21. A torsion $T \sim N(2, 0.2^2)$ kN·m is applied to a bar with a hollow round cross-section. The bar has an outside diameter of $d_o = 10$ cm and an inside diameter of $d_i = 3$ cm. If the allowable stress of the bar is $\tau_a \sim N(20, 2^2)$ MPa, estimate the probability of failure using the First Order Second Moment Method. Note that T and τ_a are independent.

Solution

The maximum shear stress is

$$\tau_{\max} = \frac{Tr}{J} = \frac{Td_o}{2J}$$

where J is the polar second moment of area, given by

$$J = \frac{\pi}{32}(d_o^4 - d_i^4)$$

The limit-state function is the maximum shear stress subtracted from the allowable stress. Failure occurs when $Y < 0$.

$$Y = g(\mathbf{X}) = \tau_a - \tau_{\max} = \tau_a - \frac{Td_o}{2\left(\frac{\pi}{32}\right)(d_o^4 - d_i^4)}$$

$$= \tau_a - 5.1345(10^3)T$$

where $\mathbf{X}$=(τ_a, T).

Using FOSM, we have

$$\mu_Y = g(\boldsymbol{\mu}_\mathbf{X}) = \mu_{\tau_a} - 5.1345(10^3)\mu_T = 20(10^6) - 5.1345(10^3)(2)(10^3) = 9.7309(10^6) \text{ Pa}$$

$$\sigma_Y = \sqrt{\left(\frac{\partial g}{\partial \tau_a}\bigg|_{\boldsymbol{\mu}_\mathbf{X}} \sigma_{\tau_a}\right)^2 + \left(\frac{\partial g}{\partial T}\bigg|_{\boldsymbol{\mu}_\mathbf{X}} \sigma_T\right)^2}$$

$$= \sqrt{\left(\sigma_{\tau_a}\right)^2 + (-5.1345(10^3)\sigma_T)^2}$$

$$= \sqrt{\left(2(10^6)\right)^2 + (-5.1345(10^3)(0.2)(10^3))^2}$$

$$= 2.2482(10^6) \text{ Pa}$$

The probability of failure is then given by

$$p_f = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-9.7309(10^6)}{2.2482(10^6)}\right) = 7.52(10^{-6})$$ **Ans.**