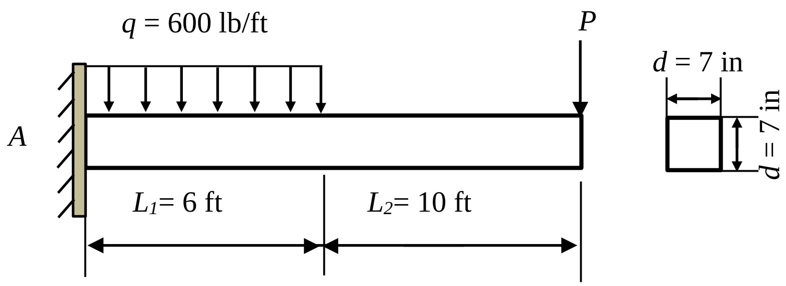
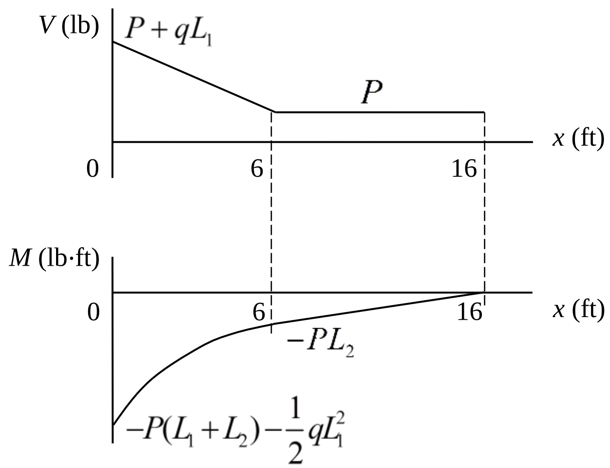
4-5. The forces acting on a beam are shown in the figure. P is a random force with $P \sim N(800, 70^2)$ lb, and q is a distributed load with $q \sim N(600, 10^2)$ lb/ft. The beam has a square cross section of 7 in on each side, and its allowable bending stress is $S_a \sim N(10, 1.2^2)$ ksi. P, q, and S_a are independent. Determine the probability of failure of the beam.



Solution:

The shear and moment diagrams are shown below.



As indicated in the moment diagram, $M_{\max} = P(L_1 + L_2) + \frac{1}{2} q L_1^2$.

The moment of inertia of the cross section about the neutral axis is $I = \frac{1}{12}(7)(7)^3 = 200.08\,\text{in}^4$.

Applying the flexure formula, we have

$$S_{\max} = \frac{M_{\max} c}{I} = \frac{\left(P(L_1 + L_2) + \frac{1}{2} q L_1^2\right) c}{I} = \frac{(16P + 18q)(12)(\frac{7}{2})}{200.08} = 3.359P + 3.778q$$

Set $Y = S_a - S_{\max}$, then $Y \sim N(\mu_Y, \sigma_Y^2)$, where

$$\mu_Y = \mu_{S_a} - \mu_{S_{\max}} = \mu_{S_a} - 3.359\mu_P - 3.778\mu_q = 10000 - 3.359(800) - 3.778(600) = 5046 \text{ psi}$$

$$\sigma_Y = \sqrt{\sigma_{S_a}^2 + (3.359)^2 \sigma_P^2 + (3.778)^2 \sigma_q^2} = \sqrt{(1200)^2 + (3.359)^2 (70)^2 + (3.778)^2 (10)^2} = 1223.4 \text{ psi}$$

The probability of failure of the beam is then given by

$$p_f = \Pr(Y < 0) = \Pr\left(\frac{Y - \mu_Y}{\sigma_Y} < \frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi(-4.125) = 1.85 \times 10^{-5}$$ **Ans.**