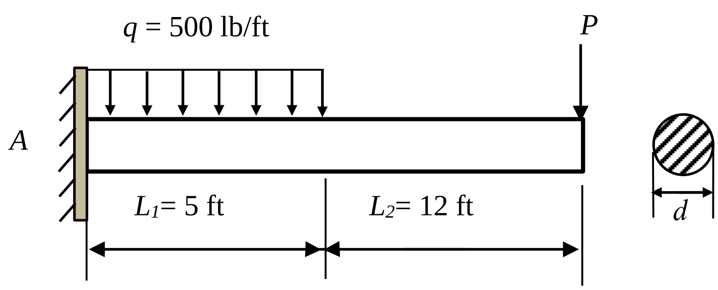
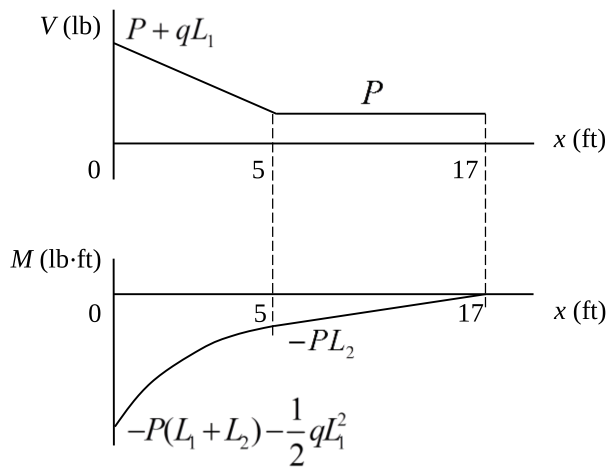
4-6. The forces acting on a shaft are shown in the figure. P is a random force with $P \sim N(1000, 85^2)$ lb, and q is a distributed load with $q \sim N(500, 10^2)$ lb/ft. The allowable bending stress of the shaft is $S_a \sim N(25, 1.5^2)$ ksi. P, q, and S_a are independent. Determine the diameter of the shaft to make sure that the probability of failure is less than 10^{-4}.



Solution:

The shear and moment diagrams are shown below.



As indicated in the moment diagram, $M_{\max} = P(L_1 + L_2) + \frac{1}{2} q L_1^2$.

The moment of inertia of the cross section about the neutral axis is

$$I = \frac{1}{64}\pi d^4 = \frac{1}{64}(3.14)d^4 = 0.049 d^4.$$

Applying the flexure formula, we have

$$S_{\max} = \frac{M_{\max} c}{I} = \frac{\left(P(L_1 + L_2) + \frac{1}{2} q L_1^2\right) c}{I} = \frac{\left(17P + 12.5q\right)(12)(\frac{d}{2})}{0.049 d^4} = 2081.6 \frac{P}{d^3} + 1530.6 \frac{q}{d^3}$$

Set $Y = S_a - S_{\max}$, then $Y \sim N(\mu_Y, \sigma_Y^2)$, where

$$\mu_Y = \mu_{S_a} - \mu_{S_{\max}} = \mu_{S_a} - 2081.6 \frac{\mu_P}{d^3} - 1530.6 \frac{\mu_q}{d^3} = 25000 - 2081.6 \frac{1000}{d^3} - 1530.6 \frac{500}{d^3}$$

$$\sigma_Y = \sqrt{\sigma_{S_a}^2 + \left(\frac{2081.6}{d^3}\right)^2 \sigma_P^2 + \left(\frac{1530.6}{d^3}\right)^2 \sigma_q^2} = \sqrt{(1500)^2 + \left(\frac{2081.6}{d^3}\right)^2 (85)^2 + \left(\frac{1530.6}{d^3}\right)^2 (10)^2}$$

The probability of failure of the shaft is then given by

$$p_f = \Pr\,(Y < 0) = \Pr\left(\frac{Y - \mu_Y}{\sigma_Y} < \frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) < 10^{-4} = \Phi\left(-3.719\right)$$

Then, $\frac{-\mu_Y}{\sigma_Y} < -3.719$. We have

$$\frac{-\left(25000 - 2081.6 \frac{1000}{d^3} - 1530.6 \frac{500}{d^3}\right)}{\sqrt{(1500)^2 + \left(\frac{2081.6}{d^3}\right)^2 (85)^2 + \left(\frac{1530.6}{d^3}\right)^2 (10)^2}} < -3.719$$

Thus, $d > 5.4\,\text{in}$. **Ans.**