39. A shaft with a circular cross section is subjected to a torque $T \sim N(1000,\ 100^2)$ N. The allowable shear stress is $\tau_a \sim N(20, 2^2)$ MPa. If the maximum probability of failure is designed to be $p_f = 10^{-5}$, determine the minimum diameter using the First Order Second Moment Method.

Solution

For a round shaft in torsion, the maximum shear stress is given by

$$\tau = \frac{16T}{\pi d^3}$$

Thus the limit-state function is the maximum stress subtracted from allowable shear stress. Failure occurs when $Y < 0$

$$Y = g(\mathbf{X}) = \tau_a - \tau = \tau_a - \frac{16T}{\pi d^3}$$

where $\mathbf{X}$=(T, τ_a).

Using FOSM, we have

$$\mu_Y = g(\boldsymbol{\mu}_{\mathbf{X}}) = \mu_{\tau_a} - \frac{16\mu_T}{\pi d^3} = 20(10^6) - \frac{16(1000)}{\pi d^3}$$

$$\sigma_Y = \sqrt{\left(\frac{\partial g}{\partial T}\bigg|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_T\right)^2 + \left(\frac{\partial g}{\partial \tau_a}\bigg|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_{\tau_a}\right)^2}$$

$$= \sqrt{\left(-\frac{16}{\pi d^3}\sigma_T\right)^2 + \left(\sigma_{\tau_a}\right)^2}$$

$$= \sqrt{\left(-\frac{16}{\pi d^3}(100)\right)^2 + \left(2(10^6)\right)^2}$$

The probability of failure is

$$p_f = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\left(20(10^6) - \frac{16(1000)}{\pi d^3}\right)}{\left(\sqrt{\left(-\frac{16}{\pi d^3}(100)\right)^2 + \left(2(10^6)\right)^2}\right)}\right) = 10^{-5}$$

Thus

$$\frac{-\mu_Y}{\sigma_Y} = \frac{-\left(20(10^6) - \frac{16(1000)}{\pi d^3}\right)}{\left(\sqrt{\left(-\frac{16}{\pi d^3}(100)\right)^2 + \left(2(10^6)\right)^2}\right)} = \Phi^{-1}(10^{-5})$$

Solving for d yields

$$d = 78.9 \text{ mm}$$

So $d = 80$ mm can be used.