7. A round shaft is subjected to an axial force $F \sim N(10, 1^2)$ kN. The length of the shaft is $l \sim N(500, 0.5^2)$ mm and the modulus of elasticity is $E = 200$ GPa. If the allowable axial extension is $\delta = 0.01$ mm and the maximum probability of failure is designed to be $p_f = 10^{-5}$, determine the minimum diameter of the shaft using the First Order Second Moment Method and then select a preferred one. Assume that F and l are independent.

Solution

The total extension of the shaft is

$$\delta = \frac{Fl}{AE} = \frac{Fl}{\frac{\pi}{4}d^2E} = \frac{4Fl}{\pi d^2 E}$$

Thus the limit-state function is the actual extension subtracted from the allowable one. Failure occurs when $Y < 0$.

$$Y = g(\mathbf{X}) = \delta_a - \delta = 0.01(10^{-3}) - \frac{4}{\pi d^2 E}Fl$$

where $\mathbf{X}$=(F, l).

Using FOSM, we have

$$\mu_Y = g(\boldsymbol{\mu}_{\mathbf{X}}) = 0.01(10^{-3}) - \frac{4}{\pi d^2 E}\mu_F\mu_l$$

$$= 0.01(10^{-3}) - \frac{4}{\pi d^2 E}\mu_F\mu_l$$

$$\sigma_Y = \sqrt{\left(\left.\frac{\partial g}{\partial F}\right|_{\boldsymbol{\mu}_{\mathbf{X}}}\sigma_F\right)^2 + \left(\left.\frac{\partial g}{\partial l}\right|_{\boldsymbol{\mu}_{\mathbf{X}}}\sigma_l\right)^2}$$

$$= \sqrt{\left(-\frac{4}{\pi d^2 E}\mu_l\sigma_F\right)^2 + \left(-\frac{4}{\pi d^2 E}\mu_F\sigma_l\right)^2}$$

The probability of failure is then given by

$$p_f = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\left(0.01(10^{-3}) - \frac{4}{\pi d^2 E}\mu_F\mu_l\right)}{\sqrt{\left(-\frac{4}{\pi d^2 E}\mu_l\sigma_F\right)^2 + \left(-\frac{4}{\pi d^2 E}\mu_F\sigma_l\right)^2}}\right) = 10^{-5}$$

Thus

$$
\begin{aligned}
\frac{-\mu_Y}{\sigma_Y} &= \frac{-\left(0.01(10^{-3}) - \frac{4}{\pi d^2 E}\mu_F \mu_l\right)}{\sqrt{\left(-\frac{4}{\pi d^2 E}\mu_l \sigma_F\right)^2 + \left(-\frac{4}{\pi d^2 E}\mu_F \sigma_l\right)^2}} \\
&= \frac{-\left(0.01(10^{-3}) - \frac{6.3662(10^{-12})}{d^2}(10)(10^3)(500)(10^{-3})\right)}{\sqrt{\left(-\frac{6.3662(10^{-12})}{d^2}(500)(10^{-3})(1)(10^3)\right)^2 + \left(-\frac{6.3662(10^{-12})}{d^2}(10)(10^3)(0.5)(10^{-3})\right)^2}} \\
&= \Phi^{-1}(10^{-5})
\end{aligned}
$$

Solving for d yields

$$d = 76.80 \text{ mm}$$

Then $d = 80$ mm can be used.

Ans.