12. A shaft with a round cross section is designed to tranismit a power at $n = 2000$ rpm. The diameter and the allowabe shear stress of the shaft are $d \sim N(50, 0.5^2)$ mm and $\tau_a \sim N(100, 10^2)$ MPa, respectively. If the maximum probability of failure is desinged to be $p_f = 10^{-5}$, estimate the maximum power that the shaft can transimit using the First Order Second Moment Method. Note that d and τ_a are independent.

Solution

The transmitted power is given by

$$H = \frac{Tn}{9.55}$$

Thus

$$T = \frac{9.55H}{n}$$

And the maximum shear stress is

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16\frac{9.55H}{n}}{\pi d^3} = \frac{152.8H}{\pi n d^3}$$

The limit-state function is the maximum shear stress subtracted from the allowable stress. Failure occurs when $Y < 0$.

$$Y = g(\mathbf{X}) = \tau_a - \tau_{max} = \tau_a - \frac{152.8H}{\pi n d^3}$$

where $\mathbf{X}=(\tau_a, d)$.

Using FOSM, we have

$$\mu_Y = g(\boldsymbol{\mu}_\mathbf{X}) = \mu_{\tau_a} - \frac{152.8H}{\pi n \mu_d^3}$$

$$\sigma_Y = \sqrt{\left(\left.\frac{\partial g}{\partial \tau_a}\right|_{\boldsymbol{\mu}_\mathbf{X}} \sigma_{\tau_a}\right)^2 + \left(\left.\frac{\partial g}{\partial d}\right|_{\boldsymbol{\mu}_\mathbf{X}} \sigma_d\right)^2}$$

$$= \sqrt{\left(\sigma_{\tau_a}\right)^2 + \left(\frac{152.8H}{3\pi n \mu_d^4}\sigma_d\right)^2}$$

The probability of failure is then given by

$$p_f = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\left(\mu_{\tau_a} - \frac{152.8H}{\pi n \mu_d^3}\right)}{\sqrt{\left(\sigma_{\tau_a}\right)^2 + \left(\frac{152.8H}{3\pi n \mu_d^3}\sigma_d\right)^2}}\right) = 10^{-5}$$

Thus

$$\frac{-\mu_Y}{\sigma_Y} = \frac{-\left(100(10^6) - \frac{152.8H}{\pi(2000)\left(50(10^{-3})\right)^3}\right)}{\sqrt{(10(10^6))^2 + \left(\frac{152.8H}{(2000)\left(50(10^{-3})\right)^4}0.5(10^{-3})\right)^2}} = \Phi^{-1}(10^{-5})$$

Solving for H yields

$$H = 291.63\ \text{kW}$$

Ans.