Uncertainty in Mechanics of Materials

MISSOURI S&T

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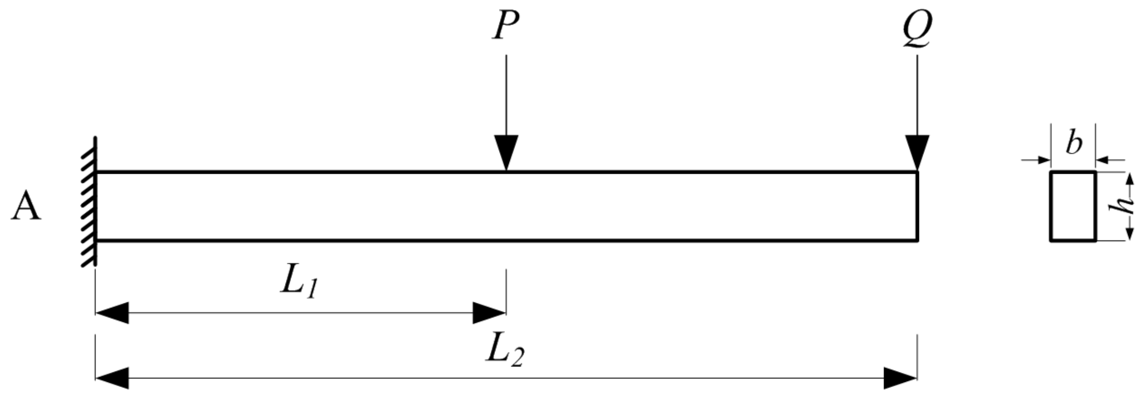
MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY

Outline

- Uncertainty in mechanics of materials
- Why consider uncertainty
- Basics of uncertainty
- Uncertainty analysis in mechanics of materials
- Examples
- Applications

Uncertainty in Mechanics of Materials

- Example
- Given: P=100 kN, Q=40 kN, L_1=0.5 m, L_2=1 m, b=0.1 m, h=0.2 m
- Find: The maximum normal stress S_{max}
- Solution: S_{max}=135 MPa
- In reality, all the input variables are random. So is S_{max}.
- S_{max} will fluctuate around 135 MPa.



Where Does Uncertainty Come From?

- Manufacturing impression
  - Dimensions of a mechanism
  - Material properties
- Environment
  - Loading
  - Temperature
  - Different users

Why Consider Uncertainty?

- We know the true solution.
- We know the effect of uncertainty.
- We can make more reliable decisions.
  - If we know uncertainties in the above beam problem, we can design a better beam so that its likelihood of failure will be low and its performance will be not affected by the uncertainties.

How Do We Model Uncertainty?

- For the beam problem, measure the force *P* (or *X*) 10 times, we get $(99.0,\ 97.6,\ 103.2,\ 103.1,\ 91.4,\ 99.7,\ 98.4,\ 106.3,\ 110.9,\ 111.1)$ kN
- How do we use the samples?
- Average

$$\mu = \frac{1}{10}(99.0 + 97.6 + 103.2 + \cdots + 110.9 + 111.1) = \frac{1}{10}\sum X_i = 102.1 \text{ kN}$$

How Do We Measure the Dispersion?

- $X = (99.0,\ 97.6,\ 103.2,\ 103.1,\ 91.4,\ 99.7,\ 98.4,\ 106.3,\ 110.9,\ 111.1)$ kN
- We could use $X_i - \mu$ and $\frac{1}{\mathrm{N}}\sum(X_i - \mu)$, $N = 10$
- But $\frac{1}{\mathrm{N}}\sum(X_i - \mu) = 0$.
- To avoid 0, we use $\frac{1}{\mathrm{N}}\sum(X_i - \mu)^2$; To have the same unit as $\bar{X}$,
- we use $\sigma = \sqrt{\frac{1}{\mathrm{N}}\sum(X_i - \mu)^2}$.
- We actually use Standard deviation: $\sigma = \sqrt{\frac{1}{\mathrm{N}-1}\sum(X_i - \mu)^2}$.
- Now we have $\sigma = 6.17$ kN, for $\mu = 102.1$ kN. (Use Excel)

More about Standard Deviation (std)

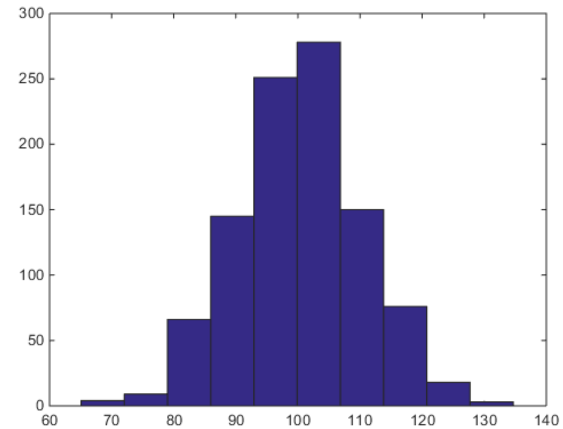
- It indicates how data spread around the mean.
- It is always non-negative.
- High std means
  - High dispersion
  - High uncertainty
  - High risk

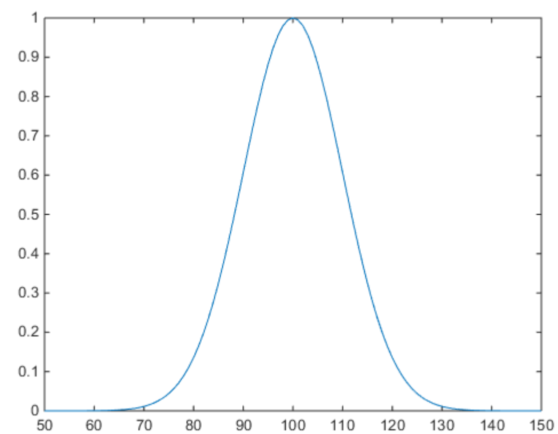
Probability Distribution

- With more samples (eg. The number of samples=1000), we can draw a histogram of $P \sim N\ (\mu_P, \sigma_P^2) = N(100, 10^2)\text{kN}$.
- If y-axis is frequency and the number of samples is infinity, we get a probability density function (PDF) .

- The probability of $a \leq X \leq b$.

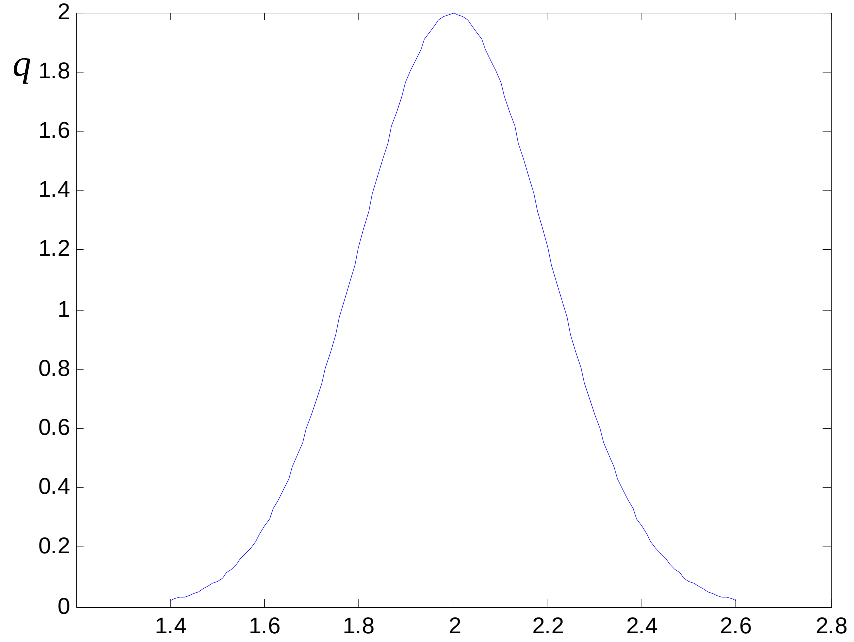
$$\Pr\{a \leq X \leq b\} = \int_a^b f(x)dx$$





Normal Distribution

- $X \sim N(\mu, \sigma^2)$
- $F(x) = \Pr\{X < x\}$ is called cumulative distribution function (CDF)
- $\Pr\{a < X < b\} = F(b) - F(a)$
- $\Pr\{X > x\} = 1 - \Pr\{X < x\} = 1 - F(x)$



How to Calculate $F(x)$?

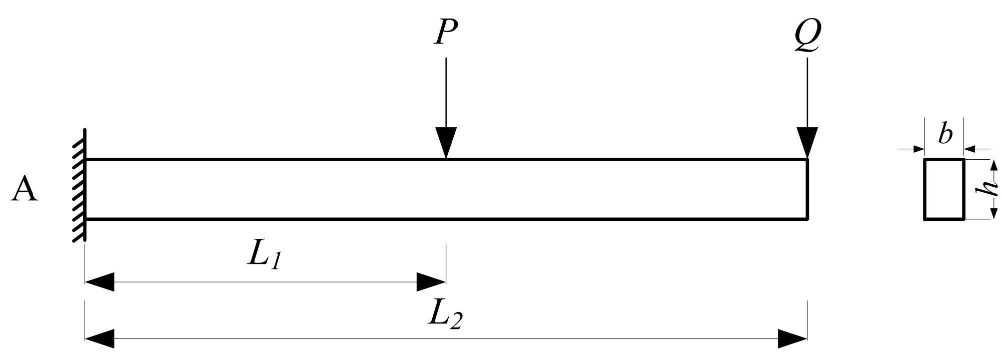
- Use Excel
  - NORMDIST(x,mean,std,cumulative)
  - cumulative=true

More Than One Random Variables

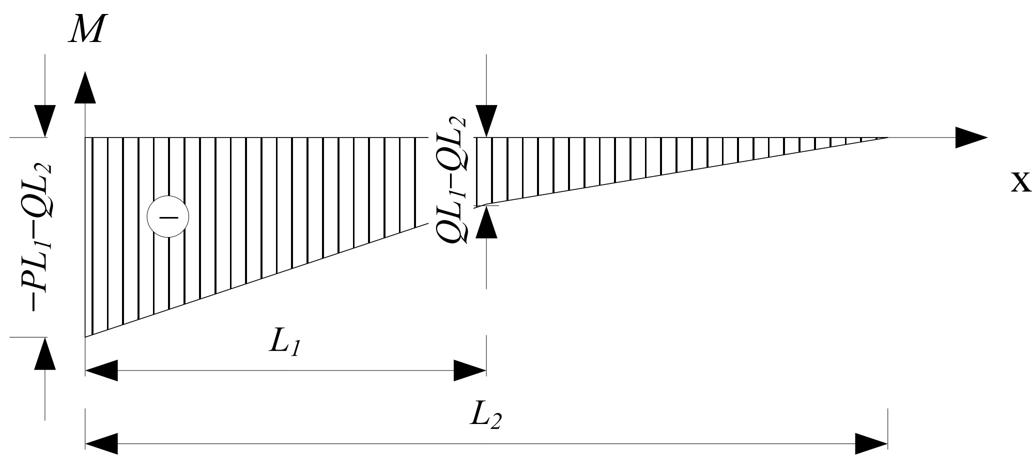
- $X_i \sim N\left(\mu_i, \sigma_i^2\right)$
- $X_i \; (i = 1,2,\cdots,n)$ are independent
- $Y = c_0 + c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$
- $c_i \; (i = 1,2,\cdots,n)$ are constants.
- Then $Y \sim N(\mu_Y, \sigma_Y^2)$
- $\mu_Y = c_0 + c_1 \mu_1 + c_2 \mu_2 + \cdots + c_n \mu_n$
- $\sigma_Y = \sqrt{c_1^2 \sigma_1^2 + c_2^2 \sigma_2^2 + \cdots + c_n^2 \sigma_n^2}$

Example

Two random forces $P \sim N(\mu_P, \sigma_P^2) = N(100, 10^2)$ kN and $Q \sim N(\mu_Q, \sigma_Q^2) = N(40, 5^2)$ kN are exerted on a cantilevered beam. The dimensions are $L_1 = 0.5$ m, $L_2 = 1$ m, $b = 0.1$ m and $h = 0.2$ m. The allowable normal stress S_a of the beam is also a random variable with $S_a \sim N(\mu_{S_a}, \sigma_{S_a}^2) = N(260, 30^2)$ MPa. *P, Q,* and S_a are independent. Determine the probability of failure of the beam.



Moment Diagram



The maximum normal stress is given by

$$S_{max} = \frac{M_{max}c}{I} = \frac{(PL_1 + QL_2)\left(\frac{h}{2}\right)}{\left(\frac{bh^3}{12}\right)} = \left(\frac{6L_1}{bh^2}\right)P + \left(\frac{6L_2}{bh^2}\right)\mathrm{Q}$$

The probability of failure is $p_f = \Pr(S_a < S_{max})$.

Let $Y = S_a - S_{max}$.

Then $Y = S_a - S_{max} = S_a - \left(\frac{6L_1}{bh^2}\right)P - \left(\frac{6L_2}{bh^2}\right)\mathrm{Q}$.

p_f is written as $p_f = \Pr(Y < 0)$. Since normal random variables *P*, *Q* and S_a are independent, *Y* also follows a normal distribution, $Y \sim N\,(\mu_Y, \sigma_Y^2)$.

Thus

$$\mu_Y = \mu_{S_a} - \left(\frac{6L_1}{bh^2}\right)\mu_P - \left(\frac{6L_2}{bh^2}\right)\mu_Q = 125 \text{ Mpa}$$

$$\sigma_Y = \sqrt{\sigma_{S_a}^2 + \left(\frac{6L_1}{bh^2}\right)^2 \sigma_P^2 + \left(\frac{6L_2}{bh^2}\right)^2 \sigma_Q^2} = 31.82 \text{ MPa}$$

Therefore,

$$p_f = \Pr(Y < 0) = \Pr\left(\frac{Y - \mu_Y}{\sigma_Y} \leq \frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right)$$

$$= 1 - \Phi(3.9283) = 1 - 0.999957 = 4.3 \times 10^{-5}$$

The probability of failure of the beam is 4.3×10^{-5}.

Conclusions

- Uncertainty is ubiquitous in engineering, including applications of mechanics of materials.
- Uncertainty is usually modeled by probability theory.
- Considering uncertainty promotes high reliability, robustness, and safety.

Assignment

- On Blackboard.
- More readings: http://web.mst.edu/~dux/repository/ce110/ce110.html