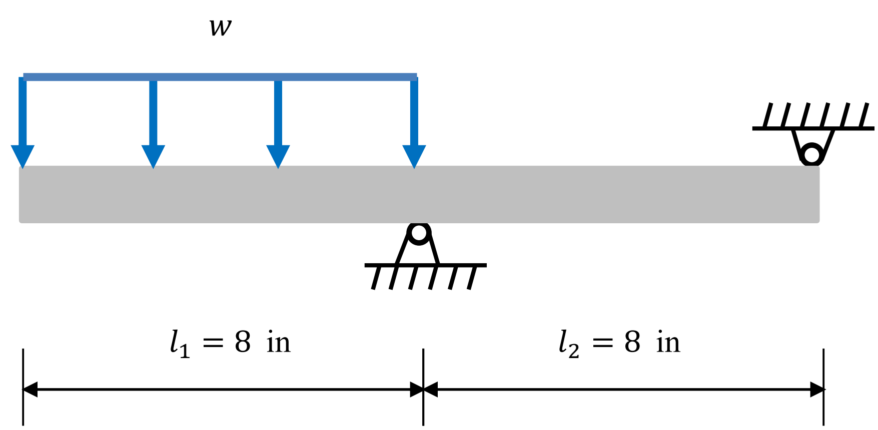
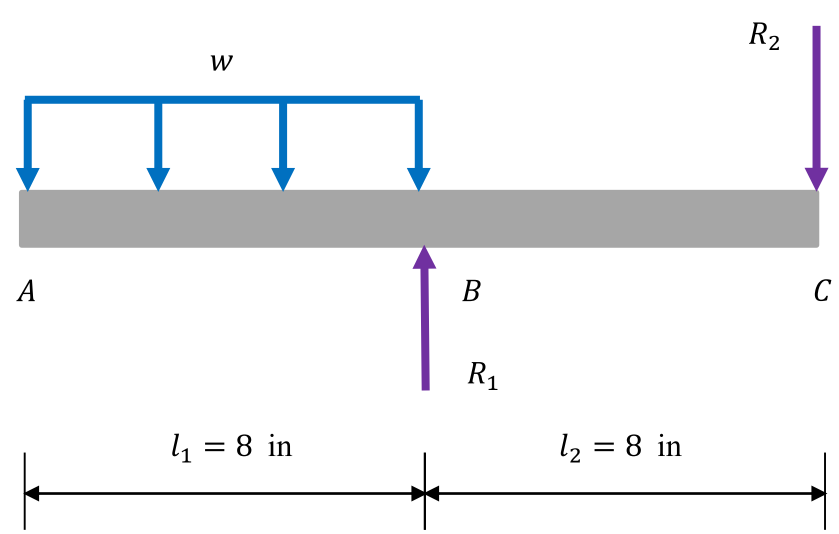
67. A uniform load of $w \sim N(200,\ 20^2)$ lbf/in is applied to a beam with a round cross section. If the allowable bending stress is $S_a \sim N(3000,\ 300^2)$ psi and the maximum probability of failure is designed to be $p_f = 10^{-5}$, estimate the minimum diameter of the beam and select a preferred one. Assume that w and S_a are independent.



Solution

Consider the free body diagram of the beam shown below.



Based on the moment equilibrium of the beam with respect to B,

$$R_2 l_2 = w l_1 \frac{l_1}{2}$$

The maximum bending moment occurs at point B and is given by

$$M = R_2 l_2 = w l_1 \frac{l_1}{2} = \frac{l_1^2}{2} w$$

Thus the maximum bending stress is

$$S = \frac{Mc}{I} = \frac{M \frac{d}{2}}{\frac{\pi}{64} d^4} = \frac{32 \frac{l_1^2}{2} w}{\pi d^3} = \frac{16 l_1^2}{\pi d^3} w$$

The limit-state function is the maximum bending stress subtracted from the allowable stress. Failure occurs when $Y < 0$.

$$Y = g(\mathbf{X}) = S_a - S = S_a - \frac{16 l_1^2}{\pi d^3} w$$

where $\mathbf{X}=(S_a, w)$.

Using FOSM, we have

$$\mu_Y = g(\boldsymbol{\mu}_{\mathbf{X}}) = \mu_{S_a} - \frac{16 l_1^2}{\pi d^3} \mu_w$$

$$\sigma_Y = \sqrt{\left(\left.\frac{\partial g}{\partial S_a}\right|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_{S_a}\right)^2 + \left(\left.\frac{\partial g}{\partial w}\right|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_w\right)^2} = \sqrt{\left(\sigma_{S_a}\right)^2 + \left(-\frac{16 l_1^2}{\pi d^3} \sigma_w\right)^2}$$

The probability of failure is then given by

$$p_f = \Phi\left(-\frac{\mu_Y}{\sigma_Y}\right) = \Phi\left(-\frac{\mu_{S_a} - \frac{16 l_1^2}{\pi d^3} \mu_w}{\sqrt{\left(\sigma_{S_a}\right)^2 + \left(-\frac{16 l_1^2}{\pi d^3} \sigma_w\right)^2}}\right) = 10^{-5}$$

Thus

$$\Phi^{-1}(10^{-5}) = -\frac{\mu_{S_a} - \frac{16 l_1^2}{\pi d^3} \mu_w}{\sqrt{\left(\sigma_{S_a}\right)^2 + \left(-\frac{16 l_1^2}{\pi d^3} \sigma_w\right)^2}} = -\frac{3(10^3) - \frac{16(8^2)}{\pi d^3}(200)}{\sqrt{(300)^2 + \left(-\frac{16(8^2)}{\pi d^3}(20)\right)^2}}$$

Solving for d yields

$$d = 3.47 \text{ in}$$

Thus $d = 3.6$ in can be used.

Ans.