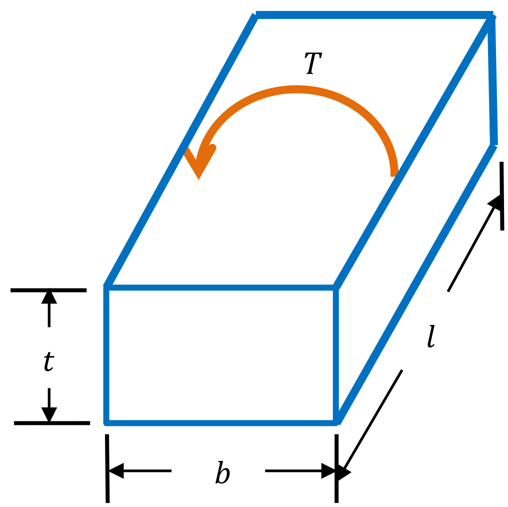
11. A rectangular shaft is designed to transimit a torque $T \sim N(2, 0.2^2)$ N·m shown in the figure. It has a rectangular cross section with a width of $b = 5$ mm and a thickness of $t = 2$ mm. The shear modulus is $G \sim N(80, 8^2)$ GPa. If the allowable angle of twist is $\theta_a = 8 \times 10^{-2}$ and the proability of failure is designed to be $p_f = 10^{-5}$, determined the maximum length of the shaft. Note that T and G are independent.



Solution

For a rectangular shaft in torsion, the maximum angle of twist is given by

$$\theta = \frac{Tl}{\beta b t^3 G}$$

where β is a function of b/t and it is 0.228 in this case.

Thus the limit-state function is the maximum angle of twist subtracted from allowable one. Failure occurs when $Y < 0$

$$Y = g(\mathbf{X}) = \theta_a - \theta = \theta_a - \frac{l}{\beta b t^3} \frac{T}{G}$$

where $\mathbf{X}$=(T, G).

Using FOSM, we have

$$\mu_Y = g(\boldsymbol{\mu}_{\mathbf{X}}) = \theta_a - \frac{l}{\beta b t^3} \frac{\mu_T}{\mu_G}$$

$$\sigma_Y = \sqrt{\left(\frac{\partial g}{\partial T}\bigg|_{\boldsymbol{\mu}_\mathbf{X}} \sigma_T\right)^2 + \left(\frac{\partial g}{\partial G}\bigg|_{\boldsymbol{\mu}_\mathbf{X}} \sigma_G\right)^2}$$

$$= \sqrt{\left(-\frac{l}{\beta b t^3 \mu_G}\sigma_T\right)^2 + \left(\frac{l\mu_T}{\beta b t^3 \mu_G^2}\sigma_G\right)^2}$$

The probability of failure is then given by

$$p_f = \Phi\left(-\frac{\mu_Y}{\sigma_Y}\right) = \Phi\left(-\frac{\theta_a - \frac{l}{\beta b t^3}\frac{\mu_T}{\mu_G}}{\sqrt{\left(-\frac{l}{\beta b t^3 \mu_G}\sigma_T\right)^2 + \left(\frac{l\mu_T}{\beta b t^3 \mu_G^2}\sigma_G\right)^2}}\right) = 10^{-5}$$

Thus

$$\Phi^{-1}(10^{-5}) = -\frac{\theta_a - \frac{l}{\beta b t^3}\frac{\mu_T}{\mu_G}}{\sqrt{\left(-\frac{l}{\beta b t^3 \mu_G}\sigma_T\right)^2 + \left(\frac{l\mu_T}{\beta b t^3 \mu_G^2}\sigma_G\right)^2}}$$

$$= -\frac{8(10^{-2}) - \frac{l}{0.228(5)(10^{-3})\left(2(10^{-3})\right)^3}\frac{2}{80(10^9)}}{\sqrt{\left(-\frac{l}{0.228(5)(10^{-3})\left(2(10^{-3})\right)^3 80(10^9)}0.2\right)^2 + \left(\frac{l(2)}{0.228(5)(10^{-3})\left(2(10^{-3})\right)^3\left(80(10^9)\right)^2}8(10^9)\right)^2}}$$

Solving for l yields

$$l = 18.20 \text{ mm}$$

Thus $l = 18$ mm can be used.

Ans.