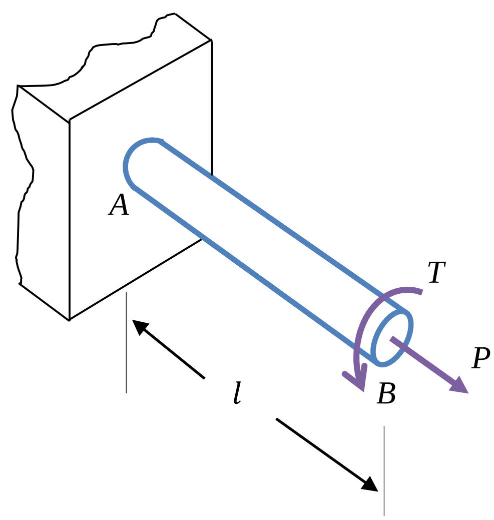
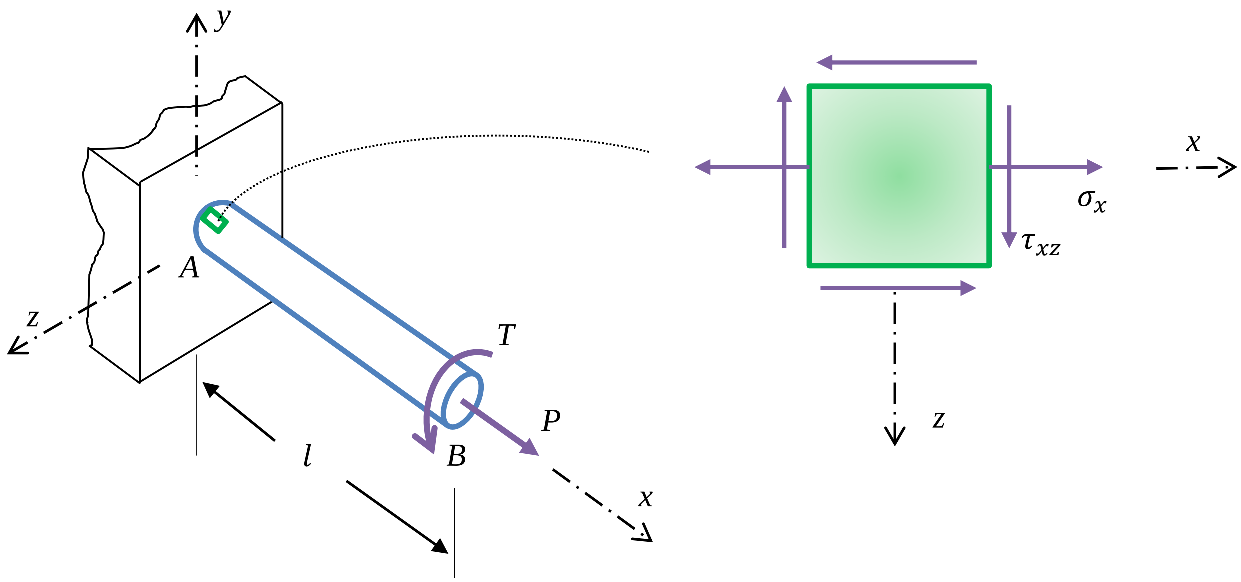
14. A cantilever with a circular cross-section is subjected to an axial tension $P \sim N(10,\ 1^2)$ kN and a torsion $T \sim N(2,\ 0.2^2)$ kN·m acting at point B. The length of the cantilever is $l = 100$ mm and the diameter is $d = 50$ mm. If the yield strength of the cantilever is $S_y \sim N(220,\ 10^2)$ MPa, and P, T and S_y are independent, determine the probability of failure using the First Order Second Moment Method. Use the distortion-energy theory.



Solution

The critical stress element appears on the top surface at point A, shown in the following figure.



The normal stress resulted from axial force is given by

$$\sigma_x = \frac{P}{A} = \frac{P}{\frac{\pi}{4}d^2} = \frac{4P}{\pi d^2}$$

where A is the area of the cross-section.

The torsion stress at the critical stress element is

$$\tau_{xz} = \frac{Tr}{J} = \frac{Td}{2J} = \frac{Td}{2\frac{\pi d^4}{32}} = \frac{16T}{\pi d^3}$$

where r is the radius of the circular cross-section, and J is the polar second moment of inertial.

According to the distortion-energy theory, the von Mises stress is found to be

$$\sigma' = (\sigma_x^2 + 3\tau_{xz}^2)^{1/2}$$

So the limit-state function is the von Mises stress subtracted from the yield strength. Failure occurs when $Y < 0$.

$$Y = g(\mathbf{X}) = S_y - \sigma' = S_y - (\sigma_x^2 + 3\tau_{xz}^2)^{1/2} = S_y - \left(\left(\frac{4P}{\pi d^2}\right)^2 + 3\left(\frac{16T}{\pi d^3}\right)^2\right)^{1/2}$$

$$= S_y - \frac{4}{\pi d^2}\left(P^2 + 48\frac{T^2}{d^2}\right)^{1/2} = S_y - 509.30(P^2 + 19200T^2)^{1/2}$$

where $\mathbf{X}$=(S_y, P, T).

Using FOSM, we have

$$\mu_Y = g(\boldsymbol{\mu}_{\mathbf{X}}) = \mu_{S_y} - 509.30(\mu_P^2 + 19200\mu_T^2)^{1/2}$$

$$= 220(10^6) - 509.30\left(\left(10(10^3)\right)^2 + 19200\left(2(10^3)\right)^2\right)^{\frac{1}{2}} = 7.88(10^7)\ \text{Pa}$$

$$\sigma_Y = \sqrt{\left(\left.\frac{\partial g}{\partial S_y}\right|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_{S_y}\right)^2 + \left(\left.\frac{\partial g}{\partial P}\right|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_P\right)^2 + \left(\left.\frac{\partial g}{\partial T}\right|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_T\right)^2}$$

$$= \sqrt{\left(\sigma_{S_y}\right)^2 + \left(-509.30(\frac{1}{2})\frac{2\mu_{S_P}}{(\mu_P^2 + 19200\mu_T^2)^{1/2}}\right)^2 \sigma_P^2 + \left(-509.30(\frac{1}{2})\frac{19200(2)\mu_{S_T}}{(\mu_P^2 + 19200\mu_T^2)^{1/2}}\right)^2 \sigma_T^2}$$

$$= \sqrt{\begin{gathered}(10(10)^6)^2 + \left(-509.30\frac{10(10)^3}{\left(\left(10(10^3)\right)^2 + 19200\left(2(10^3)\right)^2\right)^{\frac{1}{2}}}\right)\left(1(10^3)\right)^2 \\ \left(-509.30\frac{19200\left(2(10^3)\right)}{\left(\left(10(10^3)\right)^2 + 19200\left(2(10^3)\right)^2\right)^{\frac{1}{2}}}\right)(200)^2\end{gathered}}$$

$$= 1.73(10^7)\ \text{Pa}$$

The probability of failure is then given by

$$p_f = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-7.88(10^7)}{1.73(10^7)}\right) = 2.61(10^{-6})$$ **Ans.**