Machine Design Under Uncertainty

MISSOURI S&T

Founded 1870 | Rolla, Missouri

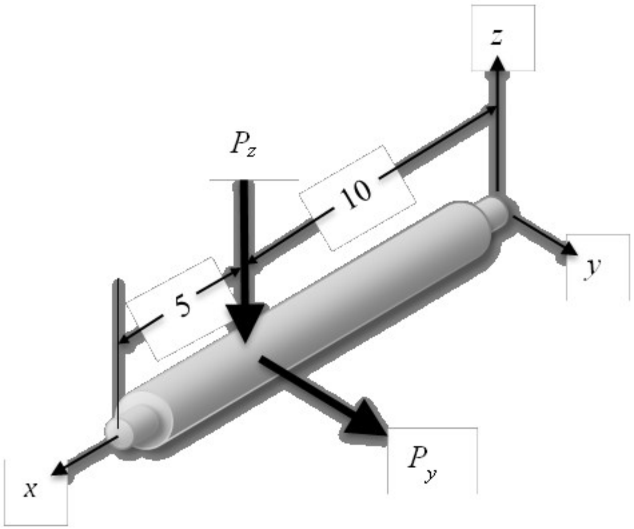
MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY

Outline

- Uncertainty in mechanical components
- Why consider uncertainty
- Basics of uncertainty
- Uncertainty analysis for machine design
- Examples
- Conclusions

Uncertainty in Mechanical Components

- A simply-supported beam has a diameter of 1.25 in. The deflection at $x = 10$ in should be less than $\delta =$0.00375 in. Can the requirement be met?
- Everything is modeled perfectly.
- In reality, the forces and dime random.
- So is the deflection.



Where Does Uncertainty Come From?

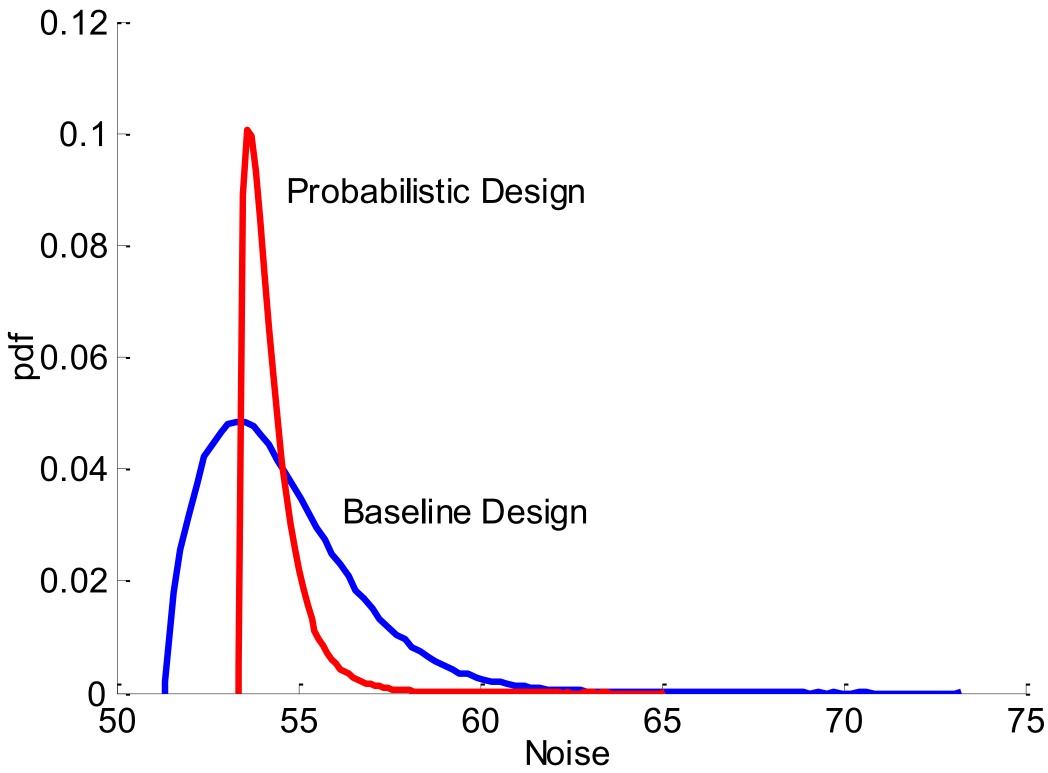
- Manufacturing impression
  - Dimensions of a component
  - Material properties
- Environment
  - Loading
  - Temperature
  - Different users

Why Consider Uncertainty?

- We know the true solution.
- We know the effect of uncertainty.
- We can make more reliable decisions.

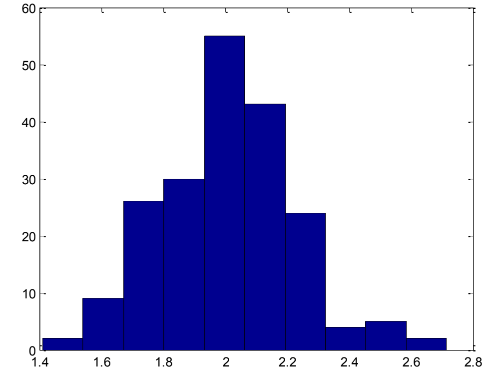
How Do We Model Uncertainty?

- We use probability distributions to model parameters with uncertainty.



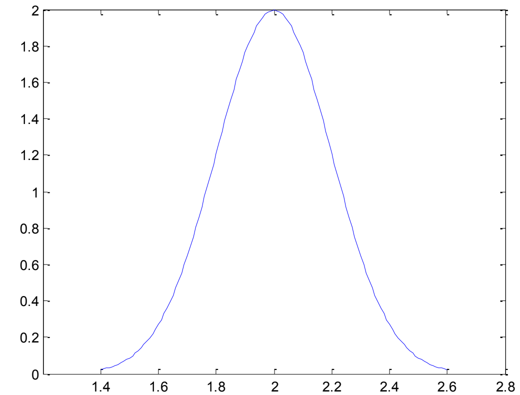
Probability Distribution

- With more samples, we can draw a histogram.



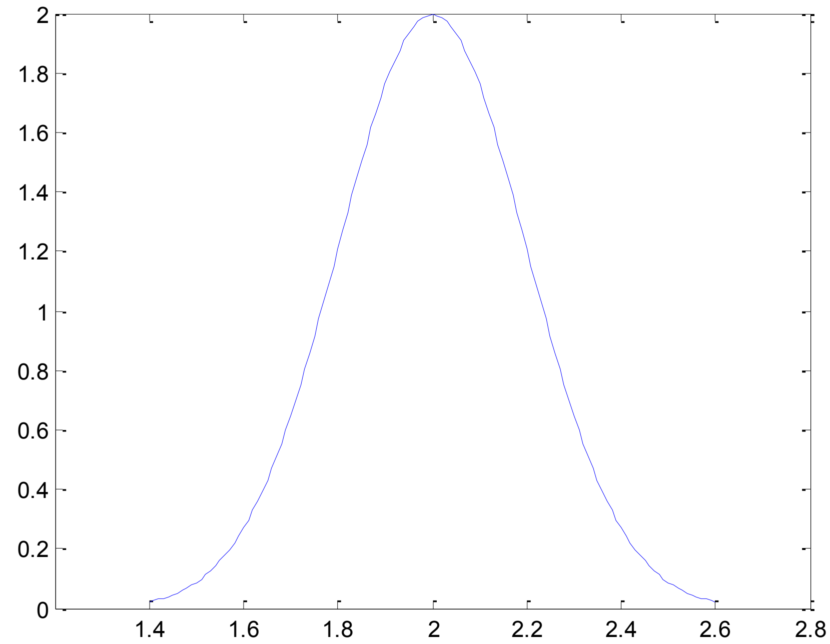
- If y-axis is frequency and the number of samples is infinity, we get a probability density function (PDF) $f(x)$.
- The probability of $a \leq X \leq b$.

$$\Pr\{a \leq X \leq b\} = \int_a^b f(x)dx$$



Normal Distribution

- $X \sim N(\mu, \sigma^2)$
- $F(x) = \Pr\{X < x\}$: cumulative distribution function (CDF)
- $\Pr\{a < X < b\} = F(b) - F(a)$
- $\Pr\{X < x\} = \Phi\left(\frac{x - \mu_Y}{\sigma_Y}\right)$
- $\Pr\{X > x\} = 1 - \Pr\{X <$



More about Standard Deviation σ (std)

- It indicates how data spread around the mean.
- It is always non-negative.
- High std means
  - High dispersion
  - High uncertainty
  - High risk

More Than One Random Variables

- If
  - $X_i \sim N(\mu_i, \sigma_i^2)$
  - X_i $(i = 1,2,\cdots,n)$ are independent
  - $Y = c_0 + c_1X_1 + c_2X_2 + \cdots + c_nX_n$
  - c_i $(i = 1,2,\cdots,n)$ are constants.
- Then
  - $Y \sim N(\mu_Y, \sigma_Y^2)$
  - $\mu_Y = c_0 + c_1\mu_1 + c_2\mu_2 + \cdots + c_n\mu_n$
  - $\sigma_Y = \sqrt{c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_n^2\sigma_n^2}$

Reliability

- Reliability is the ability of a component to perform its intended function without failure.
- Reliability is measure by the probability of such ability.
- $R = \Pr\{g(\mathbf{X}) > 0\}$
  - $\mathbf{X}$: random variables
  - $g(\cdot)$: limit-state function
  - If $g(\mathbf{X})$<0, a failure occurs
- Probability of failure $p_f = \Pr\{g(\mathbf{X}) < 0\} = 1 - R$

First Order Second Moment Method (FOSM)

- Assume $X_i \sim N\left(\mu_i, \sigma_i^2\right)$ and are independent
- First order Taylor expansion

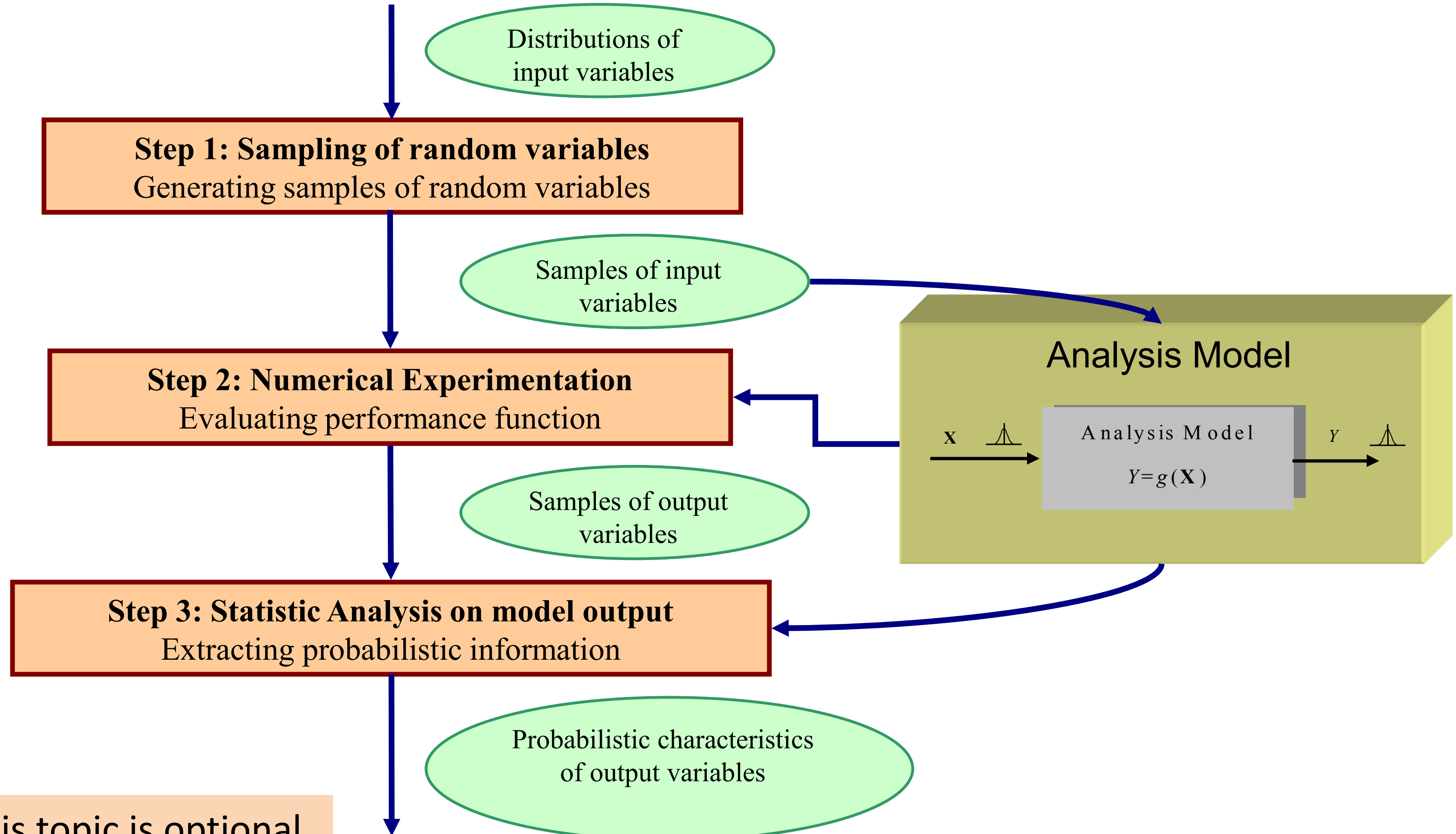
$$Y = g(\mathbf{X}) \approx g(\boldsymbol{\mu}) + \sum_{i=1}^{n} \frac{\partial g}{\partial X_i}(X_i - \mu_i)$$

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_n)$$

- $Y \sim N(\mu_Y, \sigma_Y^2)$, $\mu_Y = g(\boldsymbol{\mu})$, $\sigma_Y^2 = \sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i}\right)^2 \sigma_i^2$
- $p_f = \Pr\{g(\mathbf{X}) < 0\} = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right)$

Monte Carlo Simulation (MCS)*

A sampling-based simulation method



*This topic is optional.

Step 1: Sampling on random variables

- Generate samples of input random variables according to their distributions.
- For example, for $X_i \sim N(\mu_i, \sigma_i^2)$, samples can be generated by Matlab.
  - Matlab normrnd($\mu_i, \sigma_i, 1, N$) produces a row vector of N random samples.
  - Excel can also be used.

Step 2: Obtain Samples of Output

- Suppose *N* sets of random variables have been generated

  $\mathbf{x}_i = (x_{i1}, x_{i2}, \cdots, x_{in}), i = 1,2, \cdots, N$

  N is the number of simulations

- Then samples of output s are calculated a

  $y_i = g(\mathbf{x}_i)$

Step 3: Statistic Analysis on output

- Mean $\mu_Y = \frac{1}{N}\sum_{i=1}^{N} y_i$
- Standard deviation $\sigma_Y = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(y_i - \mu_i)^2}$
- The probability of failure $p_f = \frac{N_f}{N}$

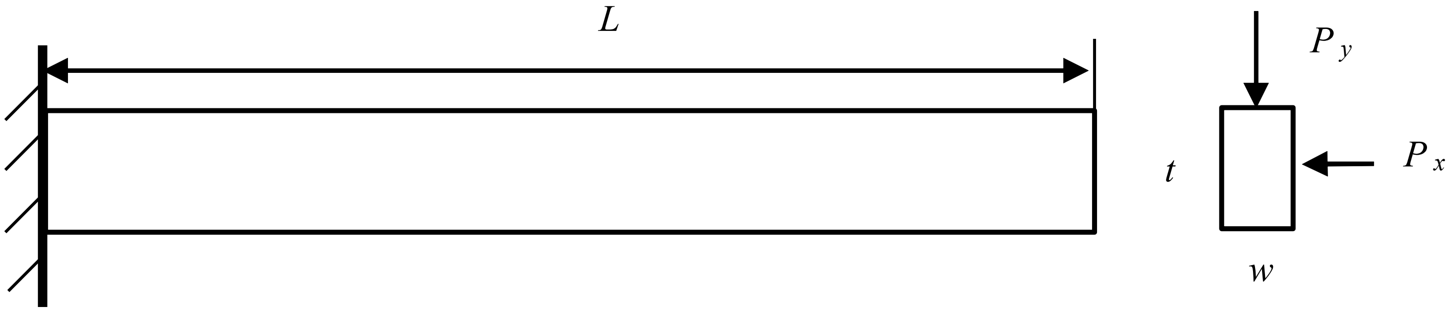
  N_f is the number of failures.

  N_f = number of y_i<0, $i = 1,2,\cdots,N$

FORM vs MCS

- FORM is more efficient
- FORM may not be accurate when a limit-state function is highly nonlinear
- MCS is very accurate if the sample size is sufficiently large
- MCS is not efficient

Example - FOSM



- $L = 100$ in, $t = 2$ in, $w = 4$ in, $E = 30\times10^6$ psi
- $\mathbf{X} = (P_x, P_y)$, $P_x \sim N(500, 60^2)$ lb, $P_y \sim N(1000, 100^2)$ lb, P_x and P_y are independent
- Allowable deflection $D_0 = 3$ in

- $Y = g(\mathbf{X}) = D_0 - \frac{4L^3}{Ewt}\sqrt{\left(\frac{P_x}{w^2}\right)^2 + \left(\frac{P_y}{t^2}\right)^2}$
- $p_f = \Pr\{Y = g(\mathbf{X}) < 0\}$

Example - FOSM

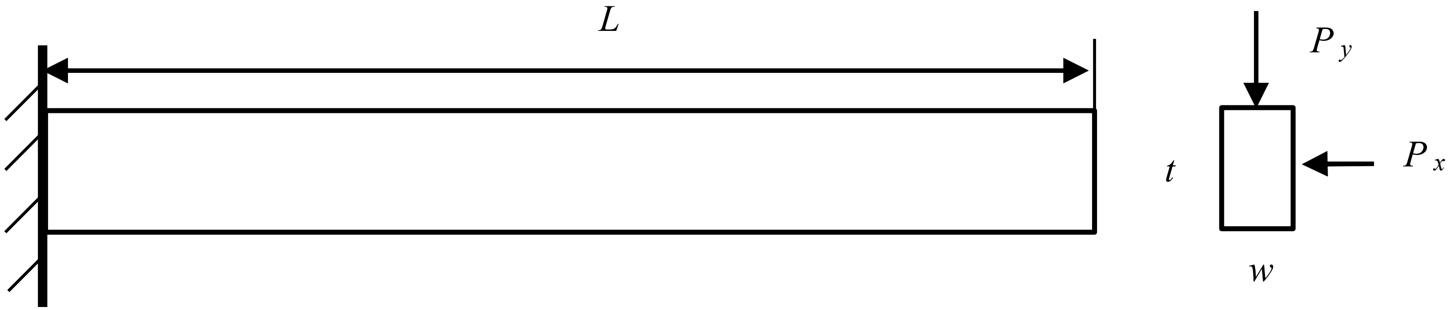
- $\mu_Y = g(\boldsymbol{\mu}) = D_0 - \frac{4L^3}{Ewt}\sqrt{\left(\frac{P_x}{w^2}\right)^2 + \left(\frac{P_y}{t^2}\right)^2} = 3 - \frac{4(100^3)}{30\times10^6(4)(2)}\sqrt{\left(\frac{500}{4^2}\right)^2 + \left(\frac{1000}{2^2}\right)^2} = 0.6708$
- $\frac{\partial g}{\partial P_x} = -\frac{4L^3}{Ewt}\frac{P_x}{w^4}\frac{1}{A} = -3.7268e-03$
- $\frac{\partial g}{\partial P_y} = -\frac{4L^3}{Ewt}\frac{P_y}{t^4}\frac{1}{A} = -4.6585e-04$

where $A = \sqrt{\left(\frac{P_x}{w^2}\right)^2 + \left(\frac{P_y}{t^2}\right)^2}$

Example - FOSM

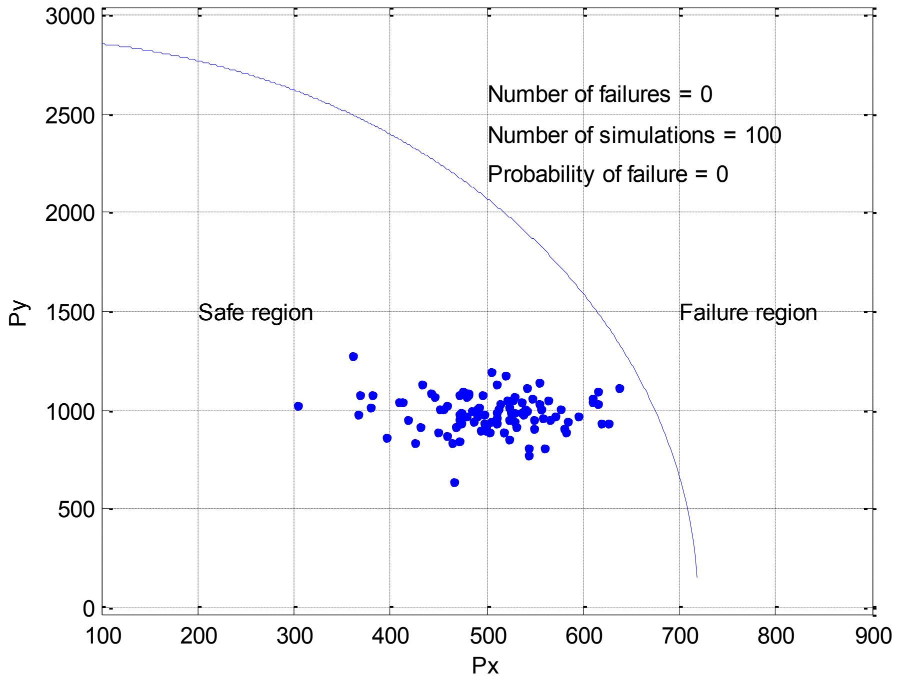
- $\sigma_Y = \sqrt{\left(\frac{\partial g}{\partial P_x}\right)^2 \sigma_{P_x}^2 + \left(\frac{\partial g}{\partial P_y}\right)^2 \sigma_{P_y}^2} = 0.2284$
- $p_f = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi\left(\frac{-0.6708}{0.2284}\right) =$ $\Phi(-2.9367) = 1.70 \times 10^{-3}$

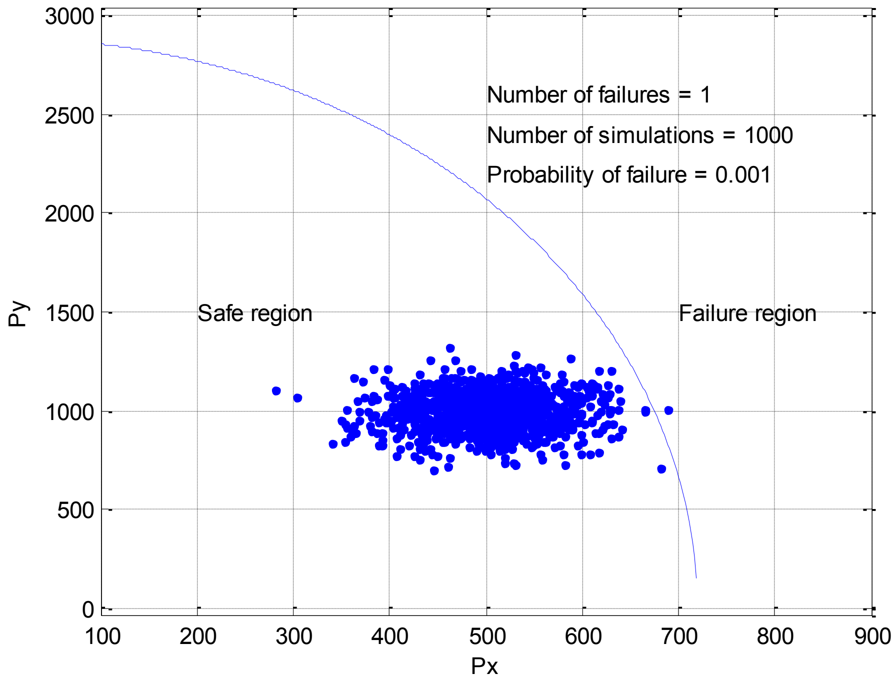
Example - MCS



- $L = 100$ in, $t = 2$ in, $w = 4$ in, $E = 30\times10^6$ psi
- $\mathbf{X} = (P_x, P_y)$, $P_x \sim N\left(500, 100^2\right)$ lb, $P_y \sim N\left(1000, 100^2\right)$ lb, P_x and P_y are independent
- Allowable deflection $D_0 = 3$ in
- $g(\mathbf{X}) = D_0 - \frac{4L^3}{Ewt}\sqrt{\left(\frac{P_x}{w^2}\right)^2 + \left(\frac{P_y}{t^2}\right)^2}$
- $p_f = \Pr\{g(\mathbf{X}) < 0\}$

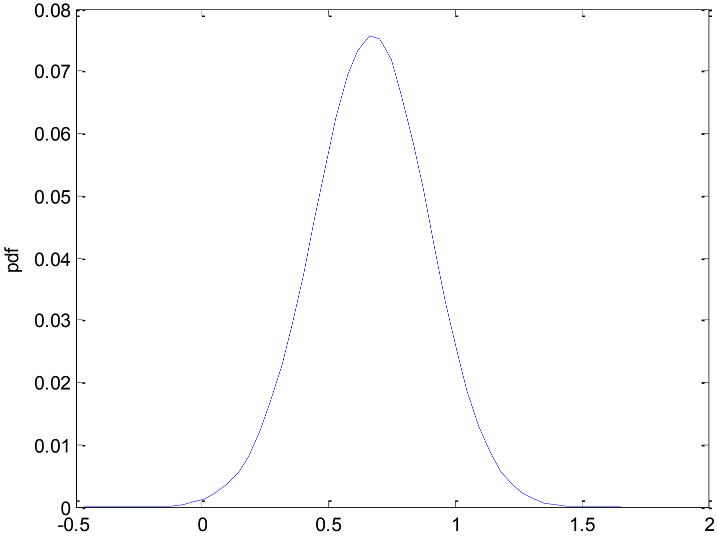
100 and 1000 Simulations

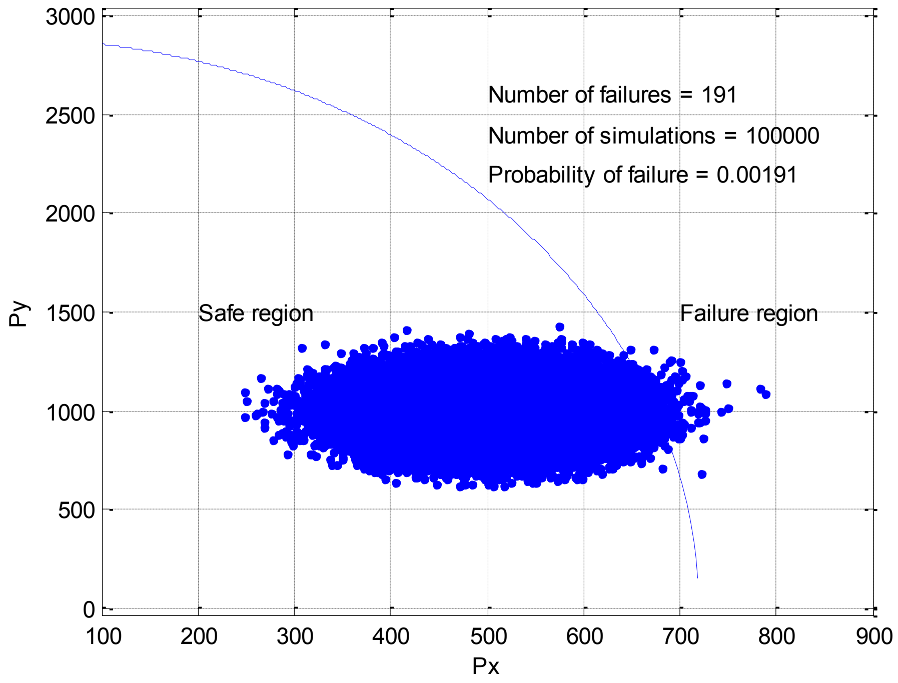




1e5 Simulations

- More simulations, More accurate result

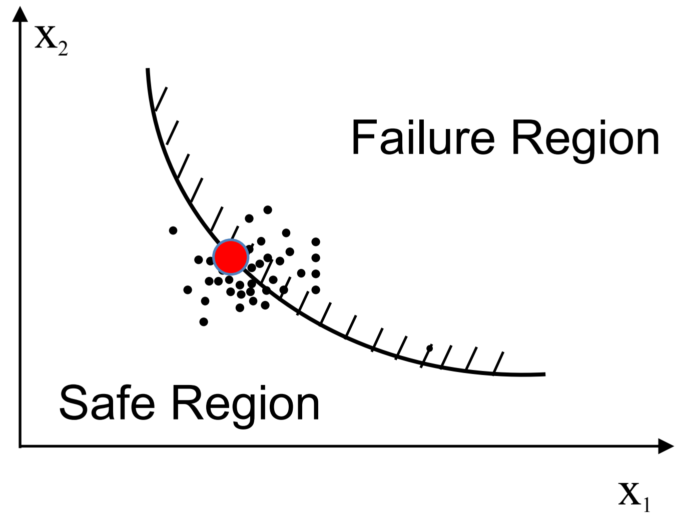




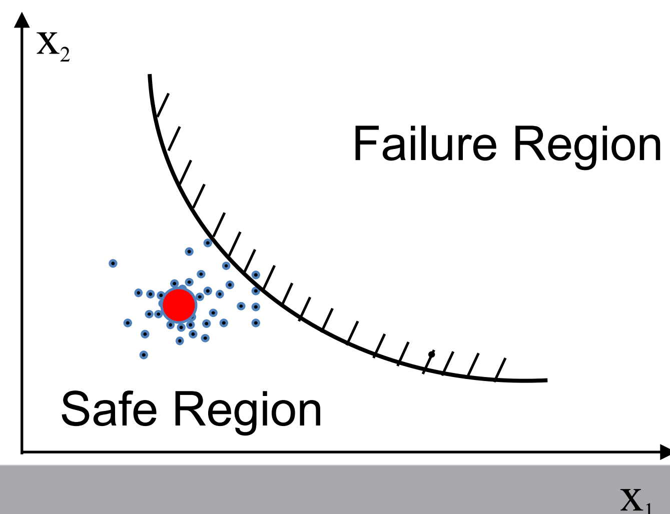
Reliability –Based Design (RBD)

Design without considering uncertainty: Low reliability

- Nominal design point
- Actual design points



Design with considering uncertainty: high reliability



RBD

- RBD ensures that a design has the probability of failure less than an acceptable level, and
- therefore ensures that events that lead to catastrophe are extremely unlikely.
- RBD is achieved by maximizing cost and maintaining reliability at a required level.

Conclusions

- For important mechanical components in important applications,
- a factor of safety may not be sufficient to account for uncertainties;
- it is imperative to consider reliability.
- Uncertainty can be modeled probabilistically.
- Reliability can be estimated by FOSM and MCS.