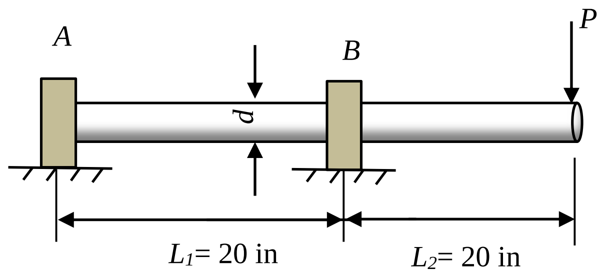
4-2. A vertical random force $P \sim N(400, 30^2)$ lb acts on a shaft as shown in the following figure. The sleeve bearings at A and B support only vertical forces. The allowable bending stress of the shaft is $S_a \sim N(30, 3^2)$ ksi. Determine the diameter of the shaft so that the probability of failure is less than 10^{-4}.



Solution

The free-body diagram of this shaft is shown in Fig. 1. The shear and moment diagrams are shown in Fig. 2.

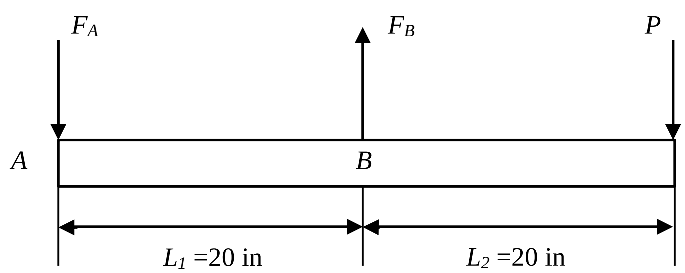


Fig. 1

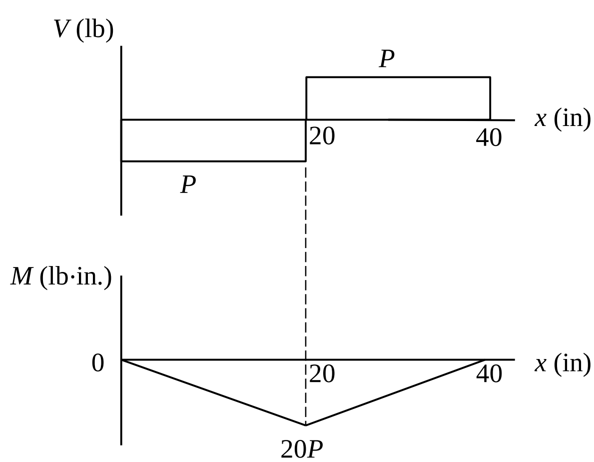


Fig. 2

$\circlearrowright +\Sigma M_A = 0$, $F_B(20) - P(40) = 0$, $F_B = 2P$

$\circlearrowright +\Sigma M_B = 0$, $F_A(20) - P(20) = 0$, $F_A = P$

As indicated in the moment diagram, $M_{\max} = 20P$.

The moment of inertia of the cross section about the neutral axis is $I = \frac{\pi}{64} d^4$.

Since $c = \frac{d}{2}$, $S_{\max} = \frac{M_{\max} c}{I} = \frac{(20P)\left(\frac{d}{2}\right)}{\left(\frac{\pi}{64} d^4\right)} = \frac{640P}{\pi d^3} = \frac{640P}{3.14 d^3} = \frac{203.82P}{d^3}$.

Set $Y = S_a - S_{\max}$, then $Y \sim N(\mu_Y, \sigma_Y^2)$, where

$$\mu_Y = \mu_{S_a} - \mu_{S_{\max}} = \mu_{S_a} - \frac{203.82}{d^3} \mu_P = 30 \times 10^3 - \frac{81528.7}{d^3}$$

$$\sigma_Y = \sqrt{\sigma_{S_a}^2 + \left(\frac{203.82}{d^3}\right)^2 \sigma_P^2} = \sqrt{(3000)^2 + \left(\frac{203.82}{d^3}\right)^2 \left(30^2\right)}$$

The probability of failure of the shaft is written as

$$p_f = \Pr(Y < 0) = \Pr\left(\frac{Y - \mu_Y}{\sigma_Y} < \frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) < 10^{-4} = \Phi\left(-3.719\right)$$

Then, $\frac{-\mu_Y}{\sigma_Y} = \frac{-\left(30 \times 10^3 - \frac{81528.7}{d^3}\right)}{\sqrt{(3000)^2 + \left(\frac{203.82}{d^3}\right)^2 \left(30^2\right)}} < -3.719$

Thus, $d > 1.661$ in. **Ans.**