10. A toruqe $T \sim N(200, 20^2)$ N·m is applied to a ductile shaft. The yield strength in tension of the shaft is $S_{yt} \sim N(180,\ 10^2)$ MPa and the one in compression is $S_{yc} \sim N(160,\ 10^2)$ MPa. If the maximum probability of failure is designed to be $p_f = 10^{-5}$, determine the minimum diameter of the shaft using the First Order Second Moment Method. Assume that T, S_{yt} and S_{yc} are independent.

Solution

According to the Coulomb-Mohr theory, the shear yield strength of ductile shaft is

$$S_{sy} = \frac{S_{yt} S_{yc}}{S_{yt} + S_{yc}}$$

The maximum shear stress is given by

$$\tau_{max} = \frac{16T}{\pi d^3}$$

So the limit-state function is the maximum shear stress subtracted from the shear yield strength. Failure occurs when $Y < 0$.

$$Y = g(\mathbf{X}) = S_{sy} - \tau_{max} = \frac{S_{yt} S_{yc}}{S_{yt} + S_{yc}} - \frac{16T}{\pi d^3}$$

where **X**=(S_{yt}, S_{yc}, T).

Using FOSM, we have

$$\mu_Y = g(\boldsymbol{\mu}_{\mathbf{X}}) = \frac{\mu_{S_{yt}} \mu_{S_{yc}}}{\mu_{S_{yt}} + \mu_{S_{yc}}} - \frac{16}{\pi d^3} \mu_T$$

$$\sigma_Y = \sqrt{\left(\left.\frac{\partial g}{\partial S_{yt}}\right|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_{S_{yt}} \right)^2 + \left(\left.\frac{\partial g}{\partial S_{yc}}\right|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_{S_{yc}} \right)^2 + \left(\left.\frac{\partial g}{\partial T}\right|_{\boldsymbol{\mu}_{\mathbf{X}}} \sigma_T \right)^2}$$

$$= \sqrt{\begin{array}{c} \left(\frac{\mu_{S_{yc}}}{\mu_{S_{yt}} + \mu_{S_{yc}}} - \frac{\mu_{S_{yt}} \mu_{S_{yc}}}{\left(\mu_{S_{yt}} + \mu_{S_{yc}}\right)^2} \right)^2 \left(\sigma_{S_{yt}}\right)^2 \\ + \left(\frac{\mu_{S_{yt}}}{\mu_{S_{yt}} + \mu_{S_{yc}}} - \frac{\mu_{S_{yt}} \mu_{S_{yc}}}{\left(\mu_{S_{yt}} + \mu_{S_{yc}}\right)^2} \right)^2 \left(\sigma_{S_{yc}}\right)^2 + \left(-\frac{16}{\pi d^3} \sigma_T\right)^2 \end{array}}$$

The probability of failure is then given by

$$p_f = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\left(\dfrac{\mu_{S_{yt}}\mu_{S_{yc}}}{\mu_{S_{yt}}+\mu_{S_{yc}}} - \dfrac{16}{\pi d^3}\mu_T\right)}{\sqrt{\begin{array}{c}\left(\dfrac{\mu_{S_{yc}}}{\mu_{S_{yt}}+\mu_{S_{yc}}} - \dfrac{\mu_{S_{yt}}\mu_{S_{yc}}}{\left(\mu_{S_{yt}}+\mu_{S_{yc}}\right)^2}\right)^2 \left(\sigma_{S_{yt}}\right)^2 \\ + \left(\dfrac{\mu_{S_{yt}}}{\mu_{S_{yt}}+\mu_{S_{yc}}} - \dfrac{\mu_{S_{yt}}\mu_{S_{yc}}}{\left(\mu_{S_{yt}}+\mu_{S_{yc}}\right)^2}\right)^2 \left(\sigma_{S_{yc}}\right)^2 + \left(-\dfrac{16}{\pi d^3}\sigma_T\right)^2\end{array}}}\right) = 10^{-5}$$

Thus

$$\Phi^{-1}(10^{-5}) = \frac{-\left(\dfrac{\mu_{S_{yt}}\mu_{S_{yc}}}{\mu_{S_{yt}}+\mu_{S_{yc}}} - \dfrac{16}{\pi d^3}\mu_T\right)}{\sqrt{\begin{array}{c}\left(\dfrac{\mu_{S_{yc}}}{\mu_{S_{yt}}+\mu_{S_{yc}}} - \dfrac{\mu_{S_{yt}}\mu_{S_{yc}}}{\left(\mu_{S_{yt}}+\mu_{S_{yc}}\right)^2}\right)^2 \left(\sigma_{S_{yt}}\right)^2 \\ + \left(\dfrac{\mu_{S_{yt}}}{\mu_{S_{yt}}+\mu_{S_{yc}}} - \dfrac{\mu_{S_{yt}}\mu_{S_{yc}}}{\left(\mu_{S_{yt}}+\mu_{S_{yc}}\right)^2}\right)^2 \left(\sigma_{S_{yc}}\right)^2 + \left(-\dfrac{16}{\pi d^3}\sigma_T\right)^2\end{array}}}$$

$$= \frac{-\left(\dfrac{180(10^6)(160)(10^6)}{180(10^6)+160(10^6)} - \dfrac{16}{\pi d^3}(200)\right)}{\sqrt{\begin{array}{c}\left(\dfrac{160(10^6)}{180(10^6)+160(10^6)} - \dfrac{180(10^6)(160)(10^6)}{(180(10^6)+160(10^6))^2}\right)^2 (10(10^6))^2 \\ + \left(\dfrac{180(10^6)}{180(10^6)+160(10^6)} - \dfrac{180(10^6)(160)(10^6)}{(180(10^6)+160(10^6))^2}\right)^2 (10(10^6))^2 + \left(-\dfrac{16}{\pi d^3}20\right)^2\end{array}}}$$

Solving for d yields

$$d = 26.3 \text{ mm}$$

Thus $d = 28$ mm can be used.

Ans.