40. A round shaft is subjected to a bending moment $M \sim N(6000,\ 300^2)$ lbf·in. The yield strength of the shaft is $S_y \sim N(60,\ 6^2)$ kpsi. If the maximum probability of failure is designed to be $p_f = 10^{-5}$, determine the minimum diameter using the First Order Second Moment Method. Note that M and S_y are independent.

Solution

For a round shaft in bending, the bending stress is given by

$$S = \frac{16M}{\pi d^3}$$

Thus the limit-state function is the bending stress subtracted from the yield strength. Failure occurs when $Y < 0$

$$Y = g(\mathbf{X}) = S_y - S = S_y - \frac{16M}{\pi d^3}$$

where **X**=(M, S_y).

Using FOSM, we have

$$\mu_Y = g(\boldsymbol{\mu}_{\mathbf{X}}) = \mu_{S_y} - \frac{16\mu_M}{\pi d^3} = 60(10^6) - \frac{16(6000)}{\pi d^3}$$

$$\sigma_Y = \sqrt{\left(\left.\frac{\partial g}{\partial M}\right|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_M\right)^2 + \left(\left.\frac{\partial g}{\partial S_y}\right|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_{S_y}\right)^2}$$

$$= \sqrt{\left(-\frac{16}{\pi d^3}\sigma_M\right)^2 + \left(\sigma_{S_y}\right)^2}$$

$$= \sqrt{\left(-\frac{16}{\pi d^3}(300)\right)^2 + \left(6(10^6)\right)^2}$$

The probability of failure is

$$p_f = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\left(60(10^6) - \frac{16(6000)}{\pi d^3}\right)}{\left(\sqrt{\left(-\frac{16}{\pi d^3}(300)\right)^2 + \left(6(10^6)\right)^2}\right)}\right) = 10^{-5}$$

Thus

$$\frac{-\mu_Y}{\sigma_Y} = \frac{-\left(60(10^6) - \frac{16(6000)}{\pi d^3}\right)}{\left(\sqrt{\left(-\frac{16}{\pi d^3}(300)\right)^2 + \left(6(10^6)\right)^2}\right)} = \Phi^{-1}(10^{-5})$$

Solving for d yields

$$d = 97.0 \text{ mm}$$

Thus $d = 100$ mm can be used.