Probabilistic Engineering Design

Chapter One Introduction

1 Introduction

This chapter discusses the basics of probabilistic engineering design. Its tutorial-style is designed to allow the reader to quickly grasp the overall picture of probabilistic engineering design without the burden of theoretical materials. For this reason, the presentation emphasizes the basic concepts and principles accompanied by simple examples. The major concepts include uncertainty, reliability, and robustness. The two commonly used design methodologies, which are reliability-based design and robust design, are also discussed briefly, with examples illustrating their typical usage.

1.1 Uncertainty

Uncertainty is something unknown, or something questionable; it also means not having sure knowledge. Uncertainty is ubiquitous in engineering. Overlooking or mistreating it may result in either overly risky or overly conservative designs. To manage uncertainty appropriately during engineering design, we need at first to model it with a suitable mathematical structure (uncertainty modeling), then understand its effects on product performance (uncertainty analysis), and ultimately minimize its effect by choosing proper design variables (design under uncertainty). Probabilistic design is a major methodology for design under uncertainty.

Let us at first understand what uncertainty is and how we model it. Uncertainty can be viewed as the difference between the present state of knowledge and the complete knowledge. Some common examples of uncertainty are listed below.

- Random part dimensions due to manufacturing imprecision. For example, suppose the length of a link of a mechanism is designed to be 1.5 m long with a tolerance of 1 mm. For the links in compliance with specification, if we measure their lengths, we may find that the actual values vary slightly around 1.5 m within the tolerance range ±1 mm. If defective links are screened out, the actual lengths of the links will always vary between 1500 – 1 mm and 1500 + 1 mm.
- Random forces acting on a structural system or a mechanical system. For example, a bridge is subject to random loading; the load of a vehicle is stochastic due to random road conditions and random vibrations.
- Random material properties. No materials are perfect. They exhibit random behaviors, such as random stiffness, random ultimate strength, and random ductility.

As engineers, why are we concerned with uncertainty? The reason is that if we ignore or inappropriately treat uncertainty during the engineering design process, we may experience the following consequences:

- Erroneous decision-making
- Low quality, robustness, reliability, safety
- High risk
- High cost of product-life cycle
- Costly warranty
- Overly conservative products
- Low customer satisfaction, and
- Catastrophe

For example, if a product is not robust against uncertainty, the product performance will be sensitive to the variation of the system inputs. As a result, small variations such as the imprecision of manufacturing may lead to large variations in product performance. Large variations in performance mean low quality and will consequently result in low customer satisfaction. Moreover, if a product is not reliable in the presence of uncertainty, the chance of failure will be high. Catastrophic events may occur when the product fails.

As a result, uncertainty should be carefully managed in engineering design. In many design problems, such as aircraft design, uncertainty has become a central consideration in performance of engineering systems.

In this book, we will discuss how to deal with uncertainty in engineering design at the following complementary levels: modeling, analysis, and design, which are the three aspects we have discussed previously. This is demonstrated by a framework of probabilistic engineering design in Fig. 1.1.

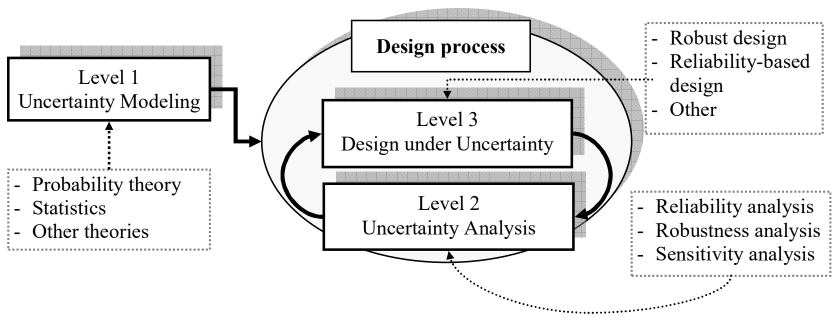

Fig. 1.1. Deal with Uncertainties at Three Levels

- Level 1 – Uncertainty modeling

The task of uncertainty modeling is to quantify uncertainty mathematically. The probability theory is commonly used for this task. An uncertain quantity is described by a random variable and is characterized by a probability distribution. Since the distribution is usually obtained from statistical data, statistics is also used to formulate the distribution. The mathematical structures of the uncertain variables at the uncertainty modeling level then provide the input to uncertainty analysis at the next level, or level 2 below.

- Level 2 – Uncertainty analysis

The task of uncertainty analysis is to quantify the uncertainty in product performance (model output) given the uncertainty in the input variables that determine the performance. The uncertainty in the input variables is modeled at the above modeling level. Uncertainty analysis helps engineers understand how uncertainty impacts design performance and also helps them evaluate important design characteristics, such as the reliability and robustness. The knowledge from uncertainty analysis will then be used at the next level, or level 3, for managing and mitigating the effects of uncertainty.

- Level 3 – Design under uncertainty

The task of design under uncertainty is to mitigate the effects of uncertainty by making appropriate decisions. Depending on design needs, the focus may be on the reliability, robustness, or quality. For making the design cost effective, the common practice is to select optimal design variables at the design stage without eliminating the causes of uncertainty. In many cases, eliminating uncertainty causes is very expensive. It requires higher precision manufacturing, stricter quality control, and higher grade materials and components. Design under uncertainty is an iterative process. During this process, the design is continuously updated until it is satisfactory. Uncertainty analysis is performed for each updated design. Therefore, the design process repeatedly calls uncertainty analysis.

Next we elaborate the three levels.

1.2 Uncertainty Modeling

There are many types of uncertainty, but the one in a form of randomness is most commonly formulated and encountered in engineering. It is the major type of uncertainty with which this book deals.

Let us use X to denote a random quantity, and we call it a random variable. As opposed to a deterministic variable, a random variable does not have a single, fixed value; it can rather take on a set of possible different values, each of which falling into a range is associated with a chance. This chance is called a probability ranging between 0 and 1. For example, the length of the mechanism link mentioned above is a random variable. It may take any values within its tolerance range. Even though we do not know the actual value of the length before measuring it, we may know the probability of possible values of the length within a certainty range in advance. The probability of X taking certain values can be described by a mathematical function, and this function is called a distribution function, or a cumulative distribution function (CDF).

Let the CDF be $F(x)$ for a particular value x. $F(x)$ is defined as the probability that the random variable X is less than or equal equation to x, or

$$F(x) = \Pr\{X \le x\} \tag{1}$$

where $\Pr(\cdot)$ stands for probability.

Knowing the CDF of X, we can find anything about X, for instance, its mean value μ_X, which tells us the average of all the possible values of X. We may also know its standard deviation σ_X, which shows the amount of variations in X or dispersion around the average. A small value of σ_X indicates that the possible values of X tend to be close to the average while a large value of σ_X suggests that the possible values vary over a large range around the average.

There are many distributions, among which the normal distribution or Gaussian distribution is the most commonly used. A normal distribution is defined by its mean μ_X and standard deviation σ_X. Its CDF is given by

$$F(x) = \Phi\left(\frac{x-\mu_X}{\sigma_X}\right) \tag{2}$$

where function $\Phi(\cdot)$ is defined by

$$\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u\right)^2 \tag{3}$$

which is usually tabulated in a statistics or probability textbook and can be computed numerically. A normally distributed random variable with μ_X and σ_X is denoted by $X \sim N(\mu_X, \sigma_X^2)$, where σ_X^2 is called variance.

For a normal distribution, there are two good futures that we will use later in this chapter. They are given below.

Feature 1: If $X \sim N(\mu_X, \sigma_X^2)$ and $Y = a + bX$ where a and b are constant, then Y is also normally distributed with $Y \sim N(\mu_Y, \sigma_Y^2)$ in which

$$\mu_Y = a + b\mu_X \tag{4}$$

and

$$\sigma_Y = b\sigma_X \tag{5}$$

Feature 2: If $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$, X and Y are independent, and $Z = aX + bY + c$ where *a*, *b*, and *c* are constant, then Z is also normally distributed with $Z \sim N(\mu_Z, \sigma_Z^2)$ in which

$$\mu_Z = a\mu_X + b\mu_Y + c \tag{6}$$

and

$$\sigma_Z = \sqrt{(a\sigma_X)^2 + (b\sigma_Y)^2} \tag{7}$$

We assumed that X and Y are independent. What does this mean? It intuitively means that the occurrence of certain values of X does not affect that of certain values of Y. For example, if two cylinders are made in two different factories with different machine tools, the diameters X and Y of the two cylinders are independent because the manufacturing imprecision in one factory has nothing to do with that in the other factory. If the cylinders are made from one product line, however, their diameters may be dependent because they may be subject to the same manufacturing imprecision.

As discussed above, uncertainty is unavoidable in engineering. To handle uncertainty, we need a mathematical tool to model uncertainty, and the probability theory is such a tool. With this tool we can quantify the effects of uncertainty and then mitigate such effects in engineering analysis and design. For this purpose, we briefly review engineering analysis and engineering design in Sec. 1.2 and then discuss uncertainty analysis and design under uncertainty.

1.2 Engineering Analysis and Engineering Design

Engineering design is a process of developing a system, component, or process to meet desired needs. It involves systematic and creative applications of basic science, mathematics, and engineering sciences to practical problems.

During a design process, there are many decisions to be made; for example, what configurations of a system should take? What materials should be used? How long will be the expected lifetime? What warranty policy should be adopted? Engineering design is therefore a decision-making process.

An engineering process is not static; it is dynamic and iterative. We may continuously refine our designs when discovering that the current design does not meet some requirements or fulfill desired functions. The iterations continue until a good design is reached within a given schedule and budget.

Due to its open-ended nature, a design problem often does not have a unique solution, and in some sense there are no correct answers to a design problem. It is the reason why so many different vehicles are running on the streets for the same need (function) – transport passengers and goods.

For example, if we would like to realize the following function: transform a rotational motion into another rotational motion, we then have a design problem. There are many potential solutions as we may realize the intended function by using a four-bar linkage, a pair of gears, a belt-pulley system, or other transmission systems. The process of designing this transmission system is iterative, and we may go back and forth before finalizing the design.

To judge if a design can meet specified needs, we must evaluate it. This is the task of engineering analysis. Engineering analysis is the evaluation of a system, component, or process under design to reveal its properties, performance, or state. Engineering analysis is typically performed for a potential design before physical prototypes are made or when the design does not meet a need. Doing engineering analysis therefore often means making predictions. The analysis result not only enables design engineers to better understand their design but also allows for improvements in the design. Table 1.1 lists the differences between design and analysis.

Table 1.1. The Differences between design and analysis

Design	Analysis
A decision making process	A problem solving process
Solutions to new problems or new solutions to existing problems	Solutions to existing problems
More than one solutions	Only one unique solution in most cases

As mentioned above, design and analysis are tied to each other. A design involves a number of analyses as shown in Fig. 1.2. After having generated a number of design concepts, engineers perform analyses on these concepts. And then they use analysis results to make decisions on selecting the best design concepts with respect to engineering requirements. After the concept selection stage, engineers make more decisions in order to detail and refine the selected design concept. If the design is not considered satisfactory, they will use analysis results to improve and update the design by making necessary changes on material selections, configurations, component interfaces, parameters, and so on. The process iterates until a satisfactory design is identified. During this process, numerous decisions are made.

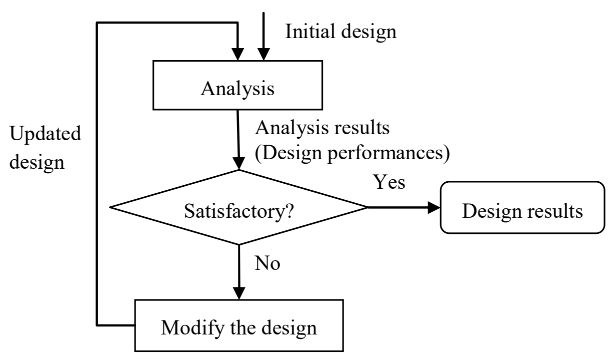

Fig. 1.2. Relationship between design and analyses

To perform an analysis, we usually rely on analysis models. A general analysis model is shown in Fig. 1.3 and is given by

$$y = g(\mathbf{x}) \tag{8}$$

in which $\mathbf{x}$ is a vector of input variables, and it may contain design variables, e.g. the diameter of a shaft, which can be controlled and changed during the design process, or design parameters, e.g. the temperature of the environment, that is out of designer's control. y is an output or response variable, which is dependent on $\mathbf{x}$. y is usually a design performance, such as the cost, maximum stress, or acceleration.

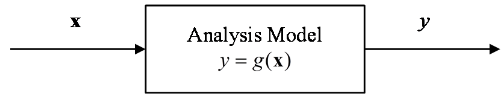

Fig. 1.3. An analysis model

$g(\cdot)$ is the functional relationship between input **x** and output y. In complex engineering design, $g(\cdot)$ may not have an analytic formula, and the output may be obtained through numerical calculations or simulations. This kind of model is often called a *black-box* model. Examples of black-box models include those of finite element analysis, dynamics simulation, and computational fluid dynamics. In product development such as automobile design, sophisticated engineering computational models are eminent. Different from a scientific model that is to fit extant data, an engineering analysis model is primarily used to predict future product performances before a physical product is made.

Analysis models are important for many reasons. (1) Significant upfront design decision-making occurs prior to the availability of physical prototypes. Such decision-making relies heavily on the predictions of design performances from the models. (2) Physical testing can be expensive, time consuming, harmful, or even in some situations prohibitive. (3) Engineers use models to gain useful insights into certain phenomena, which may be lacking from physical experiment due to measurement system limitations.

1.3 Probabilistic Analysis and Probabilistic Design

As discussed previously, uncertainty is unavoidable in engineering analysis and design. To deal with uncertainty in engineering analysis and design, we need to perform probabilistic engineering analysis and probabilistic design.

Probabilistic engineering analysis is the evaluation of the effects of uncertainty on the performance of a system, component, or process. In probabilistic engineering analysis, both model input and output are random, and we then use capital letters for them. Then the analysis model in Eq. (8) is rewritten as

$$Y = g(\mathbf{X}) \tag{9}$$

As indicated in Eq. (9), uncertainty in model input $\mathbf{X}$ can propagate to model output Y through the model $g(\cdot)$. Probabilistic analysis helps engineers understand how uncertainty in the model input impacts the uncertainty in the model output. With this understanding, engineers are able to manage and mitigate the effects of uncertainty during the design process by choosing appropriate design variables, which are part of the model input $\mathbf{X}$.

Through probabilistic analysis on an existing design, engineers can evaluate if the design satisfies all the requirements in the presence of uncertainty. For example, they will be able to know if the design is reliable so that the product has a high chance of not failing; they will also be able to know if the design is robust so that the product can function properly under various operational conditions. For this reason, next we introduce two basic concepts – reliability and robustness.

1.3.1 Reliability analysis

Reliability is intuitively the probability of success or precisely is the probability that a product properly performs its intended function without failures under specified conditions in a given period of time. Reliability tells us the likelihood of no failures. We may use the performance function $g(\cdot)$ to predict if a failure would happen. For instance, if the performance function is the difference between a strength and a stress, then a failure would occur when the strength is less than the stress, or $g(\cdot) < 0$. Hence $g(\cdot) < 0$ indicates a failure, and the probability of failure can therefore be defined by

$$p_f = \Pr\{Y = g(\mathbf{X}) < 0\} \tag{10}$$

And the associated reliability is

$$R = 1 - p_f \tag{11}$$

Or

$$R = \Pr\{Y = g(\mathbf{X}) > 0\} \tag{12}$$

The task of the reliability analysis is to find the reliability or the probability of failure given the distributions of the input variables **X**. As shown in Eq. (9), the response variable Y is also a random variable because it is a function of random variables **X**. As indicated in Eq. (10), the probability of failure is the CDF of *Y* at 0; namely

$$p_f = \Pr\{Y < 0\} = F_Y(0) \tag{13}$$

where $F_Y(\cdot)$ is the CDF of *Y*. If *Y* follows a normal distribution with a mean of μ_Y and a standard deviation of σ_Y, according to Eq. (2), we have

$$p_f = \Phi\left(-\frac{\mu_Y}{\sigma_Y}\right) \tag{14}$$

Example 1.1 Reliability analysis for a solid cylinder

The cylinder in Fig. 1.4 is subject to a random force Q, which is normally distributed. The yield strength of the component S_y is also normally distributed and is independent from *Q*. The distribution parameters are given in Table 1.2. The diameter of the cylinder is $d = 36$ mm. Determine the reliability of the component in terms of yielding failure.

In this example, for an easy demonstration, we treat the diameter d as a deterministic variable. In reality it is random, but its uncertainty is small because its tolerance is small.

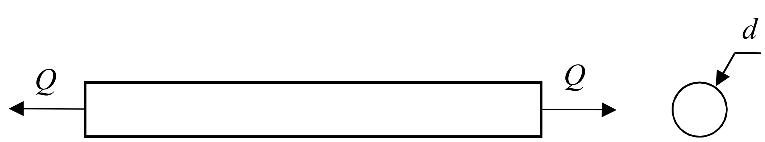

Fig. 1.4. A cylinder subject to an axial load

Table 1.2. Distributions of random variables

Variables	Mean	Standard deviation	Distribution
Q	200 kN	25 kN	Normal
S_y	250 Mpa	25 MPa	Normal

The normal stress is given by

$$S = \frac{Q}{\pi d^2 / 4} = \frac{4}{\pi d^2} Q$$

If the stress is greater than the yield strength, we consider that a failure occurs. The performance function, therefore, is define as

$$Y(\mathbf{X}) = S_y - \frac{4}{\pi d^2} Q$$

where $\mathbf{X} = (X_1, X_2) = (Q, S_y)$.

The probability of failure is then calculated by

$$p_f = \Pr\{Y < 0\} = \Pr\left\{S_y - \frac{4}{\pi d^2} Q < 0\right\}$$

Since Q and S_y are normally and independently distributed, according to Feature 2 of a normal distribution in Sec. 1.1, Y is normally distributed. Using Eqs. (6) and (7), we have the mean and standard deviation of Y as follows:

$$\mu_Y = \mu_{S_y} - \frac{4}{\pi d^2} \mu_Q = 250(10^6) - \frac{4}{\pi \left[36(10)^{-3}\right]^2} 200(10^3) = 53.51 \text{ MPa}$$

And

$$\sigma_Y = \sqrt{\sigma_{S_y}^2 + \left(\frac{4}{\pi d^2}\right)^2 \sigma_Q^2} = \sqrt{\left[25(10^6)\right]^2 + \left(\frac{4}{\pi \left[36(10)^{-3}\right]^2} \left[25(10^3)\right]\right)^2} = 25.0 \text{ MPa}$$

Then according to Eq. (14), the probability of failure is

$$p_f = \Phi\left(-\frac{\mu_Y}{\sigma_Y}\right) = \Phi\left(-\frac{53.51}{25}\right) = \Phi(-2.1405) = 0.0162$$

And the reliability is

$$R = 1 - p_f = 1 - 0.0162 = 0.9838$$

How do we interpret the result? We may say that the likelihood that the component will fail is 1.62%. Or statistically 162 components might fail if 10,000 components are put into operation.

To confirm and visualize the result, let us use a simulation. We first plot the performance function in Fig. 1.5. The straight line represents the limit of the performance function $g(\mathbf{X}) = S_y - S = S_y - \frac{4}{\pi d^2} Q = 0$ where the stress is equal to the strength. The region above the line is the domain where $g(\mathbf{X}) > 0$ or the stress is less than the strength. This domain is called a safe region. Below the line is called the failure region because $g(\mathbf{X}) < 0$ or the stress is greater than the strength.

Knowing the distributions of random variables Q and S_y, we can simulate their possible values by drawing samples from their distributions. The samples are also shown in Fig. 1.5. Most of the samples are in the safe region, and fewer samples are in the failure region. The samples in the failure region are represented as solid dots. The ratio of the number of the samples falling into the failure region over that of the total samples is an estimate of the probability of failure. When the total number of samples goes infinite, the ratio approaches the true probability of failure. This method is called Monte Carlo simulation.

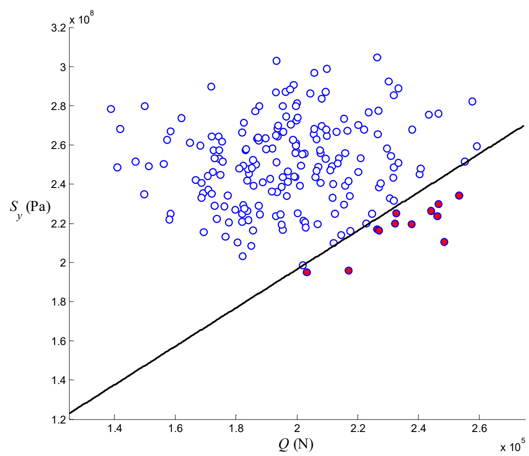

Fig. 1.5. Visualization of the reliability analysis by simulation

From the reliability analysis, we have obtained the reliability and probability of failure of the component. For a practical point of view, the probability of failure for this problem is too high, thereby not acceptable. We will then need to find a solution to reduce the probability of failure. This is the task of reliability-based design, which will be discussed in Section 1.4.

1.3.2 Robustness analysis

Robustness is used to measure the sensitivity of a product performance with respect to noises or uncertainties. A robust design ensures that the nominal (mean) performance of the product be optimal and that the variation in the performance be minimum. The purpose of robustness analysis is to evaluate the mean values and the variations of performance variables. As what we have done in reliability analysis, we also use the performance function $Y = g(\mathbf{X})$ for robust analysis, which is then concerned with the mean value of the performance variable Y for the average performance and with the standard deviation of Y for the variation in the performance. The mean and standard deviation are denoted by μ_Y and σ_Y, respectively.

Fig. 1.6 demonstrates four possible scenarios for a design with two performance variables $Y_1 = g_1(\mathbf{X})$ and $Y_2 = g_2(\mathbf{X})$ under uncertainties in input $\mathbf{X}$.

Case 1: Off target, large variation

The actual performance variables are shown as a cloud in case 1 of Fig. 1.6. The center of the cloud is far away from the target. Because of this, the average performance deviates too much from the ideal value. The size of the cloud is also too large, indicating high variations in the performance variables. Given the poor average performances and large performance variations, the design is not robust.

Case 2: Off target, small variation

The design is better than that in case 1 because its variations in the performances are smaller. But the average performances are still far away from the target, and the design is not considered robust.

Case 3: On target, large variation

The average performances are on the target, but the performance variables fluctuate dramatically around the target. We do not have a robust design because its performances vary too much, possibly from piece to piece, or from user to user.

Case 4: On target, small variation

This is a robust design because its average performances are just on the target and the variations in the performances are small.

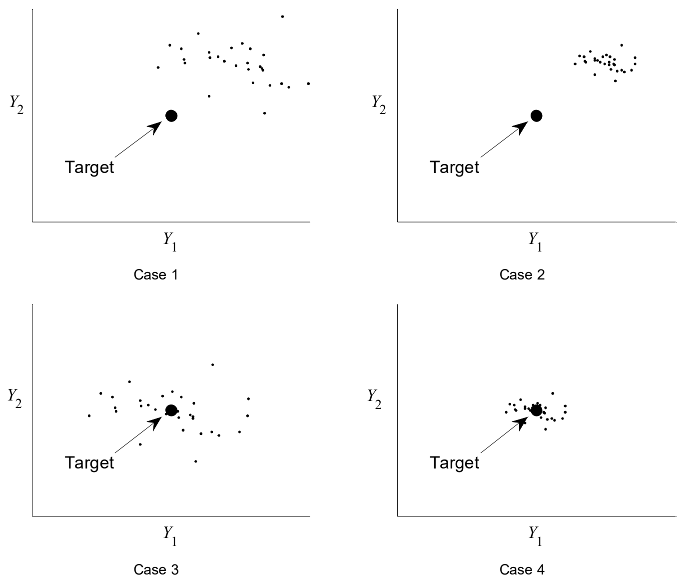

Fig. 1.6. Performance variables under uncertainty

To evaluate the robustness, we need to calculate the mean performance and the standard deviation of the performance. The robustness analysis can then be formed as follows:

Find μ_Y and σ_Y given $Y = g(\mathbf{X})$ and distributions of $\mathbf{X}$

Next, let us discuss a simple problem where the performance function is linear and $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ are normally distributed with $X_1 \sim N(\mu_i, \sigma_i)$ $(i = 1, 2, \cdots, n)$. We assume again that X_i $(i = 1, 2, \cdots, n)$ are independent. Suppose $Y = a_0 + \sum_{i=1}^{n} a_i X_i$, where a_i $(i = 0, 1, \cdots, n)$ are constant.

According to Eqs. (5) and (6), the mean of performance variable is

$$\mu_Y = a_0 + \sum_{i=1}^{n} a_i \mu_i \tag{15}$$

And the standard deviation of Y is

$$\sigma_Y = \sqrt{\sum_{i=1}^{n} a_i^2 \sigma_i^2} \tag{16}$$

Example 1.2

A steel ball is launched at point *A* with an initial speed of v and directed at an angle of θ with the horizontal as shown in Fig. 1.7. Point *B* is the target, whose distance from *A* is $r = 25.0$ m.

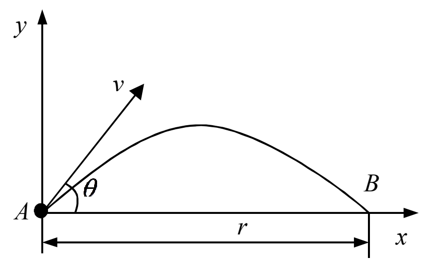

Fig. 1.7. A ball-launching machine

We can change two variables, which are v and θ, to reach the desired target. Since v and θ could not be precisely controlled, they are treated as random variables. Their distributions are normal, and their standard deviations are $\sigma_v = 0.01$ m/s and $\sigma_\theta = 0.1^\circ$. The design problem is to determine the means of v and θ, or μ_V and μ_θ, to hit the target. Two designs are generated. For Design 1, $\mu_\theta = 10^\circ$, and for Design 2, $\mu_\theta = 30^\circ$. We now evaluate the robustness of the two designs.

Assume that there is no air resistance. Then, we have the *x*- and *y*-coordinates of the ball below.

$$\begin{cases} x_{ball} = v\cos\theta - \dfrac{1}{2} g_{grav} t^2 \\ y_{ball} = 0 = v\sin\theta t \end{cases} \tag{17}$$

where t is the time, and g_{grav} is the gravitational acceleration.

Eliminating the time, we have the distance

$$x_{ball} = \frac{v^2 \sin 2\theta}{g_{grav}} \tag{18}$$

We consider the distance as the performance of the design, and then the performance function is given by

$$Y = g(\mathbf{X}) = g(v, \theta) = \frac{v^2 \sin 2\theta}{g_{grav}} \tag{19}$$

Our task now is to compute the mean and standard deviation of the performance, or μ_Y and σ_Y. To use Eqs. (5) and (6), which are for a linear performance function, we linearize $g(V, \theta)$ at μ_V and μ_θ.

$$\begin{aligned} g(v,\theta) &\approx g(\mu_v, \mu_\theta) + \left.\frac{\partial g}{\partial v}\right|_{(\mu_v,\, \mu_\theta)} (v - \mu_v) + \left.\frac{\partial g}{\partial \theta}\right|_{(\mu_v,\, \mu_\theta)} (\theta - \mu_\theta) \\ &= a_0 + a_1 v + a_2 \theta \end{aligned} \tag{20}$$

where $a_0 = g(\mu_v, \mu_\theta) - \left.\frac{\partial g}{\partial v}\right|_{(\mu_v,\, \mu_\theta)} \mu_v - \left.\frac{\partial g}{\partial \theta}\right|_{(\mu_v,\, \mu_\theta)} \mu_\theta$, $a_1 = \left.\frac{\partial g}{\partial v}\right|_{(\mu_v,\, \mu_\theta)}$, and $a_2 = \left.\frac{\partial g}{\partial \theta}\right|_{(\mu_V,\, \mu_\theta)}$.

From Eq. (17), we have

$$\left.\frac{\partial g}{\partial v}\right|_{(\mu_v,\, \mu_\theta)} = \left.\frac{2v \sin 2\theta}{g_{grav}}\right|_{(\mu_v,\, \mu_\theta)} = \left.\frac{2\mu_v \sin 2\mu_\theta}{g_{grav}}\right|_{(\mu_v,\, \mu_\theta)} \tag{21}$$

and

$$\left.\frac{\partial g}{\partial \theta}\right|_{(\mu_v,\, \mu_\theta)} = \left.\frac{2v^2 \cos 2\theta}{g_{grav}}\right|_{(\mu_v,\, \mu_\theta)} = \frac{2\mu_v^2 \cos 2\mu_\theta}{g_{grav}} \tag{22}$$

Using Eqs. (5) and (6), we obtain

$$\mu_Y = \frac{\mu_v^2 \sin 2\mu_\theta}{g_{grav}} \tag{23}$$

$$\begin{aligned} \sigma_Y &= \sqrt{\left(\left.\frac{\partial g}{\partial v}\right|_{(\mu_v,\, \mu_\theta)} \sigma_v\right)^2 + \left(\left.\frac{\partial g}{\partial \theta}\right|_{(\mu_v,\, \mu_\theta)} \sigma_\theta\right)^2} \\ &= \frac{2\mu_v}{g_{grav}} \sqrt{[\sin(2\mu_\theta)\sigma_v]^2 + [\mu_v^2 \cos(2\mu_\theta)\sigma_\theta]^2} \end{aligned} \tag{24}$$

To bring the mean performance to the target, we need set

$$\mu_Y = \frac{\mu_v^2 \sin 2\mu_\theta}{g_{grav}} = r \tag{25}$$

This gives

$$\mu_v = \sqrt{\frac{r g_{grav}}{\sin 2\mu_\theta}} \tag{26}$$

For Design 1

$$\mu_v = \sqrt{\frac{25(9.81)}{\sin[2(10^\circ)]}} = 26.78 \text{ m/s} \tag{27}$$

For Design 2

$$\mu_v = \sqrt{\frac{25(9.81)}{\sin[2(30^\circ)]}} = 16.83 \text{ m/s} \tag{28}$$

Derived from Eq. (23), both of the above mean values of the velocity guarantee that the mean distance $\mu_{Y_1} = \mu_{Y_2} = r = 25.0$ m/s.

We now calculate the standard deviations using Eq. (22).

For Design 1,

$$\sigma_Y = \frac{2(26.78)}{9.81}\sqrt{[\sin 20^\circ (0.01)]^2 + [26.78^2 (\cos 20^\circ)(0.1)]^2}$$
$$= 0.2405 \text{ m/s}$$

(29) (29)

For Design 2,

$$\sigma_Y = \frac{2(16.83)}{9.81}\sqrt{[\sin 60^\circ (0.01)]^2 + [16.83^2 (\cos 60^\circ)(0.1)]^2} \tag{30}$$
$$= 0.0585 \text{ m/s}$$

σ_Y of Design 2 is much lower than that of Design 1, and both designs produce the same average distance to its target. As a result, Design 2 is more robust than Design 1 because the former has lower variation. As what we have done for reliability analysis, we can also use a simulation to visualize the results. Figs. 1.8 and 1.9 show 50 trials of launching the ball for Design 1 and Design 2, respectively. The simulated distances from the two designs fluctuate around the desired distance. Both of the designs have the same average performance (distance). However, the simulated distances produced by Design 2 are much closer to the ideal value than those by Design 1. The simulation results clearly indicate the robustness of the two designs.

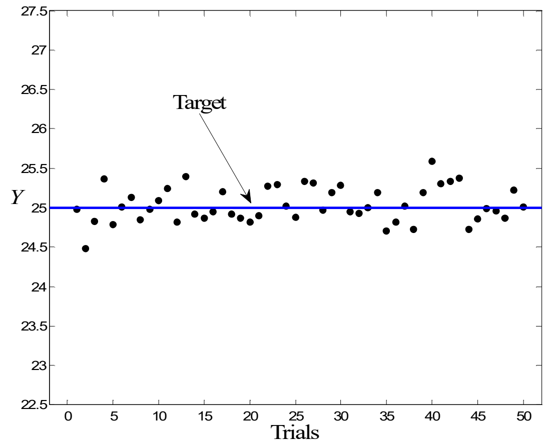

Fig. 1.8. Distance produced by Design 1

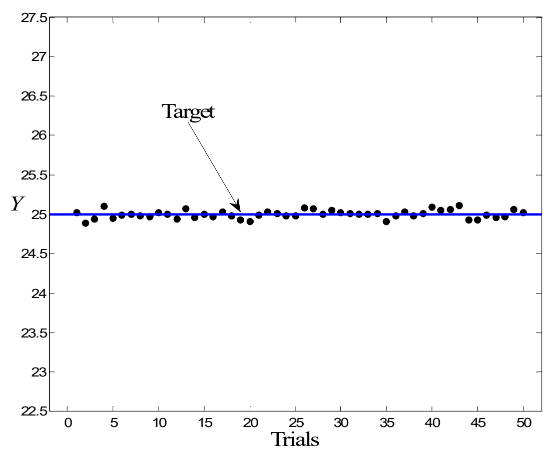

Fig. 1.9. Distance produced by Design 2

1.4 Reliability-Based Design

In Sec. 1.2 we have discussed how to evaluate the reliability for a given design. In this subsection, we look at a reverse problem – generate a design that satisfies reliability at a given reliability level. In this problem, we know the reliability target, and we choose the optimal design variables to reach the reliability target. This is the task of reliability–based design (RBD).

We usually need to tradeoff between reliability and product development cost. This is understandable because the cost normally goes up when reliability increases. The common strategy to tackle this conflict is to minimize the cost on the condition that the reliability requirement is met. Our desire or objective is then to make the development cost minimum, and at the same time we require that the actual reliability should exceed its desired level or should be at least at the desired level. In other words, we treat the reliability requirement as a design constraint. Since both the cost and reliability are determined by design variables, we can select appropriate design variables so that the cost becomes minimum while the reliability requirement is maintained. A general RBD model is therefore formatted as follows:

Find design variables $\mathbf{d}$
To minimize the cost function $C(\mathbf{d}, \mathbf{X})$
Subject to
Reliability $R(\mathbf{d}, \mathbf{X}) \geq R_{req}$
Other requirements (constraints)

In the above model, $\mathbf{d}$ is a vector of design variables, which are those variables that can be changed by designers. For example, when we design the cylinder in Example 1.1, we can change the diameter of the cylinder, and the diameter is therefore a design variable. R_{req} is the desired reliability.

The above model is in a form of optimization. It can be solved automatically by many numerical optimization algorithms. During the solution process, the design variables are continually updated until an optimal solution is reached. At each point of updated design variables, the reliability has to be calculated, which is the task of reliability analysis we have presented in Sec. 1.2. Reliability analysis is therefore an important component of RBD and is repeatedly called during the RBD process.

Example 1.3 Reliability-based design for a solid cylinder

The cylinder is the component we discussed in Example 1.1. It is shown in Fig. 1.3. This component is subject to a random force Q, and its yield strength S_y is also random. The random variables are therefore $\mathbf{X} = (X_1, X_2) = (Q, S_y)$. We have performed reliability analysis for the cylinder in Example 1.1. We now have a design problem where the required reliability is 99.99%, or $R_{reg} = 0.9999$. There is only one design variable, which is the diameter of the cylinder, and therefore $\mathbf{d} = (d)$. We choose the cross-sectional area $c = A = \pi d^2 / 4$ as our cost-type objective because it is closely related to the material cost. Then our task is to determine the value of d so that A is minimum and reliability satisfies $R > R_{reg} = 0.9999$.

In Example 1.1, we have obtained the performance function, which is given by $Y(\mathbf{X}) = S_y - 4Q/\pi d^2$. The reliability is computed by

$$R = \Pr\{Y > 0\} = \Pr\left\{S_y - \frac{4}{\pi d^2} Q > 0\right\}$$

The RBD model for this problem is then formulated as

$$\begin{cases} \min\limits_{r} C(d) = \pi d^2 / 4 \\ \text{subject to} \\ R(r, \mathbf{X}) = \Pr\left\{S_y - \dfrac{4}{\pi d^2} Q > 0\right\} \geq 0.9999 \end{cases}$$

For this simple problem, the RBD model can be solved manually. The yield stress $S = \dfrac{4}{\pi d^2} Q$ indicates that increasing d will decrease the stress, thereby improving the reliability. On the contrary, doing so would increase the cost because the cross section would become larger. If we set the reliability at its target, or $R = R_{req} = 0.9999$, we can obtain the maximum stress or the minimum d so that the reliability requirement is met. This minimum d in turn will produce a minimum cross-sectional area or cost.

We therefore use $R = 0.9999$ to determine the design variable d. From Example 1.1, we know

$$R = 1 - p_f = \Phi\left(-\frac{\mu_Y}{\sigma_Y}\right)$$

We then obtain

$$\frac{\mu_Y}{\sigma_Y} = -\Phi^{-1}(1-R) = -\Phi^{-1}(1-0.9999) = 3.7190$$

where $\Phi^{-1}(\cdot)$ is the inverse function of $\Phi(\cdot)$.

Plugging $\mu_Y = \mu_{S_y} - \frac{4}{\pi d^2}\mu_Q$, $\sigma_Y = \sqrt{\sigma_{S_y}^2 + \left(\frac{4}{\pi d^2}\right)^2 \sigma_Q^2}$, and the equations we had in Example 1.1 into the above equation yields

$$\frac{250(10^6) - \frac{4}{\pi d^2}200(10^3)}{\sqrt{\left[250(10^6)\right]^2 + \left(\frac{4}{\pi d^2}\right)^2 \left[25(10^3)\right]^2}} = 3.7190$$

Solving the equation yields $d = 0.0319$ m, and we set $d = 32$ mm. This diameter guarantees that the reliability will be at least 0.9999 and that the cross-sectional area will be at most $A = 32^2 \pi / 4 = 804.25 \text{ mm}^2$.

1.5 Robust Design

The purpose of robust design (RD) is to make product performance not sensitive to variations. As we discussed in Sec. 1.3, robustness is measured by the mean performance and the standard deviation of the performance. The major advantage of RD is that robustness can be achieved without eliminating the sources of uncertainty. (Eliminating the sources of uncertainty is costly.) This is done by just choosing appropriate design variables.

A performance variable is called a quality characteristic (QC) in RD. It is a response variable that significantly affects the product function and customer satisfaction. The efficiency, reliability, durability, and engine quietness of a vehicle, are examples of QCs. There are three types of QCs.

- Nominal-the-best QCs

There is an ideal value or target for a QC. For example, the position of the output member of a mechanism should be at a desired point.

- Smaller-the-better QCs

A smaller QC is better than a larger one, for example, the fuel consumption, the noise level of an engine, and the number of failures.

- Larger-the-better QCs

A larger QC is better than a smaller one. For example, we prefer larger yield and higher efficiency.

Since the average and the standard deviation of a QC are two major concerns, both of them appear in the objective function of RD. There are many RD models. The one below is commonly used for nominal-the-best QCs.

Find design variable $\mathbf{d}$
To minimize $w_1[\mu_Y(\mathbf{d}, \mathbf{X}) - m]^2 + w_2\sigma_Y^2(\mathbf{d}, \mathbf{X})$
Subject to
Constraints

In the above model, Y is the QC, m is its target. μ_Y and σ_Y are the mean and standard deviation of Y, respectively. They are functions of design variables $\mathbf{d}$ and random variables $\mathbf{X}$. w_1 and w_2 are the weights and $w_1 + w_2 = 1$.

The solution to the above model may bring the average QC μ_Y to its target and reduce the variation in Y, represented by σ_Y.

Example 1.4

In Example 1.2, we evaluated the robustness of two designs for a ball-launching device. Now we perform RD to find the best robust design for the device. The intended function of the device is to launch a ball to a target point. The function is realized by adjusting the initial velocity v and its direction expressed by the angle θ. Since both of the variables are random, the vector of random variables are $\mathbf{X} = (v, \theta)$, and the design variables are their nominal values (means) $\mathbf{d} = (\mu_V, \mu_\theta)$.

The QC is the horizontal distance r from the origin to the point where the ball lands. Let the QC be $Y = r$, and assume that the range of θ is $[0^\circ, 40^\circ]$ and that the target is $m = 25$ m.

The RD model is then formulated as

$$\begin{cases} \min\limits_{\mathbf{d}=(\mu_V,\, \mu_\theta)} w_1[\mu_Y - 25]^2 + w_2\sigma_Y^2 \\ \text{subject to} \\ 0^\circ \le \mu_\theta \le 40^\circ \end{cases}$$

For this specific problem, the values of w_1 and w_2 do not matter.

For this simple problem, to avoid using an optimization algorithm, we find the solution with a graphical method.

We at first bring the mean of the QC to its target. This gives

$$\mu_Y - 25 = 0$$

Using the equation for μ_Y obtained in Example 1.2, we have

$$\frac{\mu_V^2 \sin 2\mu_\theta}{g_{grav}} - 25 = 0$$

which yields

$$\mu_v = \sqrt{\frac{25g}{\sin 2\mu_\theta}}$$

From Example 1.2, we also know

$$\mu_Y = \frac{\mu_v^2 \sin 2\mu_\theta}{g_{grav}}$$

$$\sigma_Y = \frac{2\mu_v}{g_{grav}} \sqrt{[\sin(2\mu_\theta)\sigma_v]^2 + [\mu_v^2 \cos(2\mu_\theta)\sigma_\theta]^2}$$

Using the above three equations, by changing μ_θ over $[0^\circ, 40^\circ]$, we obtain design results that are tabulated in Table 1.3.

Table 1.3. RD Results

Design	μ_θ (°)	μ_v (m/s)	μ_Y (m)	σ_Y (m)
Design 1	10	26.7780	25.0	0.2405
Design 2	15	22.1472	25.0	0.1528
Design 3	20	19.5331	25.0	0.1071
Design 4	25	17.8928	25.0	0.0784
Design 5	30	16.8283	25.0	0.0585
Design 6	35	16.1552	25.0	0.0443
Design 7	40	15.7808	25.0	0.0352

All the designs can bring the QC to its target with $\mu_Y = 25$ m, but they produce different standard deviations σ_Y of the QC. As shown in the table and Fig. 1.10, as μ_θ increases, σ_Y decreases. Design 7 is the best because it brings the mean QC to its target and also produces the minimum standard deviation of the QC.

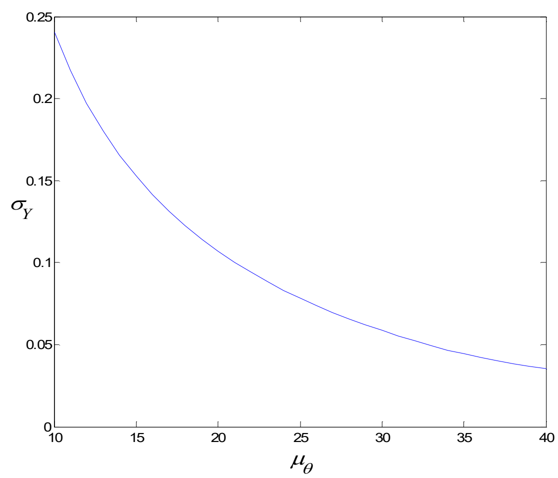

Fig. 1.10. The means and standard deviations of the QC

1.6 Organization of the Book

In addition to this introduction chapter, this book consists of three parts. Part one is for uncertainty modeling. It has just three chapters, which focus on modeling uncertainty with the probability theory. Basics of how to use the probability theory for qualifying uncertainty will be discussed from an engineer's aspect.

Part two is for probabilistic analysis and consists of Chapters 5 through 9. It concentrates on evaluating the effects of uncertainty of model input on model output. Reliability analysis and robustness analysis are the focus.

Part three deals with probabilistic design, particularly reliability-based design and robust design. Since both design methodologies are based on optimization, the first chapter, Chapter 10, in this part, is dedicated to a brief introduction to optimization. Reliability-based design and robust design are then discussed in Chapters 11 and 12.

1.7 Conclusions

In this tutorial-style introduction, we have reviewed the general engineering design process and introduced important concepts of uncertainty, including reliability and robustness. We also discussed two basic probabilistic design methodologies – reliability-based design and robust design. Below are listed some of the important points we have presented.

- Uncertainty is the gap between the present state of knowledge and the complete knowledge.
- An uncertain variable is commonly treated as a random variable, which can be fully represented by its cumulative distribution function (CDF).
- The effects of uncertainty include quality loss, risk, large variation, low customer satisfaction, and high cost.
- Uncertainty should be treated as a core element in engineering design.
- Reliability and robustness are two primary measures of product performance in the presence of uncertainty. The former is the

probability of no failure and the latter is the insensitivity to uncertainty.

- Reliability-based design and robust design make a product perform its intended function at a desired level of probability and make the product performance stable under uncertainty, respectively.

Probabilistic engineering design can be used in all stages of engineering design, including the stages of conceptual design, preliminary design, and detail design. Since probabilistic design has benefits on product reliability, robustness, safety, quality, and customer satisfaction, engineers should have sufficient knowledge about probabilistic design.