48. A bar has a hollow round cross section and an inner diameter of $d_i = 3$ cm. The bar is subjected to a torsion $T \sim N(5, 0.5^2)$ kN·m. And the allowable stress of the bar is $\tau_a \sim N(20, 2^2)$ MPa. If the maximum probability of failure is designed to be $p_f = 10^{-5}$, determine the minimum outer diameter of the bar using the First Order Second Moment Method. Note that T and τ_a are independent.

Solution

The maximum shear stress is

$$\tau_{\max} = \frac{Tr}{J} = \frac{Td_o}{2J}$$

where J is the polar second moment of area, given by

$$J = \frac{\pi}{32}(d_o^4 - d_i^4)$$

The limit-state function is the maximum shear stress subtracted from the allowable stress. Failure occurs when $Y < 0$.

$$Y = g(\mathbf{X}) = \tau_a - \tau_{\max} = \tau_a - \frac{Td_o}{2J}$$

$$= \tau_a - \frac{d_o}{\frac{\pi}{16}\left(d_o^4 - \left(3(10^{-2})\right)^4\right)} T$$

where $\mathbf{X}=(\tau_a, T)$.

Using FOSM, we have

$$\mu_Y = g(\boldsymbol{\mu}_\mathbf{X}) = \mu_{\tau_a} - \frac{d_o}{\frac{\pi}{16}\left(d_o^4 - \left(3(10^{-2})\right)^4\right)}\mu_T = 20(10^6) - \frac{d_o}{\frac{\pi}{16}\left(d_o^4 - \left(3(10^{-2})\right)^4\right)}(5)(10^3)$$

$$\sigma_Y = \sqrt{\left(\left.\frac{\partial g}{\partial \tau_a}\right|_{\boldsymbol{\mu}_\mathbf{X}} \sigma_{\tau_a}\right)^2 + \left(\left.\frac{\partial g}{\partial T}\right|_{\boldsymbol{\mu}_\mathbf{X}} \sigma_T\right)^2}$$

$$= \sqrt{\left(\sigma_{\tau_a}\right)^2 + \left(-\frac{d_o}{\frac{\pi}{16}\left(d_o^4 - \left(3(10^{-2})\right)^4\right)}\sigma_T\right)^2}$$

$$= \sqrt{\left(2(10^6)\right)^2 + \left(-\frac{d_o}{\frac{\pi}{16}\left(d_o^4 - \left(3(10^{-2})\right)^4\right)}(0.5)(10^3)\right)^2}$$

The probability of failure is then given by

$$p_f = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\left(20(10^6) - \frac{d_o}{\frac{\pi}{16}\left(d_o^4 - \left(3(10^{-2})\right)^4\right)}(5)(10^3)\right)}{\sqrt{\left(2(10^6)\right)^2 + \left(-\frac{d_o}{\frac{\pi}{16}\left(d_o^4 - \left(3(10^{-2})\right)^4\right)}(0.5)(10^3)\right)^2}}\right) = 10^{-5}$$

Thus

$$\frac{-\mu_Y}{\sigma_Y} = \frac{-\left(20(10^6) - \frac{d_o}{\frac{\pi}{16}\left(d_o^4 - \left(3(10^{-2})\right)^4\right)}(5)(10^3)\right)}{\sqrt{\left(2(10^6)\right)^2 + \left(-\frac{d_o}{\frac{\pi}{16}\left(d_o^4 - \left(3(10^{-2})\right)^4\right)}(0.5)(10^3)\right)^2}} = \Phi^{-1}(10^{-5})$$

Solving for d_o yields

$$d_o = 134.9 \text{ mm}$$

Thus $d_o = 140$ mm can be used. **Ans.**