2. A steel bar is used to be as a torsion spring. The diameter and length of the bar are $d \sim N(0.75, 0.01^2)$ in and $l = 24$ in, respectively. The shear modulus is $G = 11.5(10^6)$ kpsi and the allowable torsional stress is $\tau_a \sim N(30, 3^2)$ kpsi. If the maximum probability of failure is desinged to be $p_f = 10^{-5}$, determine the maximum angle of twist of the bar. Note that d and τ_a are independent.

Solution

The angle of twist is given by

$$\theta = \frac{Tl}{JG} = \frac{32Tl}{\pi d^4 G}$$

Thus

$$T = \frac{\pi d^4 G\theta}{32l}$$

The maximum shear stress is

$$\tau_{max} = \frac{Tr}{J} = \frac{16T}{\pi d^3} = \frac{16}{\pi d^3}\frac{\pi d^4 G\theta}{32l} = \frac{d\theta G}{2l}$$

Thus the limit-state function is the maximum stress subtracted from allowable shear stress. Failure occurs when $Y < 0$

$$Y = g(\mathbf{X}) = \tau_a - \tau_{max} = \tau_a - \frac{\theta G}{2l} d$$

where $\mathbf{X}=(\tau_a, d)$.

Using FOSM, we have

$$\mu_Y = g(\boldsymbol{\mu}_\mathbf{X}) = \mu_{\tau_a} - \frac{\theta G}{2l}\mu_d$$

$$\sigma_Y = \sqrt{\left(\left.\frac{\partial g}{\partial \tau_a}\right|_{\boldsymbol{\mu}_\mathbf{X}} \sigma_{\tau_a}\right)^2 + \left(\left.\frac{\partial g}{\partial d}\right|_{\boldsymbol{\mu}_\mathbf{X}} \sigma_d\right)^2}$$

$$= \sqrt{\left(\sigma_{\tau_a}\right)^2 + \left(\frac{\theta G}{2l}\sigma_d\right)^2}$$

The probability of failure is then given by

$$p_f = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-\left(\mu_{\tau_a} - \frac{\theta G}{2l}\mu_d\right)}{\sqrt{\left(\sigma_{\tau_a}\right)^2 + \left(\frac{\theta G}{2l}\sigma_d\right)^2}}\right) = 10^{-5}$$

Thus

$$\frac{-\mu_Y}{\sigma_Y} = \frac{-\left(30(10^3) - \frac{\theta 11.5(10^6)}{2(24)}0.75\right)}{\sqrt{(3(10^3))^2 + \left(\frac{\theta 11.5(10^6)}{2(24)}0.01\right)^2}} = \Phi^{-1}(10^{-5})$$

Solving for θ yields

$$\theta = 0.0955 \text{ rad}$$

Ans.