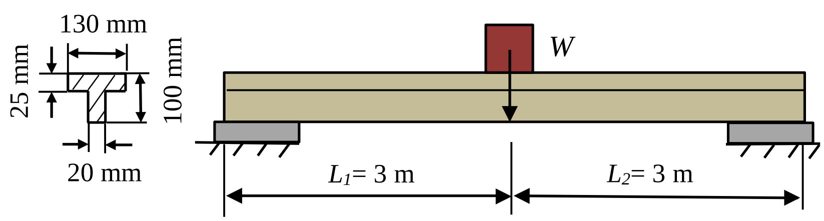
7-3. A box with a weight of $W \sim N(600, 50^2)$ N is placed in the center of a T-shape beam as shown in the figure. The ends support only vertical forces. If the allowable bending stress of the beam is $S_a \sim N(27, 2^2)$ MPa and the allowable shear stress is $\tau_a \sim N(0.4, 0.05^2)$ MPa, determine the probabilities of failure of the beam caused by excessive bending stress and shear stress. Assume W, S_a, τ_a are independent.



Solution:

The neutral axis can be calculated by

$$\bar{y} = \frac{0.0125(0.025\times0.13)+0.0625(0.02\times0.075)}{(0.025\times0.13)+(0.02\times0.075)} = 0.0283\,\text{m}$$

$$I = \frac{1}{12}(0.13)(0.025)^3 + (0.025)(0.13)(0.0283-0.0125)^2 + \frac{1}{12}(0.02)(0.075)^3 + (0.02)(0.075)(0.0283-0.0625)^2 = 3.469\times10^{-6}\,\text{m}^4$$

Maximum moment develops at the center of the beam:

$$M_{\max} = \frac{WL_1}{2} = 1.5W$$

Thus, the maximum bending stress is

$$S_{\max} = \frac{M_{\max}c}{I} = \frac{1.5W(0.1-0.0283)}{3.469\times10^{-6}} = 3.1\times10^4 W$$

Set $Y_1 = S_a - S_{\max}$, then $Y_1 \sim N(\mu_{Y_1}, \sigma_{Y_1}^2)$, where

$$\mu_{Y_1} = \mu_{S_a} - \mu_{S_{\max}} = \mu_{S_a} - 3.1\times10^4\mu_W = 27\times10^6 - 3.1\times10^4\times600 = 8.4\times10^6\ \text{Pa}$$

$$\sigma_{Y_1} = \sqrt{\sigma_{S_a}^2 + (3.1\times10^4)^2 \sigma_W^2} = \sqrt{(2\times10^6)^2 + (3.1\times10^4)^2 (50)^2} = 2.53\times10^6 \text{ Pa}$$

The probability of failure of the beam caused by excessive bending stress is

$$p_{f1} = \Pr(Y_1 < 0) = \Pr\left(\frac{Y_1 - \mu_{Y_1}}{\sigma_{Y_1}} < \frac{-\mu_{Y_1}}{\sigma_{Y_1}}\right) = \Phi\left(\frac{-\mu_{Y_1}}{\sigma_{Y_1}}\right) = \Phi(-3.32) = 4.5\times10^{-4}$$ **Ans.**

(2) The maximum shear stress can be calculated by

$$\tau_{\max} = \frac{V_{\max} Q}{It} = \frac{\left(\frac{W}{2}\right)\left(\frac{1}{2}(0.1-0.0283)(0.02)(0.1-0.0283)\right)}{3.469\times10^{-6}(0.02)} = 370.5W$$

Set $Y_2 = \tau_a - \tau_{\max}$, then $Y_2 \sim N(\mu_{Y_2}, \sigma_{Y_2}^2)$, where

$$\mu_{Y_2} = \mu_{\tau_a} - \mu_{\tau_{\max}} = \mu_{\tau_a} - 370.5\mu_W = 0.4\times10^6 - 370.5\times600 = 0.178\times10^6 \text{ Pa}$$

$$\sigma_{Y_2} = \sqrt{\sigma_{\tau_a}^2 + (370.5)^2 \sigma_W^2} = \sqrt{(0.05\times10^6)^2 + (370.5)^2 (50)^2} = 0.053\times10^6 \text{ Pa}$$

The probability of failure of the beam caused by excessive shear stress is

$$p_{f2} = \Pr(Y_2 < 0) = \Pr\left(\frac{Y_2 - \mu_{Y_2}}{\sigma_{Y_2}} < \frac{-\mu_{Y_2}}{\sigma_{Y_2}}\right) = \Phi\left(\frac{-\mu_{Y_2}}{\sigma_{Y_2}}\right) = \Phi(-3.36) = 3.9\times10^{-4}$$ **Ans.**