13. A ductile shaft is subjected to a toruqe $T \sim N(200,\ 20^2)$ N·m. The diameter of the shaft is $d =$ 25 mm. If the yield strength in tension of the shaft is $S_{yt} \sim N(180,\ 10^2)$ MPa and the yield strength in compression is $S_{yc} \sim N(160,\ 10^2)$ MPa, and T, S_{yt} and S_{yc} are independent, determine the probability of failure using the First Order Second Moment Method.

Solution

Based on the Coulomb-Mohr theory for ductile materials, the shear yield strength is given by

$$S_{sy} = \frac{S_{yt} S_{yc}}{S_{yt} + S_{yc}}$$

The maximum shear stress is

$$\tau_{max} = \frac{16T}{\pi d^3}$$

So the limit-state function is the maximum shear stress subtracted from the shear yield strength. Failure occurs when $Y < 0$.

$$Y = g(\mathbf{X}) = S_{sy} - \tau_{max} = \frac{S_{yt} S_{yc}}{S_{yt} + S_{yc}} - \frac{16T}{\pi d^3} = \frac{S_{yt} S_{yc}}{S_{yt} + S_{yc}} - 3.26(10^5)T$$

where $\mathbf{X}$=(S_{sy}, S_{st}, T).

Using FOSM, we have

$$\mu_Y = g(\boldsymbol{\mu}_{\mathbf{X}}) = \frac{\mu_{S_{yt}} \mu_{S_{yc}}}{\mu_{S_{yt}} + \mu_{S_{yc}}} - 3.26(10^5)\mu_T$$

$$= \frac{180(10^6)(160)(10^6)}{180(10^6) + 160(10^6)} - 3.26(10^5)(200) = 1.95(10^7) \text{ Pa}$$

$$\sigma_Y = \sqrt{\left(\left.\frac{\partial g}{\partial S_{yt}}\right|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_{S_{yt}}\right)^2 + \left(\left.\frac{\partial g}{\partial S_{yc}}\right|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_{S_{yc}}\right)^2 + \left(\left.\frac{\partial g}{\partial T}\right|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_T\right)^2}$$

$$= \sqrt{\begin{array}{c} \left(\frac{\mu_{S_{yc}}}{\mu_{S_{yt}} + \mu_{S_{yc}}} - \frac{\mu_{S_{yt}} \mu_{S_{yc}}}{\left(\mu_{S_{yt}} + \mu_{S_{yc}}\right)^2}\right)^2 \left(\sigma_{S_{yt}}\right)^2 \\ + \left(\frac{\mu_{S_{yt}}}{\mu_{S_{yt}} + \mu_{S_{yc}}} - \frac{\mu_{S_{yt}} \mu_{S_{yc}}}{\left(\mu_{S_{yt}} + \mu_{S_{yc}}\right)^2}\right)^2 \left(\sigma_{S_{yc}}\right)^2 + (-3.26(10^5)\sigma_T)^2 \end{array}}$$

$$= \sqrt{\begin{array}{c}\left(\frac{160(10^6)}{180(10^6)+160(10^6)} - \frac{180(10^6)(160)(10^6)}{(180(10^6)+160(10^6))^2}\right)^2 (10(10^6))^2 \\ + \left(\frac{180(10^6)}{180(10^6)+160(10^6)} - \frac{180(10^6)(160)(10^6)}{(180(10^6)+160(10^6))^2}\right)^2 (10(10^6))^2 + (-3.26(10^5)20)^2\end{array}}$$

$= 4.84(10^6)$ Pa

The probability of failure is then given by

$$p_f = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-1.95(10^7)}{4.84(10^6)}\right) = 2.72(10^{-5})$$ **Ans.**