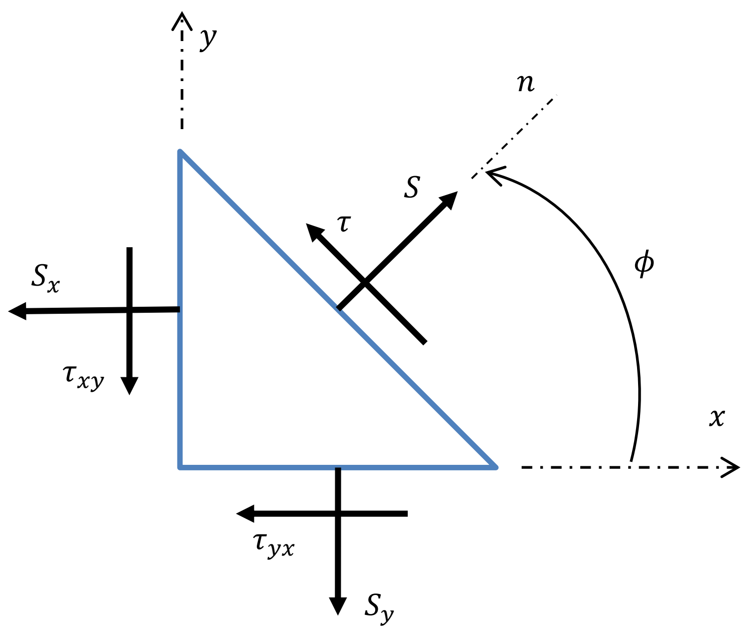
7. A stress element is cut by an oblique plane with a normal n at an angle $\phi = 45^0$ counterclockwise from the x axis. If $S_x \sim N(80, 8^2)$ MPa, $S_y \sim N(60, 6^2)$ MPa, $\tau_{xy} \sim N(50, 5^2)$ MPa, and S_x, S_y and τ_{xy} are independent, what is the distribution of stresses S and τ ?



Solution

Considering the force equilibrium with respect to the x axis, we have

$$S\cos\phi - \tau\sin\phi - S_x - \tau_{yx} = 0 \tag{1}$$

Considering the force equilibrium with respect to the y axis, we have

$$S\sin\phi + \tau\cos\phi - S_y - \tau_{xy} = 0 \tag{2}$$

The cross shears are equal, and hence

$$\tau_{xy} = \tau_{yx} \tag{3}$$

Combining Eqs. (1) - (3), we find the stresses S and τ given below.

$$S = \frac{S_x + S_y}{2} + \frac{S_x - S_y}{2}\cos 2\phi + \tau_{xy}\sin 2\phi = \frac{S_x + S_y}{2} + \tau_{xy}$$

$$\tau = -\frac{S_x - S_y}{2}\sin 2\phi + \tau_{xy}\cos 2\phi = -\frac{S_x - S_y}{2}$$

Because S_x, S_y and τ_{xy} are independently and normally distributed, their linear combinations, S and τ, are also normally distributed. The mean and standard deviation of S and τ are given

$$\mu_S = \frac{1}{2}\mu_{S_x} + \frac{1}{2}\mu_{S_y} + \mu_{\tau_{xy}} = \frac{1}{2}(80) + \frac{1}{2}(60) + 50 = 120 \text{ MPa}$$

$$\sigma_S = \sqrt{\left(\frac{\partial S}{\partial S_x}\right)^2 \sigma_{S_x}^2 + \left(\frac{\partial S}{\partial S_y}\right)^2 \sigma_{S_y}^2 + \left(\frac{\partial S}{\partial \tau_{xy}}\right)^2 \sigma_{\tau_{xy}}^2} = \sqrt{\left(\frac{1}{2}\right)^2 \sigma_{S_x}^2 + \left(\frac{1}{2}\right)^2 \sigma_{S_y}^2 + \sigma_{\tau_{xy}}^2}$$

$$= \sqrt{\left(\frac{1}{2}\right)^2 8^2 + \left(\frac{1}{2}\right)^2 6^2 + 5^2} = 7.07 \text{ MPa}$$

$$\mu_\tau = -\frac{1}{2}\mu_{S_x} + \frac{1}{2}\mu_{S_y} = -\frac{1}{2}(80) + \frac{1}{2}(60) = -10 \text{ MPa}$$

$$\sigma_\tau = \sqrt{\left(\frac{\partial \tau}{\partial S_x}\right)^2 \sigma_{S_x}^2 + \left(\frac{\partial \tau}{\partial S_y}\right)^2 \sigma_{S_y}^2} = \sqrt{\left(-\frac{1}{2}\right)^2 \sigma_{S_x}^2 + \left(\frac{1}{2}\right)^2 \sigma_{S_y}^2} = \sqrt{\left(-\frac{1}{2}\right)^2 8^2 + \left(\frac{1}{2}\right)^2 6^2} = 5 \text{ MPa}$$

So the distributions of the stresses are $S \sim N(120, 7.07^2)$ MPa and $\tau \sim N(-10, 5^2)$ MPa. **Ans.**