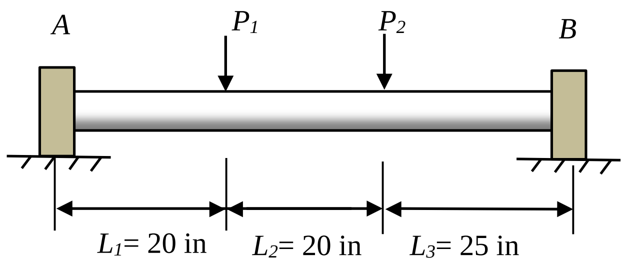
4-3. Two independent vertical random forces $P_1 \sim N(450, 50^2)$ lb and $P_2 \sim N(700, 50^2)$ lb act on a shaft as shown in the following figure. The sleeve bearings at *A* and *B* support only vertical forces. The allowable bending stress of the shaft is $S_a \sim N(45, 3^2)$ ksi. Determine the diameter of the shaft so that the probability of failure is less than 10^{-4}.



Solution

The free-body diagram of this shaft is shown in Fig. 1. The shear and moment diagrams are shown in Fig. 2.

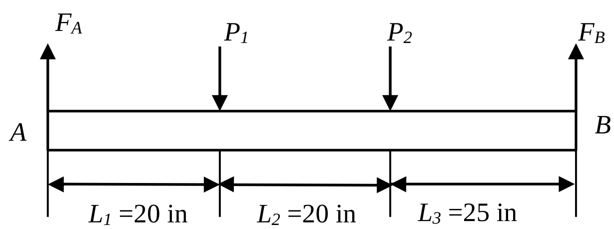


Fig. 1

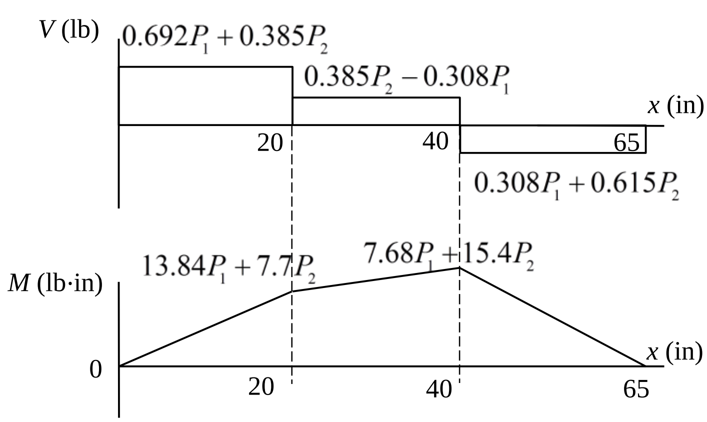


Fig. 2

$\curvearrowleft +\Sigma M_A = 0$, $\quad F_B(65) - P_1(20) - P_2(40) = 0$, $\quad F_B = 0.308P_1 + 0.615P_2$

$\curvearrowleft +\Sigma M_B = 0$, $\quad -F_A(65) + P_1(45) + P_2(25) = 0$, $\quad F_A = 0.692P_1 + 0.385P_2$

As indicated in the moment diagram, $M_{\max} = 7.68P_1 + 15.4P_2$.

The moment of inertia of the cross section about the neutral axis is $I = \dfrac{\pi}{64} d^4$.

Given $c = \dfrac{d}{2}$, we have

$$S_{\max} = \frac{M_{\max} c}{I} = \frac{(7.68P_1 + 15.4P_2)\left(\dfrac{d}{2}\right)}{\left(\dfrac{\pi}{64} d^4\right)} = \frac{(10.19)(7.68P_1 + 15.4P_2)}{d^3} = \frac{78.26P_1 + 156.93P_2}{d^3}.$$

Set $Y = S_a - S_{\max}$, then $Y \sim N(\mu_Y, \sigma_Y^2)$, where

$$\mu_Y = \mu_{S_a} - \mu_{S_{\max}} = \mu_{S_a} - \frac{78.26\mu_{P_1} + 156.93\mu_{P_2}}{d^3} = 45000 - \frac{78.26(450) + 156.93(700)}{d^3}$$

$$\sigma_Y = \sqrt{\sigma_{S_a}^2 + \left(\frac{78.26}{d^3}\right)^2 \sigma_{P_1}^2 + \left(\frac{156.93}{d^3}\right)^2 \sigma_{P_2}^2} = \sqrt{(3000)^2 + \left(\frac{78.26}{d^3}\right)^2 (50)^2 + \left(\frac{156.93}{d^3}\right)^2 (50)^2}$$

The probability of failure of the shaft is computed by

$$p_f = \Pr(Y < 0) = \Pr\left(\frac{Y - \mu_Y}{\sigma_Y} < \frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) < 10^{-4} = \Phi(-3.719)$$

Then, $$\frac{-\mu_Y}{\sigma_Y} = \frac{-\left(45000 - \dfrac{78.26(450) + 156.93(700)}{d^3}\right)}{\sqrt{(3000)^2 + \left(\dfrac{78.26}{d^3}\right)^2 (50)^2 + \left(\dfrac{156.93}{d^3}\right)^2 (50)^2}} < -3.719$$

Thus, $d > 1.6593$ in. **Ans.**