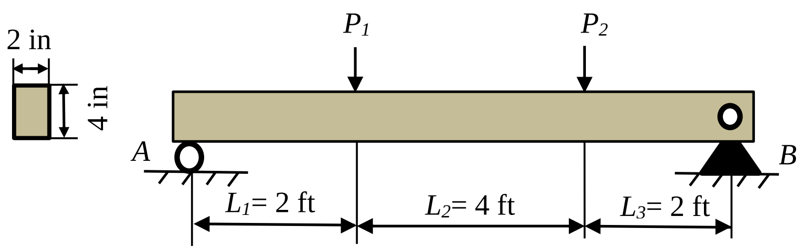
7-1. Two independent random forces $P_1 \sim N(100,10^2)$ lb and $P_2 \sim N(350,20^2)$ lb act on the beam shown in the figure. If the allowable normal stress of the beam is $S_a \sim N(1.8, 0.1^2)$ ksi and the allowable shear stress is $\tau_a \sim N(120,10^2)$ psi, determine the probabilities of failure of the beam caused by excessive bending stress and shear stress. Assume F, S_a, τ_a are independent.



Solution:

The free-body diagram of this shaft is shown in Fig. 1. The shear and moment diagrams are shown in Fig. 2.

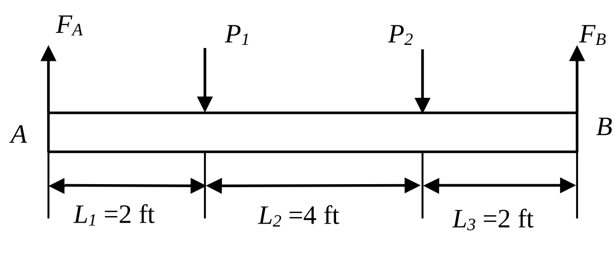


Fig. 1

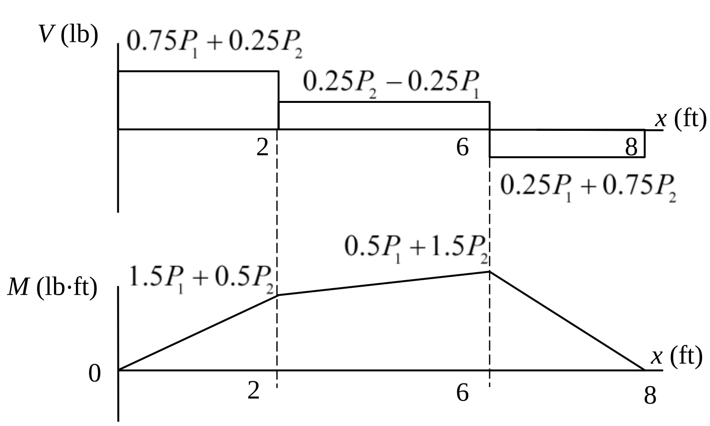


Fig. 2

$\circlearrowleft +\Sigma M_A = 0$, $F_B(8) - P_1(2) - P_2(6) = 0$, $F_B = 0.25P_1 + 0.75P_2$

$\circlearrowleft +\Sigma M_B = 0$, $-F_A(8) + P_1(6) + P_2(2) = 0$, $F_A = 0.75P_1 + 0.25P_2$

As indicated in the figure $M_{\max} = 0.5P_1 + 1.5P_2$, $V_{\max} = 0.75P_1 + 0.25P_2$

The moment of inertia of the cross section about the neutral axis is $I = \frac{1}{12}2(4)^3 = 10.67 \text{ in}^4$.

(1) The maximum bending stress can be calculated by

$$S_{\max} = \frac{M_{\max}c}{I} = \frac{(0.5P_1 + 1.5P_2)\left(\frac{4}{2}\right)(12)}{(10.67)} = \frac{(0.5P_1 + 1.5P_2)(2)(12)}{10.67} = 1.12P_1 + 3.37P_2 .$$

Set $Y_1 = S_a - S_{\max}$, then $Y_1 \sim N(\mu_{Y_1}, \sigma_{Y_1}^2)$, where

$$\mu_{Y_1} = \mu_{S_a} - \mu_{S_{\max}} = \mu_{S_a} - 1.12\mu_{P_1} - 3.37\mu_{P_2} = 506.25 \text{ psi}$$

$$\sigma_{Y_1} = \sqrt{\sigma_{S_a}^2 + (1.12)^2 \sigma_{P_1}^2 + (3.37)^2 \sigma_{P_2}^2} = 121.17 \text{ psi}$$

The probability of failure of the bean caused by excessive bending stress is

$$p_{f1} = \Pr(Y_1 < 0) = \Pr\left(\frac{Y_1 - \mu_{Y_1}}{\sigma_{Y_1}} < \frac{-\mu_{Y_1}}{\sigma_{Y_1}}\right) = \Phi\left(\frac{-\mu_{Y_1}}{\sigma_{Y_1}}\right) = \Phi(-4.18) = 1.47 \times 10^{-5}$$ **Ans.**

(2) The maximum shear stress can be calculated by

$$\tau_{\max} = \frac{VQ_{\max}}{It} = \frac{(0.75P_1 + 0.25P_2)(4)}{10.67(2)} = 0.14P_1 + 0.047P_2$$

in which $Q_{\max} = \overline{y}'A' = 1 \times (2 \times 2) = 4 \text{ in}^3$.

Set $Y_2 = \tau_a - \tau_{\max}$, then $Y_2 \sim N(\mu_{Y_2}, \sigma_{Y_2}^2)$, where

$$\mu_{Y_2} = \mu_{\tau_a} - \mu_{\tau_{\max}} = \mu_{\tau_a} - 0.14\mu_{P_1} - 0.047\mu_{P_2} = 89.53 \text{ psi}$$

$$\sigma_{Y_2} = \sqrt{\sigma_{\tau_a}^2 + (0.14)^2 \sigma_{P_1}^2 + (0.047)^2 \sigma_{P_2}^2} = 19.27 \text{ psi}$$

The probability of failure of the beam caused by excessive shear stress is

$$p_{f2} = \Pr(Y_2 < 0) = \Pr\left(\frac{Y_2 - \mu_{Y_2}}{\sigma_{Y_2}} < \frac{-\mu_{Y_2}}{\sigma_{Y_2}}\right) = \Phi\left(\frac{-\mu_{Y_2}}{\sigma_{Y_2}}\right) = \Phi(-4.647) = 1.68 \times 10^{-6}$$ **Ans.**