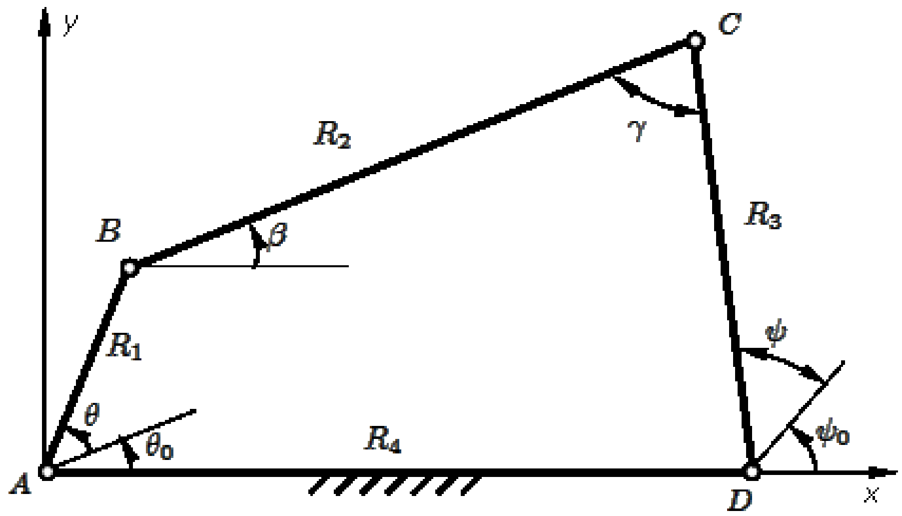
Probabilistic Mechanism Analysis

Outline

- Uncertainty in mechanisms
- Why consider uncertainty
- Basics of uncertainty
- Probabilistic mechanism analysis
- Examples
- Probabilistic mechanism synthesis
- Conclusions

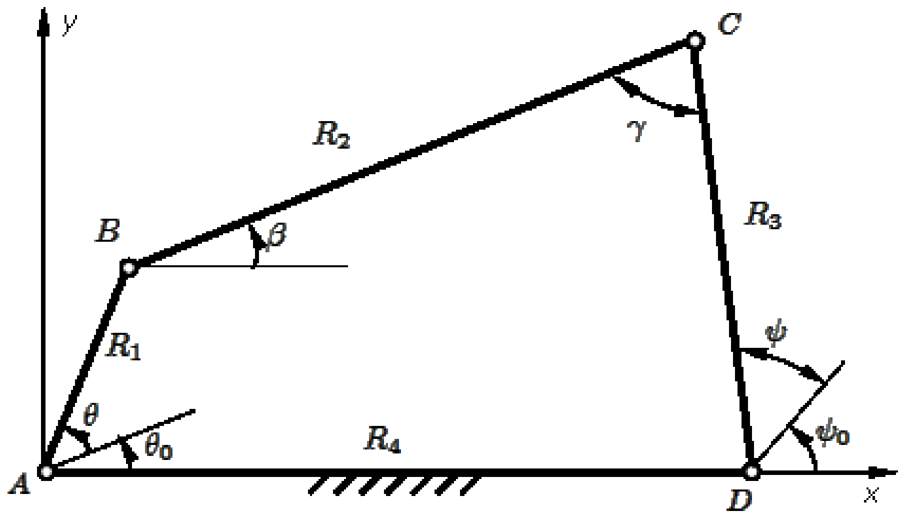
Uncertainty in Mechanisms

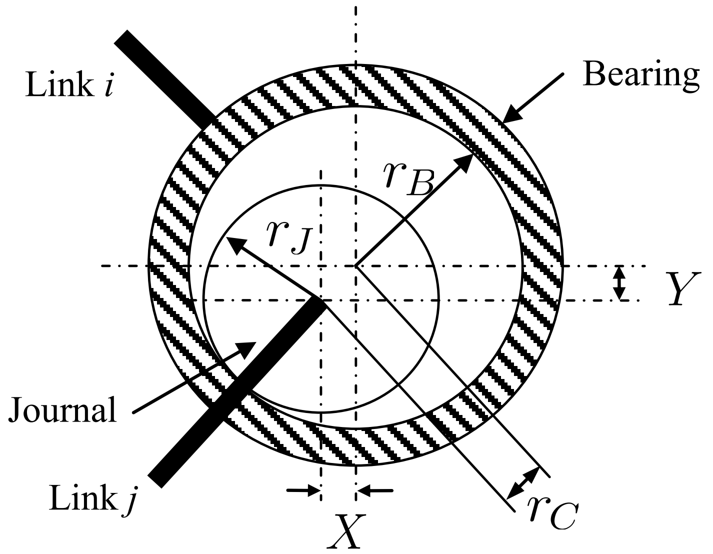
- Lengths R_1, R_2, R_3, and R_4 are random variables due to manufacture imprecision



Uncertainty in Mechanisms

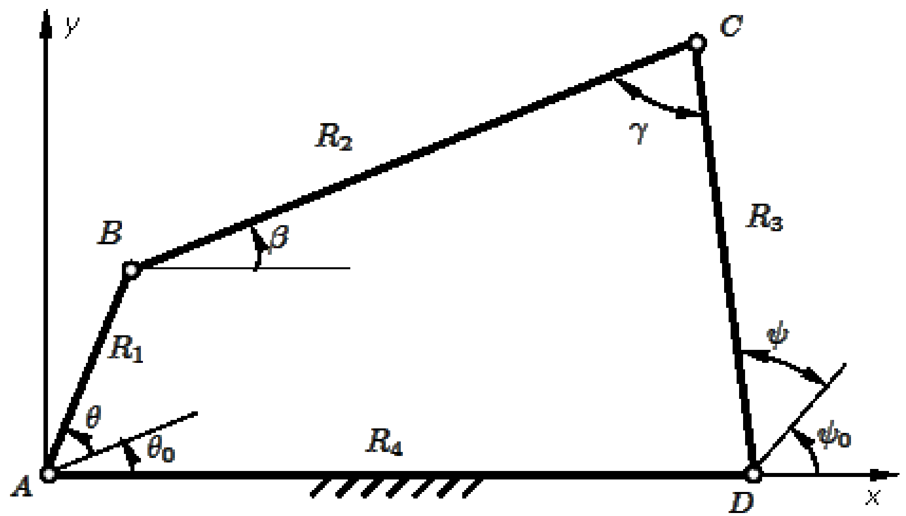
- The joint clearances at *A*, *B*, *C*, and *D* are also random due to manufacture imprecision and installation errors.





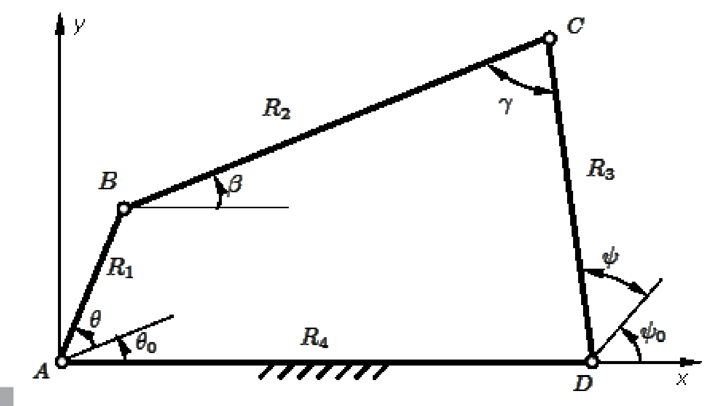
Uncertainty in Mechanisms

- Loads and material properties are also random.



Impact of Uncertainty

- If the above mechanism generates a functional relationship $\psi = f(\mathbf{R})$ and the required motion output is ψ_r, then the motion error is $\varepsilon = f(\mathbf{R}) - \psi_r$, where $\mathbf{R} = (R_1, R_2, R_3, R_4)$
- ε is also a random variable.
- Required motion output may not be realized with certainty.



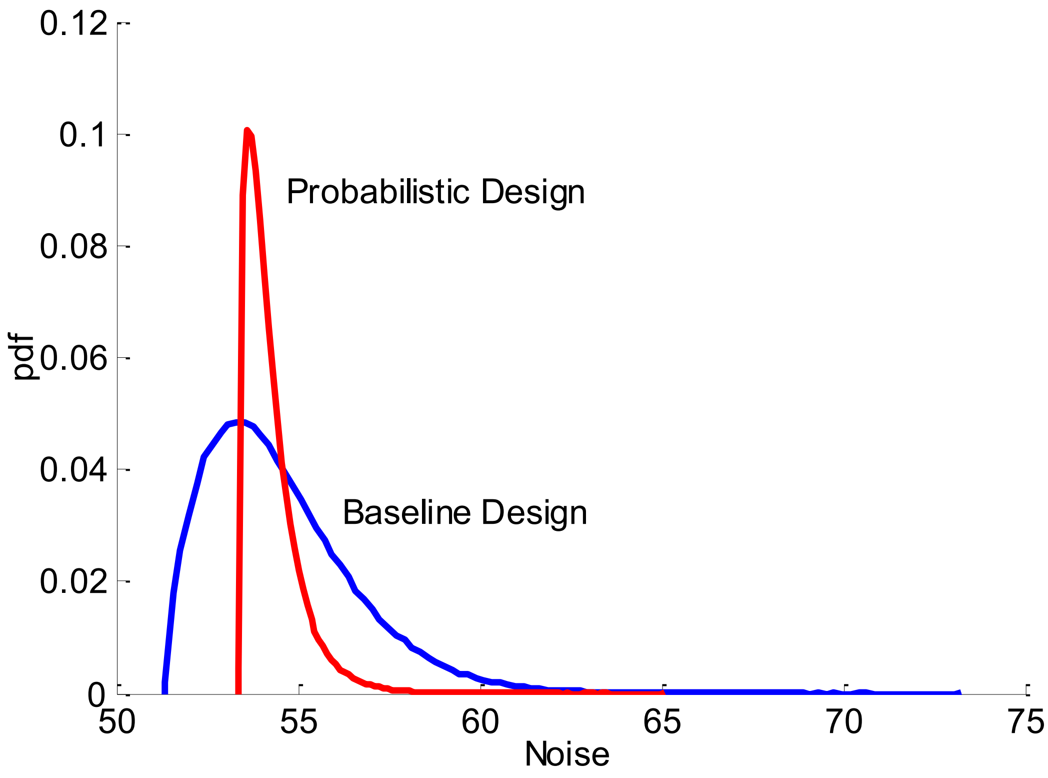
Why Consider Uncertainty?

- We know the true solution.
- We know the effect of uncertainty.
- We can design mechanisms whose performance is not sensitive to uncertainty
- We can make more reliable decisions.

In this presentation, we focus on uncertainty only in **R**.

How Do We Model Uncertainty?

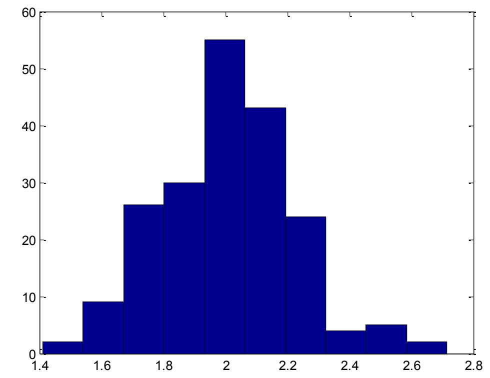
- We use probability distributions to model parameters with uncertainty.

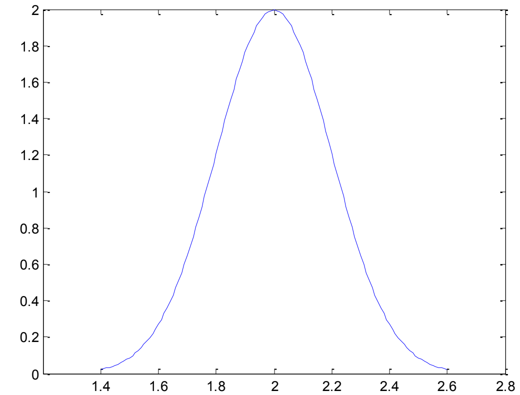


Probability Distribution

- With random samples, we can draw a histogram.
- If y-axis is frequency and the number of samples is infinity, we get a probability density function (PDF) $f(x)$ for random variable X.
- The probability of $a \leq X \leq b$ is

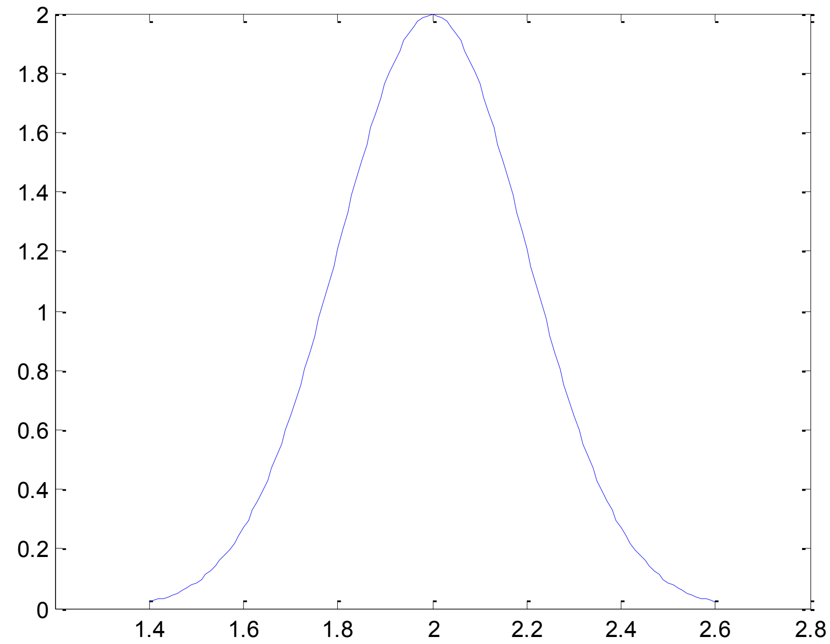
$$\Pr\{a \leq X \leq b\} = \int_a^b f(x)dx$$





Normal Distribution

- $X \sim N(\mu, \sigma^2)$
- $F(x) = \Pr\{X < x\}$: cumulative distribution function (CDF)
- $\Pr\{a < X < b\} = F(b) - F(a)$
- $\Pr\{X < x\} = \Phi\left(\frac{x-\mu_Y}{\sigma_Y}\right)$
- $\Pr\{X > x\} = 1 - \Pr\{X <$



More about Standard Deviation σ (std)

- It indicates how data spread around the mean.
- It is always non-negative.
- High std means
  - High dispersion
  - High uncertainty
  - High risk

More Than One Random Variables

- If all lengths are normally distributed
  - $R_i \sim N(\mu_i, \sigma_i^2)$
  - R_i $(i = 1,2,\cdots,n)$ are independent
  - $Y = c_0 + c_1X_1 + c_2X_2 + \cdots + c_nX_n$
  - c_i $(i = 1,2,\cdots,n)$ are constants.
- Then
  - $Y \sim N(\mu_Y, \sigma_Y^2)$
  - $\mu_Y = c_0 + c_1\mu_1 + c_2\mu_2 + \cdots + c_n\mu_n$
  - $\sigma_Y = \sqrt{c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_n^2\sigma_n^2}$

Mechanism Reliability

- Let the required motion error be δ.
- Mechanism reliability is $R = \Pr\{|f(\mathbf{R}) - \psi_r| < \delta\}$
- Probability of failure $p_f = 1 - R = \Pr\{|f(\mathbf{R}) -$

First Order Second Moment Method (FOSM)

- Assume lengths $R_i \sim N\left(\mu_i, \sigma_i^2\right)$ $(i = 1,2,\cdots,n)$ are independent
- First order Taylor expansion for motion output $\psi = f(\mathbf{R})$

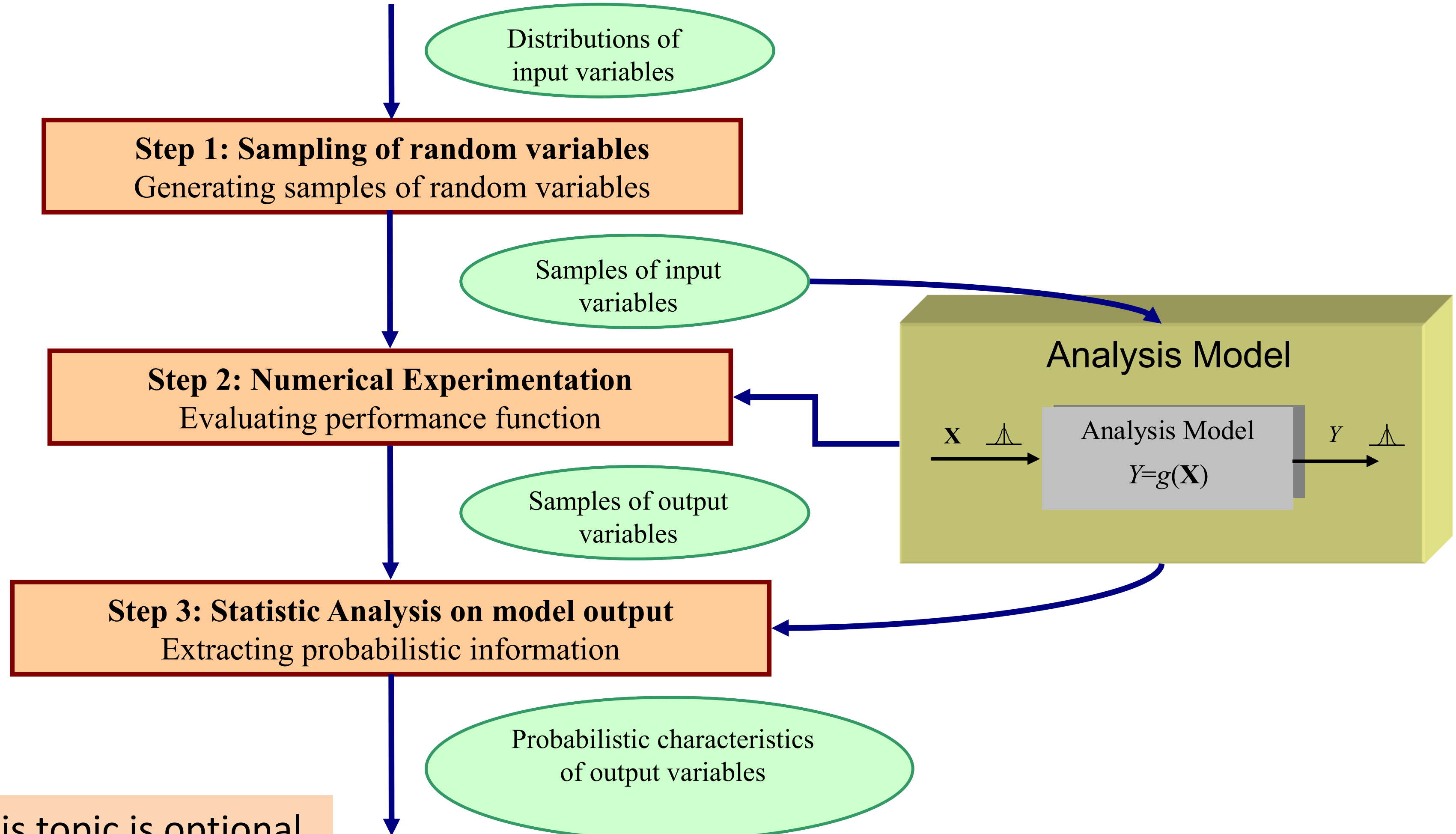
$$\psi = f(\mathbf{R}) \approx f(\boldsymbol{\mu}) + \sum_{i=1}^{n} \frac{\partial f}{\partial R_i}(R_i - \mu_i)$$

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_n)$$

- Motion error $\varepsilon = \psi - \delta$
- $\varepsilon \sim N\left(\mu_\psi - \delta, \sigma_\psi^2\right)$, $\mu_\psi = f(\boldsymbol{\mu})$, $\sigma_\psi^2 = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial R_i}\right)^2 \sigma_i^2$
- $p_f = \Pr\{\psi - \psi_r > \delta\} + \Pr\{\psi - \psi_r < -\delta\}$
- $p_f = 1 - \Pr\{\psi - \psi_r < \delta\} + \Pr\{\psi - \psi_r < -\delta\}$
- $p_f = 1 - \Phi\left(\frac{\delta - (\mu_\psi - \psi_r)}{\sigma_\psi}\right) + \Phi\left(\frac{-\delta - (\mu_\psi - \psi_r)}{\sigma_\psi}\right)$

Monte Carlo Simulation (MCS)*

A sampling-based simulation method



*This topic is optional.

Step 1: Sampling on random variables

- Generate samples of input random variables according to their distributions.
- For example, for $X_i \sim N\left(\mu_i, \sigma_i^2\right)$, samples can be generated by Matlab.
  - Matlab normrnd($\mu_i, \sigma_i, 1, N$) produces a row vector of N random samples.
  - Excel can also be used.

Step 2: Obtain Samples of Output

- Suppose *N* sets of random variables have been generated

  $\mathbf{x}_i = (x_{i1}, x_{i2}, \cdots, x_{in}), i = 1,2, \cdots, N$

  N is the number of simulations

- Then samples of output s are calculated a

  $y_i = g(\mathbf{x}_i)$

Step 3: Statistic Analysis on output

- Mean $\mu_Y = \frac{1}{N}\sum_{i=1}^{N} y_i$
- Standard deviation $\sigma_Y = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(y_i - \mu_i)^2}$
- The probability of failure $p_f = \frac{N_f}{N}$

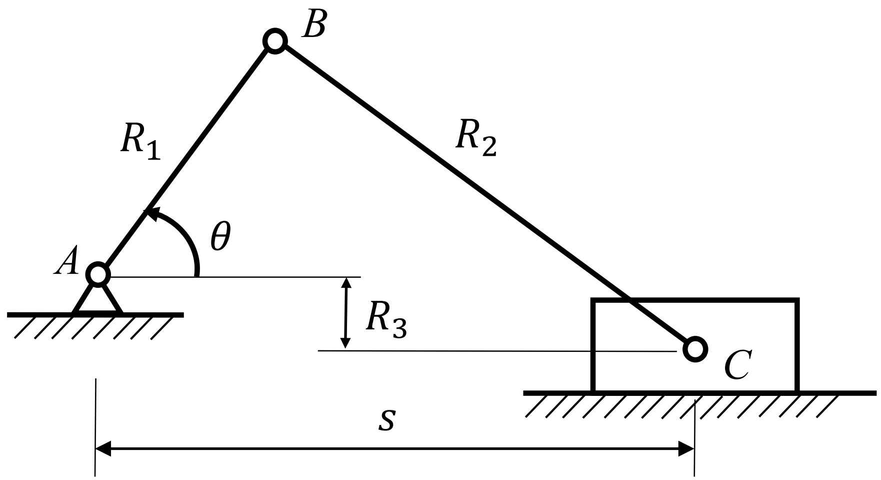
  N_f is the number of failures.

  N_f = number of y_i<0, $i = 1,2,\cdots,N$

FORM vs MCS

- FORM is more efficient
- FORM may not be accurate when a limit-state function is highly nonlinear
- MCS is very accurate if the sample size is sufficiently large
- MCS is not efficient

Example - FOSM



- $R_1 \sim N(120, 0.1^2)$ mm, $R_2 \sim N(200, 0.1^2)$ mm, $R_3 \sim N(20, 0.1^2)$ mm; they are independent.
- Requirement: $s_r = 287$ mm when $\theta = 30°$ (The motion output here is displacement s instead of an angle ψ.)
- Allowable motion error: $\delta = 0.7$ mm
- $s = f(\mathbf{R}) = R_1 \cos\theta + \sqrt{R_2^2 - (R_3 + R_1 \sin\theta)^2}$

Example - FOSM

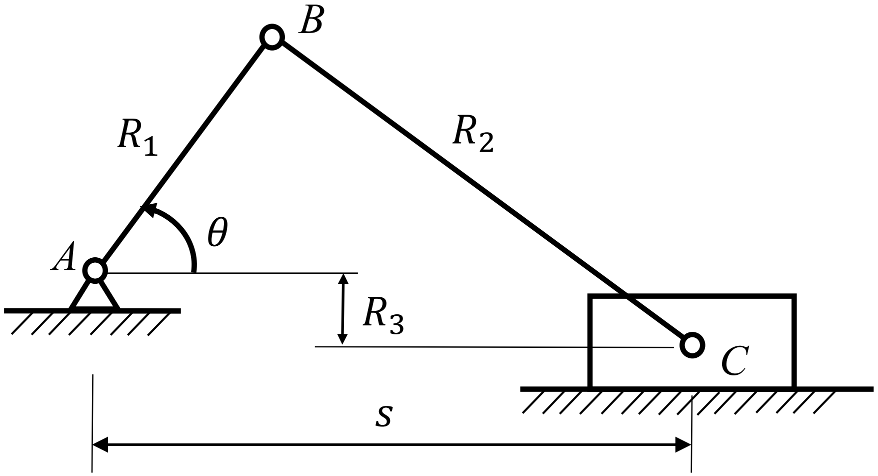
- $\mu_s = f(\boldsymbol{\mu}) = R_1\cos\theta + \sqrt{R_2^2 - (R_3 + R_1\sin\theta)^2} = 120\cos30°\sqrt{200^2 - (20 + 120\sin30°)^2} = 287.2261$ mm
- $\frac{\partial f}{\partial R_1} = \cos\theta - \frac{(R_3+R_1\sin\theta)\sin\theta}{A} = 0.6478$
- $\frac{\partial f}{\partial R_2} = \frac{R_2}{A} = 1.0911$
- $\frac{\partial f}{\partial R_3} = -\frac{R_3+R_1\sin\theta}{A} = -0.4364$

where $A = \sqrt{R_2^2 - (R_3 + R_1\sin\theta)^2}$

Example - FOSM

- $\sigma_s = \sqrt{\left(\frac{\partial s}{\partial R_1}\right)^2 \sigma_{R_1}^2 + \left(\frac{\partial s}{\partial R_2}\right)^2 \sigma_{R_2}^2 + \left(\frac{\partial s}{\partial R_3}\right)^2 \sigma_{R_3}^2} = 0.1342 \text{ mm}$
- $p_f = 1 - \Phi\left(\frac{\delta-(\mu_s - s_r)}{\sigma_s}\right) + \Phi\left(\frac{-\delta-(\mu_s - s_r)}{\sigma_s}\right) = 1 - \Phi\left(\frac{0.7-(287.2261-287)}{0.1342}\right) + \Phi\left(\frac{-0.7-(287.2261-287)}{0.1342}\right) = 2.0635 \times 10^{-4}$

Example - MCS



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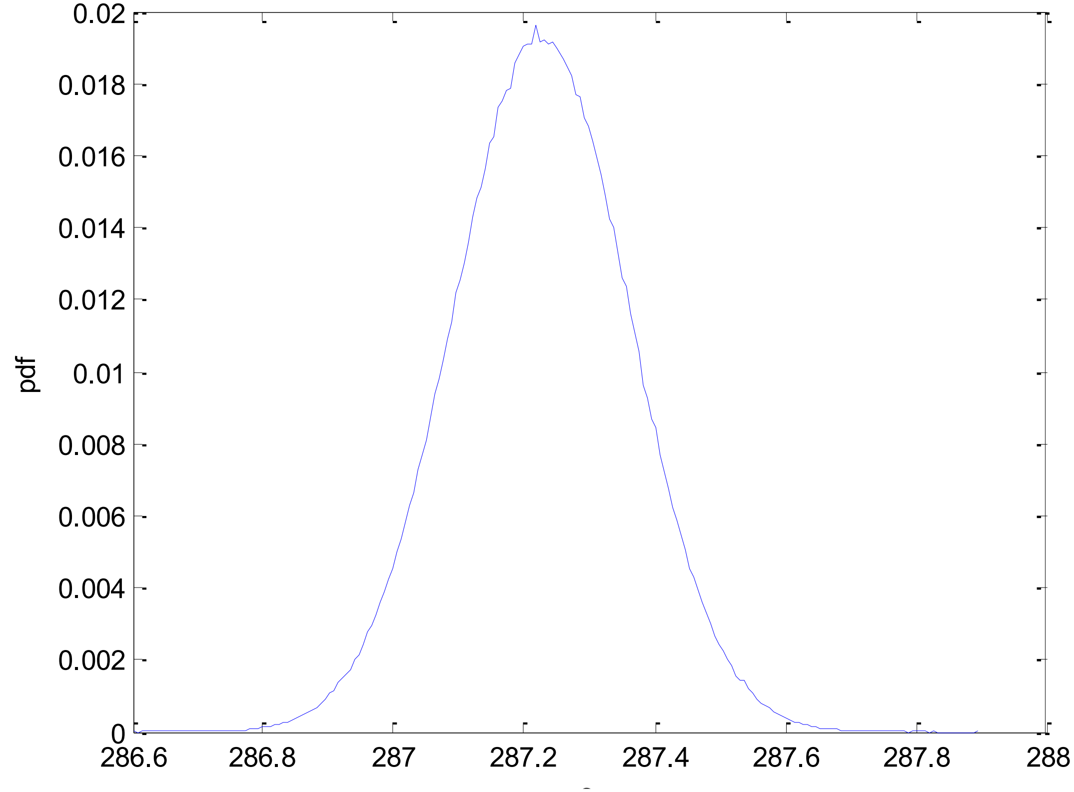
1e6 Simulations

- $p_f = 2.120 \times 10^{-4}$



Reliability–Based Mechanism Synthesis

- Find: Design variables (average mechanism dimensions) $\boldsymbol{\mu}_{\mathrm{R}}$
- Minimize: average motion error $\varepsilon = |\mu_{\psi} - \psi_r|$ or other objective
- Subject to reliability constraint $\Pr\{|\psi -$

Conclusions

- For important mechanisms in important applications, it is imperative to consider reliability.
- Uncertainty can be modeled probabilistically.
- Reliability can be estimated by FOSM and MCS.
- Same methodologies can also be used for cams, gears, and other mechanisms.