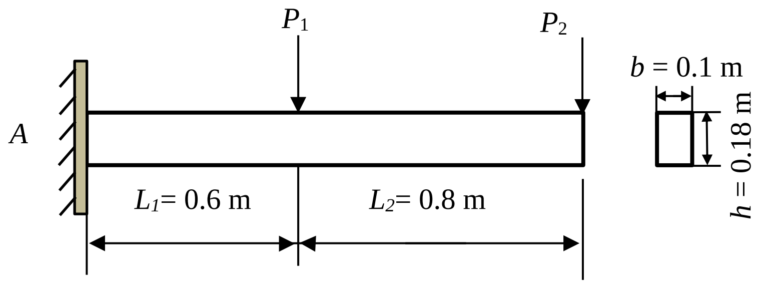
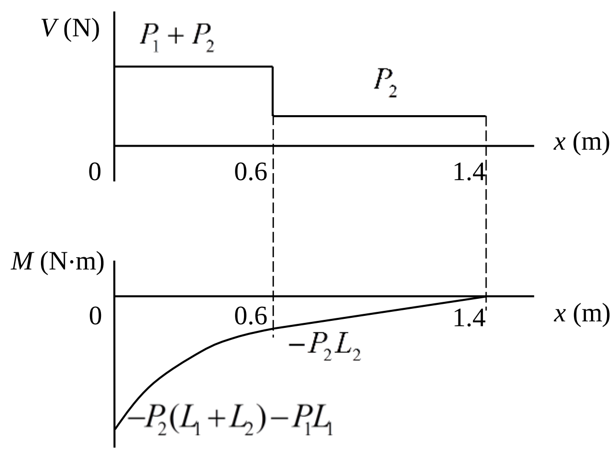
7. Two random forces $P_1 \sim N(60, 7^2)$ kN and $P_2 \sim N(35, 4^2)$ kN act on a beam in the figure. The allowable bending stress of the beam is $S_a \sim N(285, 25^2)$ MPa . P_1, P_2 and S_a are independent. Determine the probability of failure of the beam.



Solution

The shear and moment diagrams are shown below.



As indicated in the moment diagram, $M_{\max} = P_2(L_1 + L_2) + P_1 L_1$.

The moment of inertia of the cross section about the neutral axis is

$$I = \frac{1}{12} bh^3 = \frac{1}{12}(0.1)(0.18)^3 = 4.86 \times 10^{-5}$$

Applying the flexure formula, we have

$$S_{\max} = \frac{M_{\max} c}{I} = \frac{\left(P_2(L_1 + L_2) + P_1 L_1\right) c}{I} = \frac{\left(1.4P_2 + 0.6P_1\right)(\frac{0.18}{2})}{4.86 \times 10^{-5}} = 1.1111 \times 10^3 P_1 + 2.5926 \times 10^3 P_2$$

Set $Y = S_a - S_{\max}$, then $Y \sim N(\mu_Y, \sigma_Y^2)$, where

$$\mu_Y = \mu_{S_a} - \mu_{S_{\max}} = \mu_{S_a} - 1.1111 \times 10^3 \mu_{P_1} - 2.5926 \times 10^3 \mu_{P_2}$$
$$= 285 \times 10^6 - 1.1111 \times 10^3 (60 \times 10^3) - 2.5926 \times 10^3 (35 \times 10^3) = 127.59 \times 10^6 \text{ Pa}$$

$$\sigma_Y = \sqrt{\sigma_{S_a}^2 + \left(1.1111 \times 10^3\right)^2 \sigma_{P_1}^2 + \left(2.5926 \times 10^3\right)^2 \sigma_{P_2}^2}$$
$$= \sqrt{(25 \times 10^6)^2 + \left(1.1111 \times 10^3\right)^2 (7 \times 10^3)^2 + \left(2.5926 \times 10^3\right)^2 (4 \times 10^3)^2} = 28.16 \times 10^6 \text{Pa}$$

The probability of failure of the beam is then given by

$$p_f = \Pr\,(Y < 0) = \Pr\left(\frac{Y - \mu_Y}{\sigma_Y} < \frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(-4.5309\right) = 2.9366 \times 10^{-4}$$ **Ans.**