and $\tau_a \sim N\left(20, 4^2\right)$ psi are independent, Y also follows a normal distribution. $Y \sim N(\mu_Y, \sigma_Y^2)$.

$$\mu_Y = \mu_{\tau_a} - \mu_{\tau_{\max}} = 20 - \frac{400}{A} \text{ psi} \quad \textbf{(4)}$$

$$\sigma_Y = \sqrt{\sigma_{\tau_a}^2 + \sigma_{\tau_{\max}}^2} = \sqrt{4^2 + \left(\frac{50}{A}\right)^2} \text{ psi} \qquad \textbf{(5)}$$

Equation (3) can be written as

$$p_f = \Pr(Y < 0) = \Pr\left(\frac{Y - \mu_y}{\sigma_y} < \frac{-\mu_y}{\sigma_y}\right) = \Phi\left(\frac{-\mu_y}{\sigma_y}\right) = \Phi(-3.719) = 10^{-4}$$

From (3), (4), (5), and some algebra we obtain

$$\frac{-\left(20 - \frac{400}{A}\right)}{\sqrt{4^2 + \left(\frac{50}{A}\right)^2}} = -3.719 \Rightarrow A > 32.9973 \text{ in}^2 \sim 33 \text{ in}^2 \qquad \textbf{Ans.}$$

Assume square cross sectional area: $b = \sqrt{A}$ and $h = \sqrt{A}$.

Moment of inertia: Calculate the moment of inertia about the *x* axis.

$$I = \frac{bh^3}{12} = 90.74 \text{ in}^4$$

Bending stress: Calculate the distribution of stress that is developed in the beam.

$$\sigma_{\max} = \frac{Mc}{I};$$

$$\mu_{\sigma_{\max}} = \frac{c}{I}\mu_{M_A} = \left(\frac{\frac{h}{2}}{90.74}\right)(|\text{-}47980|) = \left(\sqrt{33}\right)(264.4) = 1518.84 \text{ psi} \qquad \textbf{(6)}$$

$$\sigma_{\sigma_{\max}} = \frac{c}{I}\sigma_F = \left(\frac{\frac{h}{2}}{90.74}\right)(|\text{-}5996|) = \left(\sqrt{33}\right)(33.04) = 189.81 \text{ psi} \qquad \textbf{(7)}$$

$\sigma_{\max} \sim N\left(1518.84, 189.81^2\right)$ psi

Probability of failure:

$$p_f = \Pr(\sigma_{\max} > \sigma_a) = \Pr(Y = \sigma_a - \sigma_{\max} < 0) \quad \textbf{(8)}$$

Since $\sigma_{\max} \sim N\left(1518.84, 189.81^2\right)$ psi and $S_a \sim N\left(1900, 200^2\right)$ psi are independent, Y also follows a normal distribution. $Y \sim N(\mu_Y, \sigma_Y^2)$.

$$\mu_Y = \mu_{S_a} - \mu_{\sigma_{\max}} = 381.16 \text{ psi} \quad \textbf{(9)}$$

$$\sigma_Y = \sqrt{\sigma_{S_a}^2 + \sigma_{\sigma_{\max}}^2} = \sqrt{200^2 + 189.81^2} = 275.73 \text{ psi} \quad \textbf{(10)}$$

Equation (8) can be written as

$$p_f = \Pr\left(Y < 0\right) = \Pr\left(\frac{Y - \mu_y}{\sigma_y} < \frac{-\mu_y}{\sigma_y}\right) = \Phi\left(\frac{-\mu_y}{\sigma_y}\right) = \Phi\left(-1.3826\right) = 0.0834 \quad \textbf{Ans.}$$