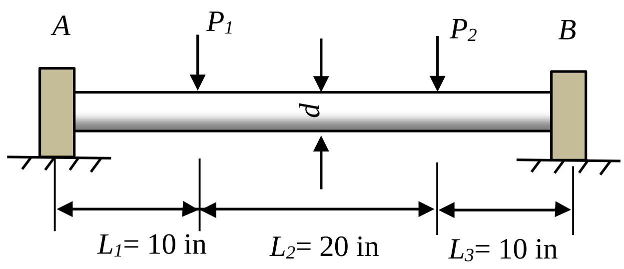
4-4. Two vertical random forces $P_1 \sim N(500, 50^2)$ lb and $P_2 \sim N(850, 60^2)$ lb act on a shaft shown in the following figure. The sleeve bearings at *A* and *B* support only vertical forces. The diameter of the shaft $d = 2.5$ in, and the allowable bending stress of the shaft is $S_a \sim N(18, 3^2)$ ksi. P_1 and P_2, and S_a are independent. Determine the probability of failure of the shaft.



Solution:

The free-body diagram of this shaft is shown in Fig. 1. The shear and moment diagrams are shown in Fig. 2.

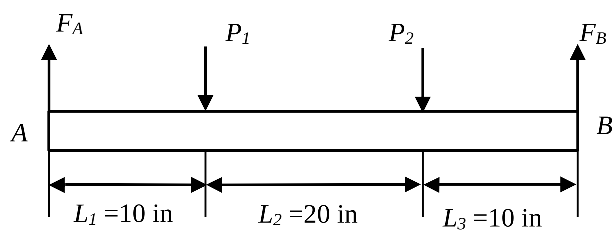


Fig. 1

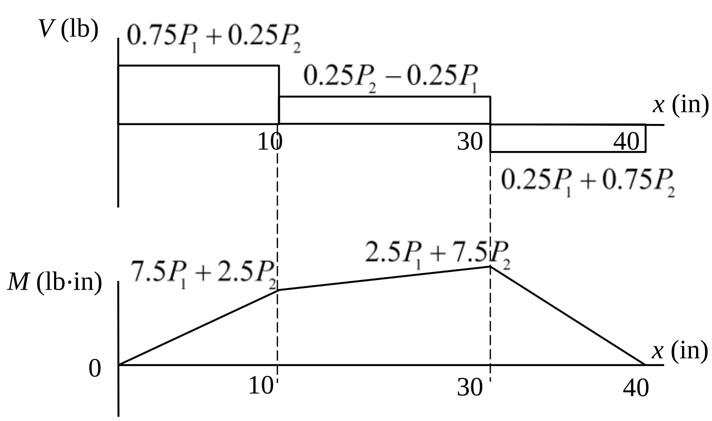


Fig. 2

$$\circlearrowleft +\Sigma M_A = 0, \quad F_B(40) - P_1(10) - P_2(30) = 0, \quad F_B = 0.25P_1 + 0.75P_2$$

$$\circlearrowleft +\Sigma M_B = 0, \quad -F_A(40) + P_1(30) + P_2(10) = 0, \quad F_A = 0.75P_1 + 0.25P_2$$

As indicated on the moment diagram, $M_{\max} = 2.5P_1 + 7.5P_2$.

The moment of inertia of the cross section about the neutral axis is $I = \frac{\pi}{64}d^4$.

Given $c = \frac{d}{2}$, we have

$$S_{\max} = \frac{M_{\max}c}{I} = \frac{(2.5P_1 + 7.5P_2)\left(\frac{d}{2}\right)}{\left(\frac{\pi}{64}d^4\right)} = \frac{(10.19)(2.5P_1 + 7.5P_2)}{2.5^3} = 1.63P_1 + 4.89P_2 .$$

Set $Y = S_a - S_{\max}$, then $Y \sim N(\mu_Y, \sigma_Y^2)$, where

$$\mu_Y = \mu_{S_a} - \mu_{S_{\max}} = \mu_{S_a} - 1.63\mu_{P_1} - 4.89\mu_{P_2} = 18000 - 1.63(500) - 4.89(850) = 13028.5 \text{ psi}$$

$$\sigma_Y = \sqrt{\sigma_{S_a}^2 + (1.63)^2\sigma_{P_1}^2 + (4.89)^2\sigma_{P_2}^2} = \sqrt{(3000)^2 + (1.63)^2(50)^2 + (4.89)^2(60)^2} = 3015.4 \text{ psi}$$

The probability of failure of the shaft is then given by

$$p_f = \Pr(Y < 0) = \Pr\left(\frac{Y - \mu_Y}{\sigma_Y} < \frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi(-4.32) = 7.8\times10^{-6}$$ **Ans.**